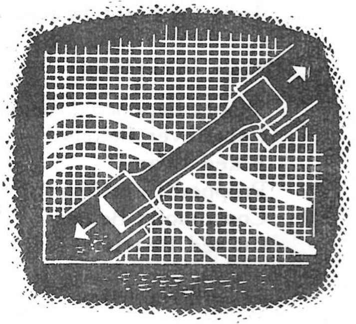
Stress



SECTION V

STRESS

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December 7, 1954

AIRCRAFT ENGINEERING MANUAL

VOLUME 1 (DESIGN)

ERRATA - SECTION V (Stress)

The following errors have been noted in the Stress Manual and should be corrected pending re-issue of the relevant pages.

(1) Sub-Sect. 2.2.2, Table 2.2.2-3, Row 14, column "Reference":

Figure 2.2.2-3 should read 2.2.2-6

Rows 35, 36, 37, column "Reference":

Sec. 6.1 should be deleted.

Figure 2.2.2-13 - Indexing of ordinates R_s:

For 1.1 read 1.2; for 1.2 read 1.4.

(2) Sub-Sect. 4.2.1. At D = 1 1/4 in., t = .049 in., column ρ_x:

Value .1250 should read .4250.

(3) Sub-Sect. 6.5.7, Fig. 6.5.7-1. Curves "B" and "D" are incorrect and should not be used.

This page should be inserted at the front of your copy of the Stress Section.

C L Moon

C.L. Moon,
Standards Supervisor.

SECTION V

STRESS

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2.2.2 PLANE TENSION FIELD BEAMS

2.2.2.1 General

Beam Analysis

The following sections present a method of analysis for partial tension field beams with parallel flanges and perpendicular uprights. Information, largely in the form of graphs, is provided for the analysis of beams constructed of 24S-T and 75S-T alloys.

Beam Design

For convenience, a table (Table 2.2.2-2) giving properties of standard extruded angles used as single uprights is included. A table of design information (Table 2.2.2-1) is presented which should enable the designer to select approximate web gauge, upright size, rivet sizes, etc. For a given shear flow, the necessary web gauge is listed, along with a large and small upright spacing, thus allowing the designer to determine the effect of this spacing on the remaining design variables. The information in this table applies only when the beam flanges are very stiff; nevertheless it should be useful as a first approximation for tension field beam design.

Effective Beam Depth

It is considered sufficiently accurate to take the distance between flange centroids, h, as a common dimension for the effective depth of the beam, depth of the web, and length of the upright.

Limitations

The curves given herein are believed to give reasonable assurance of conservative strength predictions provided that normal design practices and proportions are used. The most important points are:

(i) The upright should not be too thin, keep $\frac{t_u}{t} > 0.6$

(ii) The upright spacing should be in the range $0.2 < \frac{d}{h} < 1.0$

(iii) The method of analysis presented here is applicable only to beams with webs in the range $115 < \frac{h}{t} < 1500$.

References

1. "Strength Analysis of Stiffened Beam Webs", by Paul Kuhn and James P. Peterson, NACA, T.N. 1364, July 1947.
2. "Ultimate Stresses Developed by 24S-T and Alclad 75S-T Aluminum-Alloy Sheet in Incomplete Diagonal Tension", by L. Ross Levin, NACA, T.N. 1756, November 1948.
3. "Strength Analysis of Stiffened Thick Beam Webs", by L. Ross Levin and Charles W. Sandlin, Jr. NACA, T.N. 1820, March 1949.
4. A summary of Diagonal Tension Part I by Kuhn, Peterson, and Levin, NACA T.N. 2661.
5. Strength of Aircraft Elements ANC-5, June 1951.

Symbols

A_f	Average area of tension and compression flanges, in.^{2}
A_u	Actual area of upright, in.^{2}
A_{u_e}	Effective area of upright, in.^{2}
C_1, C_2, C_3	Stress concentration factors
C_r	Rivet factor $= \frac{p - D}{p}$
D	Rivet diameter, in.
E	Modulus of elasticity, p.s.i.
F_{co}	Column yield stress, p.s.i.
F_{col}	Allowable column stress, p.s.i.
F_{max}	Ultimate allowable compressive stress for natural crippling, p.s.i.
F_o	Ultimate allowable compressive stress for forced crippling, p.s.i.
F_s	Ultimate allowable web shear stress, p.s.i.
$F_{s_{cr}}$	Critical shear stress for initial buckling of web, p.s.i.
I_f	Average moment of inertia of beam flanges, in.^{4}
M_{sb}	Secondary bending moment in the flange, in. lb.
P	Applied shear load, lb.

