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CF-105

WING INFLUENCE COEFFICIENTS

A. Thomann [signature] June 1954.

A. V. ROE CANADA LIMITED
MALTON - ONTARIO
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A METHOD OF WING DEFLECTION ANALYSIS

SUMMARY

This is a method for calculating influence coefficients in multi-spar wings of any plan form. Shear deflection, chordwise bending and taper are taken into account. All the bending material is concentrated at the rib and spar booms and the skin is assumed to carry only shear. The method is particularly useful in the early stages of design as it is rapid and gives a good internal load distribution.

INTRODUCTION

The present trend towards swept thin wings and multispar structures, which are often designed from stiffness considerations more than by strength, has prompted the development of new methods of analysis which give good results whereas the conventional methods are unsatisfactory.

These methods must be fairly rapid, give accurate influence coefficients and at the same time yield an internal load distribution good enough to base upon it the preliminary design of most of the scantlings.

IDEALIZATION OF THE STRUCTURE

The point of intersection between two or more beams will be called a node and the portion of a beam between two nodes, a segment.

The structure is reduced to a system of beams (spars, ribs) and torque boxes. The latter being connected to the network of beams by vertical shear at their corner nodes. As each node represents an equation in the final matrix, it is profitable to simplify the system as much as possible, within the accuracy required. This is usually done by lumping together ribs and/or spars.

The ability of the cover skin to carry shear and end load is obtained by concentrating its effective bending material at the rib and spar booms and then assuming that the skin carries only shear.

NOTATION

E	Young's modulus (lb./in.^{2})
G	Shear modulus of rigidity (lb./in.^{2})
I	Moment of inertia (in.^{4})
ℓ	Length of segment (in.)
c	Depth of segment (in.)
t	Skin thickness (in.)
T	Torque box vertical shear force (lb.)
V	Segment vertical shear force (lb.)
M	Segment end bending moments (lb.in.)
F	External applied shear force (lb.)
y_x	Vertical displacement of node "x" (in.)
ϕ_x, ψ_x	Angular rotations of node "x" (radians)
ϕ_{xy}	Angular rotation at end "x" of segment "xy" (radians)
α, β, λ, μ	Segment stiffness coefficients
xy	Suffixes; "x" being node at which action takes place in member "xy"
A	Web shear area = ct
P	Load transferred to a beam at a node.
$\bar{q}$	Average shear flow (lb./in.)
x, z	Co-ordinates of skewed shear panels.
k	Shear panel load distribution constant
B	Web or panel area.
K	Strain energy coefficient due to end load.

NOTATION (Continued)

U	Strain energy
Δ	Rows of stiffness matrix representing angular displacement coefficients.
h	$\sqrt[4]{z_A z_B z_C z_D}$
a	Length between poles of quadrilateral panels.
x, y	Cotangents of angles in quadrilateral panels.
α, β	Angles in quadrilateral shear panels.
ϕ, θ	Angles in trapezoidal panels.

METHOD

The internal forces are first expressed as functions of the displacements at the nodes by means of stiffness coefficients. Next the equations of equilibrium, at each node, are established and then the conditions of compatibility of deformation are brought in. The solution of this system of equations gives the influence coefficients.

This method could be applied to each node in turn but it would produce a very large stiffness matrix. Fortunately, the equilibrium of the straight portion of any beam can be considered independently from the remainder of the structure, providing no external moments are applied at its central nodes. Therefore a "submatrix" is formed from the equilibrium relationships of the beam and as the moments at the central nodes are zero, the corresponding rotation terms can be eliminated. The submatrix being "reduced" by these rows and columns. The shear equations will be put equal to an externally applied load which represents the load transferred to the beam at these nodes. At this stage the end conditions are brought in, which often allows further reduction.

The procedure then follows the outline of the first paragraph. The internal loads can be obtained by substituting the influence coefficients back into the submatrices and considering the equilibrium of the beams.

SEGMENTS STIFFNESS COEFFICIENTS

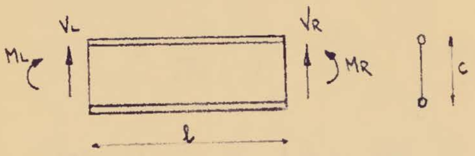


FIG. 1 SEGMENT LOADS

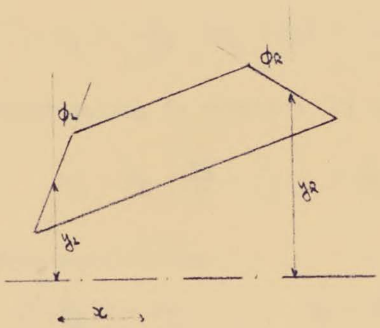


FIG. 2 SEGMENT DEFORMATIONS

Consider a parallel segment loaded and deforming as shown in Figs. 1 & 2 respectively.

At a point "x" we have:

Bending moment $= M_x = M_L + V_L . x$

Shear force $= S_x = V_L$

~~Strain energy due to bending~~

Strain energy due to bending

$$= U_B = \int_0^{\ell} \frac{M_x^2}{2EI} dx$$

Strain energy due to shear

$$= U_S \int_0^{\ell} \frac{S_x^2}{2AG} dx$$

Total strain energy $= U = U_B + U_S$

By Castigliano's First Theorem $\frac{\partial U}{\partial V_L} = (y_L - y_R) + \phi_R \ell \qquad \frac{\partial U}{\partial M_L} = (\phi_L + \phi_R)$

Differentiating the equations for the strain energy and equating to the displacements gives:

$$\frac{\ell^2}{EI}\left\{\frac{M_L}{2} + \frac{V_L \ell}{3}\right\} + \frac{V_L \ell}{AG} = (y_L - y_R) + \phi_R \ell \qquad (1)$$

$$\frac{\ell}{EI}\left\{M_L + \frac{V_L \ell}{2}\right\} = (\phi_L + \phi_R) \qquad (2)$$

Multiplying equation (2) by "$\ell/2$ and subtracting from (1) gives:

$$V_L \frac{\ell^3}{4EI}\left(\frac{1}{3} + \frac{4EI}{GA\ell^2}\right) = (y_L - y_R) + \frac{\ell}{2}(\phi_R - \phi_L)$$

Let $\omega = \frac{\ell}{EI}\left(\frac{1}{3} + \frac{4EI}{GA\ell^2}\right)$

$$\therefore \quad V_L = \frac{4}{\omega\ell^2}(y_L - y_R) + \frac{2}{\omega\ell}(\phi_R - \phi_L) \qquad (3)$$

Substituting in equation (2) and simplifying

$$M_L = \phi_L\left(\frac{EI}{\ell} + \frac{1}{\omega}\right) + \phi_R\left(\frac{EI}{\ell} - \frac{1}{\omega}\right) - \frac{2}{\omega\ell}(y_L - y_R) \qquad (4)$$

From equilibrium

$$V_R = -V_L \qquad M_R = M_L + V_L \cdot \ell$$

Substituting equations (3) and (4) in the equation for "M_R", we obtain:

$$M_R = \phi_L\left(\frac{EI}{\ell} - \frac{1}{\omega}\right) + \phi_R\left(\frac{EI}{\ell} + \frac{1}{\omega}\right) + \frac{2}{\omega\ell}(y_L - y_R) \qquad (5)$$

The following symbols are introduced to simplify the notation:

$$\alpha = \left(\frac{EI}{\ell} + \frac{1}{\omega}\right) \quad \beta = \left(\frac{EI}{\ell} - \frac{1}{\omega}\right) \quad \lambda = \frac{2}{\omega\ell} \quad \mu = \frac{4}{\omega\ell^2} \qquad (6)$$

where $\omega = \frac{\ell}{EI}\left(\frac{1}{3} + \frac{4EI}{GA\ell^2}\right)$

The forces and moments are now written in the general form:

$$V_{xy} = \mu_1 (y_x - y_y) - \lambda_1(\phi_{xy} - \phi_{yx}) \qquad (7)$$

$$M_{xy} = \alpha_1 \phi_{xy} + \beta_1 \phi_{yx} - \lambda_1(y_x - y_y) \qquad (8)$$

The stiffness coefficients require only a single index for identification. - (e.g. 1 as above)

<u>Note:</u>

When the segment is tapered and/or the section properties vary along the length, it is usually sufficient to take the average values. Should it be felt that more accuracy is required, each segment can be broken up into smaller segments to form a segment submatrix which can then be solved to give the stiffness coefficients.

TORQUE BOX STIFFNESS COEFFICIENTS

Consider a torque box loaded by external loads as shown in Fig. 3. The top and bottom skin panels are quadrilaterals and the webs are tapered in depth.