P_u	Upright end load, lb.
d	Spacing of uprights, in.
e	Distance from upright centroid to web, in.
f_{cent}	Compressive stress at centroid of upright, p.s.i.
f_f	Compressive stress in flange due to horizontal component of diagonal tension, p.s.i.
f_s	Shear stress in web, p.s.i.
f_u	Average lengthwise compressive stress in upright, p.s.i.
$f_{u\,max}$	Maximum compressive stress in upright, p.s.i.
h	Effective depth of beam, centroid of compression flange to centroid of tension flange, in.
k	Diagonal tension factor
p	Rivet pitch, in.
q	Beam shear flow, lb/in.
t	Thickness of web, in.
t_u	Thickness of upright, in.
α	Angle of diagonal tension, degrees
ρ	Radius of gyration of upright with respect to its centroidal axis parallel to web, in. (No portion of web to be included.)

2.2.2.2 Analysis of Web

Web Shear Stress

The web shear flow can be closely approximated by, $q = \frac{P}{h}$

From this it follows that the web shear stress is,

$$f_s = \frac{q}{t}$$

Diagonal Tension Factor

The diagonal tension field factor, k, can be found from Figure 2.2.2-4 as a function of the web shear stress, f_s, and the ratios d/t and d/h. This factor specifies the portion of the total shear that is carried by the diagonal tension action of the web. The curves contain a practical approximation for values in the plastic range and the simplifying assumption is made that the shear panel has simply supported edges. The more exact method which attempts to account for varying degrees of edge restraint, as employed for shear resistant webs, is considered an unnecessary refinement in the case of tension field webs because the effect of a more exact value of k is small. Moreover, the effect of edge restraint is considered in the allowable web shear stresses where the difference is appreciable.

It is recommended that the diagonal tension factor at ultimate load be limited to a maximum value,

$$k_{max} = .75 - 2.6t$$

in order to avoid excessive wrinkling and permanent set at limit load and thereby inviting fatigue failure.

Allowable Web Shear Stresses

The allowable web shear stress, F_s, can be obtained from Figure 2.2.2-11 for 75S-T Alclad sheet or Figure 2.2.2-12 for 24S-T Alclad sheet. These values are based on tests of long webs subjected to loads approximating pure shear, and contain an allowance for the rivet factor, C_r. This factor is included because tests have shown that the ultimate allowable shear stress based on the gross section is almost constant in the normal range of the rivet factor. ($C_r > 0.6$). The values for F_s are given as a function of the stress concentration factor, C_2, which can be found from Figure 2.2.2-6 as a function of ωd. Unfortunately the higher values of C_2 are largely theoretical since very few tests, if any, were made with beams having extremely flexible flanges. On the basis of a few scattered tests, the indications are that the values of C_2 become increasingly conservative for the higher values of the flange flexibility parameter. However, until more test data is available, the present values remain the closest conservative approximation.

The NACA also gives an additional stress concentration factor, C_1, due to the diagonal tension angle differing from the 45° angle associated with the state of pure diagonal tension. However, this correction is always small and is felt to be unwarranted in view of the conservatism employed throughout the analysis.

2.2.2.3 Analysis of Uprights

Effective Area of Upright

The total cross sectional area of the upright for double or single uprights is given as A_u, and no part of the web is to be included in any case. However, if the upright is attached to a rib or bulkhead an effective width of the bulkhead web may be included. In order to make the design charts apply to both single and double uprights the following effective upright areas, A_{u_e}, are to be used in the analysis.

(i) For double upright symmetrical with respect to web,

$$A_{u_e} = A_u$$

(ii) For single uprights,

$$A_{u_e} = \frac{A_u}{1 + (e/\rho)^2}$$

See Table 2.2.2-2 for properties of standard extruded angles commonly used as uprights.

Design Criteria

The uprights or web stiffeners must have sufficient moment of inertia to prevent buckling of the web panel as a whole before formation of the tension field, and to prevent column failure under the loads imposed on the upright by the tension field. Furthermore, the upright must be thick enough to prevent forced crippling failure caused by the waves of the buckled web. The latter mode of failure is almost always the most critical. From the values given in Table 2.2.2-1 a good first approximation for A_u seems to be in the range A_u equals 0.6 dt to 0.8 dt.

Moment of Inertia

The required moment of inertia of the upright about its own neutral axis parallel to the web is obtained from Figure 2.2.2-7 and is a function of ht^3 and the ratio d/h. Essentially, the curve is derived by equating the critical buckling stress of the sheet between stiffeners to the general instability stress of the stiffened web as a whole. As the stiffener spacing decreases, the required moment of inertia increases as a consequence of the higher critical buckling stress of the sheet.