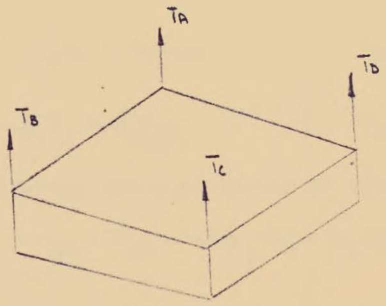


FIG. 3 EXTERNAL LOADS.

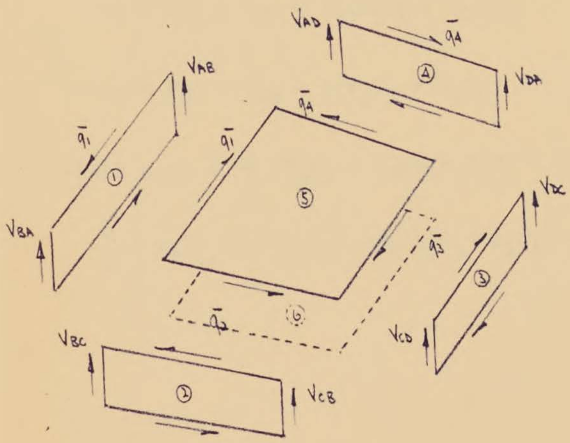


FIG. 4 INTERNAL LOADS.

The internal loads are:

Cover skin:

$$\bar{q}_1 = k_1 q \qquad \bar{q}_2 = k_2 q \qquad \bar{q}_3 = k_3 q \qquad \bar{q}_4 = k_4 q \tag{9}$$

where "k's" vary with the geometry of the panel and are defined in the appendix.

Webs:

$$V_{AB} = -\bar{q}_1 \cdot C_B = -k_1 \cdot C_B \cdot q \qquad V_{BA} = \bar{q}_1 \cdot C_A = k_1 C_A \cdot q$$

$$V_{BC} = \bar{q}_2 \cdot C_C = k_2 C_C \cdot q \qquad V_{CB} = -\bar{q}_2 C_B = -k_2 C_B \cdot q \tag{10}$$

$$V_{CD} = -\bar{q}_3 C_D = -k_3 C_D q \qquad V_{DC} = \bar{q}_3 C_C = k_3 C_C q$$

$$V_{DA} = \bar{q}_4 C_A = k_4 C_A q \qquad V_{AD} = -\bar{q}_4 C_D = -k_4 C_D q$$

Considering the equilibrium of internal and external loads:

$$T_A = V_{AB} + V_{AD} = -q(k_1 \cdot C_B + k_4 \cdot C_D)$$

$$T_B = V_{BA} + V_{BC} = q(k_2 C_C + k_1 C_A) \tag{11}$$

$$T_C = V_{CB} + V_{CD} = -q(k_3 C_D + k_2 C_B)$$

$$T_D = V_{DC} + V_{DA} = q(k_4 C_A + k_3 C_C)$$

$$\text{Work Done} = \frac{1}{2}(T_A y_A + T_B y_B + T_C y_C + T_D y_D) =$$

$$-\frac{q}{2}\left\{ y_A(k_1 \cdot C_B + k_4 C_D) - y_B(k_2 C_C + k_1 C_A) + y_C(k_3 C_D + k_2 C_B) - y_D(k_4 C_A + k_3 C_C) \right\} \tag{12}$$

EQUILIBRIUM EQUATIONS

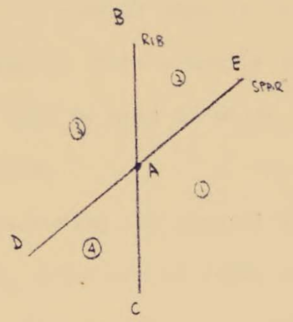


FIG. 5 STRAIGHT BEAM

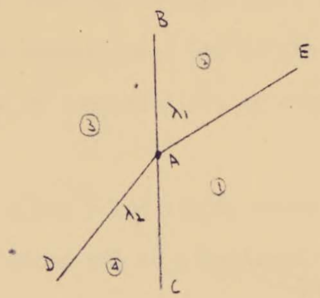


FIG. 6. KINKED BEAM

Consider the junction "A" of two beams, one of which is kinked at that node (see Fig. 6). It follows that there will be a transfer of bending moment and for that reason the bending equilibrium of that ~~of that~~ node cannot be dissociated from the remainder of the structure. Similarly for any node which is loaded by an externally applied moment. Therefore, we have one equation of vertical equilibrium for each node and two bending equilibrium equation for nodes of the type outlined above.

General Case (See Fig. 5)

$$F_A = P_{A\,spar} + P_{A\,rib} + T_{A1} + T_{A2} + T_{A3} + T_{A4} \qquad (16)$$

Special Case (See Fig. 6)

$$F_A = P_{AB} + P_{AC} + P_{AD} + P_{AE} + T_{A1} + T_{A2} + T_{A3}\ T_{A4}$$

$$- M_{AE} \sin \lambda_1 + M_{AD} \sin \lambda_2 = 0 \qquad (16a)$$

$$M_{AE} \cos \lambda_1 + M_{AB} - M_{AC} - M_{AD} \cos \lambda_2 = 0$$

Should there be an applied moment "M_A", one of the moment equations would be equal to "M_A" and not zero.

COMPATIBILITY OF ANGULAR DEFORMATION:

In the general case it is easy to show that $\phi_{AE} = -\phi_{AD}$ (Ref. Fig. 5) for instance. This property is used in the submatrices, but in the special case we need to write the equations of compatibility of angular deformation.

Considering the example shown in Fig. 6 and 7 and taking rotations ϕ_A and ψ_A which are at right angle to each other and in directions corresponding to the moment equilibrium, we have:

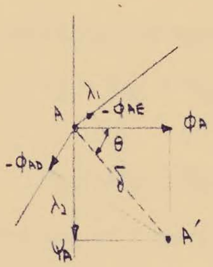


FIG. 7. ANGULAR DEFORMATIONS.

Take point "A" as moving to "A'" and introduce the unknown "δ" and "θ".

$\phi_A = \delta \cos \theta \qquad \psi_A = \delta \sin \theta$

$\phi_{AB} = \psi_{18} \qquad \phi_{AC} = -\psi_{18}$

$$\phi_{AE} = -\delta \cos(90-\theta-\lambda_1) = \phi_A \sin \lambda_1 + \psi_A \cos \lambda_1 \qquad (17)$$

$$\phi_{AD} = -\delta \cos(90+\lambda_2-\theta) = \phi_A \sin \lambda_2 - \psi_A \cos \lambda_2$$

The rotations are shown as the action they produce on the under surface of the wing.

SUBMATRICES

Consider the straight portion of a beam, as shown in Fig. 8. Only three segments are taken, for simplification and as the extension to more segments is simple.

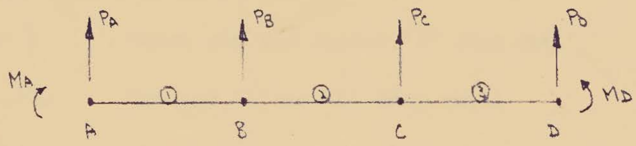


FIG. 8. STRAIGHT BEAM LOADS.

Writing the equations of equilibrium:

$$P_A = V_{AB} \qquad P_B = V_{BA} + V_{BC} \qquad P_C = V_{CB} + V_{CD} \qquad P_D = V_{DC}$$

$$M_A = M_{AB} \qquad M_B = M_{BC} - M_{BA} \qquad M_C = M_{CD} - M_{CB} \qquad -M_D = -M_{DC} \tag{18}$$

In matrix form and as functions of displacements, using equation (7) and (8) See Table I.

TABLE I

			1	2	3	4	5	6	7	8		
1	P_A		μ_1	$-\lambda_1$	$-\mu_1$	$-\lambda_1$						y_A
2	M_A		$-\lambda_1$	α_1	λ_1	$-\beta_1$						ϕ_{AB}
3	P_B		$-\mu_1$	λ_1	$(\mu_1+\mu_2)$	$(\lambda_1-\lambda_2)$	$-\mu_2$	$-\lambda_2$				y_B
4	M_B	=	$-\lambda_1$	$-\beta_1$	$(\lambda_1-\lambda_2)$	$(\alpha_1+\alpha_2)$	λ_2	$-\beta_2$			X	ϕ_{BC}
5	P_C				$-\mu_2$	λ_2	$(\mu_2+\mu_3)$	$(\lambda_2-\lambda_3)$	$-\mu_3$	$-\lambda_3$		y_C
6	M_C				$-\lambda_2$	$-\beta_2$	$(\lambda_2-\lambda_3)$	$(\alpha_2+\alpha_3)$	λ_3	$-\beta_3$		ϕ_{CD}
7	P_D						$-\mu_3$	λ_3	μ_3	λ_3		y_D
8	$-M_D$						$-\lambda_3$	$-\beta_3$	λ_3	α_3		$-\phi_{DC}$

STIFFNESS COEFFICIENTS MATRIX

Once the submatrices have been reduced and the angular deformation relationship brought in they, together with the torque box forces, can be substituted in the equilibrium equations to give the stiffness coefficients matrix. Now, as for the submatrices, we can reduce the columns corresponding to the rows which are equal to zero. These rows "Δ" which usually represent the angular deformation coefficients are then removed and recorded. We now have the stiffness matrix in the form:

$$\left[\delta\right] \left\{y\right\} \qquad (19)$$

where $\left[\delta\right]$ = stiffness matrix $\left\{y\right\}$ = unknown deflections.

INFLUENCE COEFFICIENT MATRIX

This is imply the inverse of the stiffness matrix - i.e.

$$\left[\delta\right]^{-1}$$

INTERNAL LOADS DISTRIBUTION

Having the inverse we can readily obtain the deflections for any loading by using the equation:

$$\left\{ y_F \right\} = \left[\delta \right]^{-1} \left\{ F \right\} \qquad (20)$$

where F = an external loading case.

The angular displacements are calculated:

$$\left\{ \phi_F \right\} = \left[\Delta \right] \left\{ y_F \right\} \qquad (21)$$

By substituting into the reduced submatrices we can now calculate the loads transferred to the beams at each node "P". The shear and bending moments in the beams are easily obtained by considering its equilibrium.

From the torque box equations (15) by substituting for the appropriate "y", we obtain "q" which is the average shear flow in the cover skin panel. Next the additional shear flows in the webs are calculated using "q" and equations (10) and these are applied as corrections to the shears in the beams.

COMPUTATION

The work involved in an analysis of this type can be divided under the following main headings:

1. Formation of the Stiffness Matrix.
2. Inversion of the Stiffness Matrix.
3. Investigation of Various Loading Cases.
4. Structural Changes and Parameter Studies.

MACHINERY

For simple structures with few nodes, the work can be handled by desk calculators, but when the structure is complex and has many nodes a computing machine is necessary so as to keep to a minimum the time required for the analysis.

STRUCTURAL CHANGES AND CORRECTION OF ERRORS

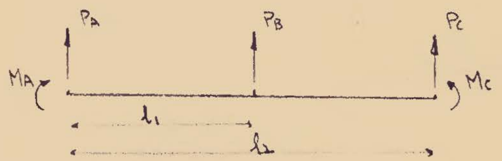
Both structural changes and errors in forming the stiffness matrix have the same result, that is, a number or a group of numbers in the stiffness matrix has to be changed and a new matrix of influence coefficients must be obtained. The method given in Ref. 2 is very useful at this point, to quote:

Given: Original matrix $= A_o = \begin{vmatrix} B & C \\ D & E \end{vmatrix}$ $\qquad A_o^{-1} = \begin{vmatrix} K & L \\ M & N \end{vmatrix}$

New matrix $= A = A_o + a = \begin{vmatrix} B_1 & C \\ D & E \end{vmatrix}$ where $a = \begin{vmatrix} b & 0 \\ 0 & 0 \end{vmatrix}$

For the submatrices before reduction the symmetry check is sufficient providing the geometry and stiffness coefficients are correct. After reduction we have, say -

$$\begin{vmatrix} M_A \\ P_A \\ P_B \\ P_C \\ M_C \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \begin{vmatrix} \phi_{AB} \\ y_A \\ y_B \\ y_C \\ \phi_{CB} \end{vmatrix}$$



Summing up the vertical forces

$$(a_{2j} + a_{3j} + a_{4j}) = 0$$

FIG. 9. BEAM REPRESENTED BY SUBMATRIX

Taking moments $(-a_{1j} + a_{3j} \cdot \ell_1 + a_{4j} \cdot \ell_2 + a_{5j}) = 0$

Symmetry i.e. $a_{ij} = a_{ji}$

It must be emphasized that the checks for torque boxes and submatrices must agree absolutely, even if it means adjusting a few of the figures.

To check after the inversion of the matrix we take any loading case and find the internal loads and moments transferred at the nodes. These are then substituted numerically in the equilibrium equations and a reasonable agreement should be expected.

FORM 1319A

If the errors are found too large and can be traced to the inversion of the matrix the following corrective procedure from Reference 3, para. 4.11, Page 120 can be adopted.

$$A^{-1} = \eta\,(2I - A\eta) \qquad (23)$$

where A = given matrix η = approximate inverse.

APPENDIX

SHEAR PANELS

The derivation of the equilibrium conditions and the strain energy of quadrilateral shear panels is given in Reference 1; the results are quoted here for convenience.

To obtain the relationships between the average shear flows, it is advisable to draw the panel to scale and measure off the constants required; this is sufficiently accurate. For calculating the strain energy the exact formula may be used, but when two sides are nearly parallel the evaluation is difficult. In most cases sufficient accuracy is obtained by using the formula for trapezoidal panels.

The shears relationships and the strain energy for each type of panel is now given.

PARALLELOGRAM

Taking $\frac{x_1}{x_2} = 1$ and $\theta_1 = \theta_2 = \theta$ gives:

$$k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 1 \qquad K_5 = K_6 = \cot an^2 \theta$$

RECTANGLE:

Taking $\theta = 90^\circ$

$$k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 1 \qquad K_5 = K_6 = 0$$

TRIANGULAR PANELS

The accurate analysis of triangular panels is involved and not easily adaptable to this type of analysis. An approximation is therefore desired.

In general triangular panels are lightly loaded in average shear and for that reason could be omitted altogether. The difficulty could also be by-passed, see Fig. 13, by lumping together the two panels "1.2.4.5 and 2.3.5" to form the torque box "1.3.4.5" which is then considered attached at these points.

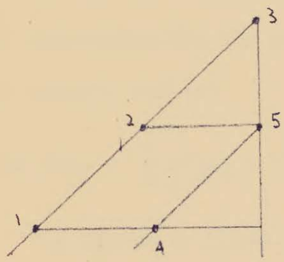


FIG. 13 STRUCTURE WITH TRIANGULAR PANELS.

However, these approximations are not sufficient in all cases, say, for instance at the root triangle of a swept wing where the more accurate approximation is essential.

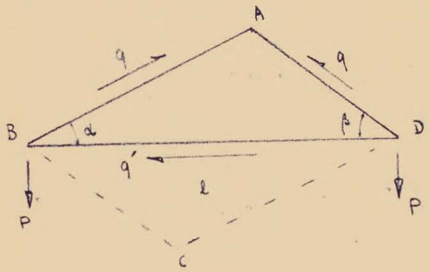


FIG. 14 APPLIED LOADS ON TRIANGLE.

The triangle is assumed to be half of a parallelogram. The applied loads being as shown in Fig. 14, "P" representing end loads on the face "BD" which are concentrated at the points "B and D".

From equilibrium $q' = -\dfrac{q \sin(\alpha - \beta)}{\sin(\alpha + \beta)}$ $\quad P = \dfrac{q\,\ell \sin\alpha \sin\beta}{\sin(\alpha + \beta)}$

Taking half the strain energy of a parallelogram gives:

$$U = \frac{q^2 B}{2Gt}\left\{1 + \frac{4G}{E}\cot^2(\alpha + \beta)\right\}$$

As $\cot\theta = \cot\left\{180 - (\alpha + \beta)\right\} = -\cot(\alpha + \beta)$

and where B = area of triangle.

To find the stiffness coefficients the method is the same as the one already described for the torque box,

The expression for the work done is modified to take into account the moments and is:

$$\text{Work done} = \frac{1}{2}\left\{T_A y_A + T_B y_B + T_C y_C + P(C_B \cdot \phi_B + C_D \cdot \phi_D)\right\}$$

It should be noted that the moments will affect the segments attached at those nodes, which means that these rotation terms cannot be eliminated in the submatrices. These nodes are treated as special cases, see "Equilibrium Equations" and " Compatibility of Angular Deformation".

CONDITION FOR PANEL TO BE IN A PLANE

It was stated in the main part of the report that the top and bottom panels are flat and that the depths are not independent. With reference to Fig. 10, the necessary conditions are as follows:

The equation of the plane is:

$$\begin{vmatrix} x_A & z_A & C_A & 1 \\ x_B & z_B & C_B & 1 \\ x_C & z_C & C_C & 1 \\ x_D & z_D & C_D & 1 \end{vmatrix} = 0$$

Now $x_A = z_A . x_1$ $\qquad z_A = \dfrac{a}{x_1 + y_1}$ $\qquad$ etc....

Substituting in the determinant and simplifying we obtain:

$$\frac{C_A}{z_A} - \frac{C_B}{z_B} + \frac{C_C}{z_C} - \frac{C_D}{z_D} = 0$$

Unless this condition is satisfied the sum of the torque boxes vertical shears is not zero.

A. V ROE CANADA LIMITED
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