Stresses

The lengthwise average stress in the upright at the faying surface of the leg attached to the web, f_u, is obtained from Figure 2.2.2-5 as a function of k, $\frac{A_{u_e}}{dt}$, and f_s as

$$f_u = f_s \, (f_u / f_s)$$

The upright stress varies from a maximum at the neutral axis of the beam to a minimum at the ends of the upright due to the gusset effect. The maximum stress, $f_{u_{max}}$, is obtained from Figure 2.2.2-5 as a function of k, d/h, and f_u as,

$$f_{u_{max}} = f_u \, (f_{u_{max}} / f_u)$$

Allowables

The maximum upright stress, $f_{u_{max}}$, should be checked against an allowable forced crippling stress, F_o, which is obtained from Figure 2.2.2-10. In the case of double uprights only, $f_{u_{max}}$ should also be checked against an allowable natural crippling stress, F_{max}, as calculated from the information given in Section 3.1.1 of this Manual.

The stress f_u should be no greater than the column yield stress, F_{co}.

The centroidal upright stress, $f_{cent} = f_u \frac{A_{u_e}}{A_u}$ should be checked against an allowable column stress, F_{col}, which is obtained from Figure 2.2.2-10 for the slenderness ratio, h'/ρ.

For simplicity, h' may be taken as h, since this effect is rarely critical. However if necessary, an effective column length h', may be used where,

$$h' = \frac{h}{\sqrt{1 + k^2 \, (3 - \frac{2d}{h})}} \text{ ....for double uprights}$$

$$h' = \frac{h}{2} \text{ ....for single uprights}$$

The theoretical "column" types of failure are greatly at variance with test results but strength checks given are the best available to date.

2.2.2.4 Analysis of Rivets

Rivets: Web-to-Flange

The shear load per inch acting on the web-to-flange rivets is given in Figure 2.2.2-8. It should be pointed out that the rivet factor, C_r, should be not less than 0.6 in order to justify the allowable web stresses used and to avoid undue stress concentration.

Rivets: Web-to-Upright

The tensile load per inch acting on the web-to-upright is given in Figure 2.2.2-9. The tensile loads on the rivets are exerted by the prying action of the buckled web. The criteria used is a variation of that suggested by the NACA because the degree of diagonal tension is accounted for which is considered to be a more reasonable and realistic approximation. However, these loads should be considered only tentative because of the meager test data available at this time, but nevertheless, they do reflect time tested practice.

The tensile load criteria is believed to insure a satisfactory design as far as shear strength is concerned. The NACA has suggested a lesser tensile load criteria for double uprights but added a formula to check the total rivet strength in shear. This latter formula does not account for the restraining action of the web among other things and consequently is not recommended.

Rivets: Upright-to-Flange

The shear load on the upright to flange rivets is given by the formula,

$$P_u = (f_u)\,(A_{u_e})$$

(Note: $A_{u_e} = A_u$ for double uprights)

This formula ignores the gusset effect in order to be conservative and also neglects certain other forces, the net effect of which makes the above formula a close approximation.

2.2.2.5 Analysis of Flange

The total stress in the flanges is the result of the superposition of three individual stresses; the primary beam stress, the compressive stress caused by the horizontal component of the diagonal tension in the web, and the secondary bending stress, caused by the distributed vertical component of the diagonal tension.

Compressive stresses due to the horizontal component of diagonal tension

$$f_f = \frac{kqh}{2A_f + .5\,(1 - k)\,th}$$

Secondary bending moment due to the distributed vertical component of diagonal tension:

$$M_{sb} = \frac{1}{12}\, k\, f_s\, td^2\, C_3$$

where C_3 is given in Figure 2.2.2-6 (Compression is on the innermost fiber of the flange). M_{sb} is the maximum moment in the bay and exists at the ends of the bay over the uprights. If C_3 and k are near unity, the moment in the middle of the bay is half as large as that given and of opposite sign. Shear beam flanges are often attached to upper and lower surfaces etc. The effects of these webs must be computed in the same manner and superimposed upon those from the main shear web.

2.2.2.6 Combined Loading

Interaction Curves

This section may be used to solve plane rectangular panels and cylinders subjected to shear and normal stress or shear and transverse stress. Reference N.A.C.A. T.N. 1223.

The combination of shear and axial or transverse stress that causes plane panels to buckle may be obtained from Figures 2.2.2-13 and 2.2.2-14. When axial or transverse stress are known to exist in a plane shear web, it is necessary to alter the beam analysis in a manner similar to the solution of combined loadings shown in Section 2.2.5-1.

TABLE 2.2.2.-1

TENSION FIELD BEAM DESIGN DATA

75S-T6 Alclad Webs - 75S-T6 Extruded Angle Uprights

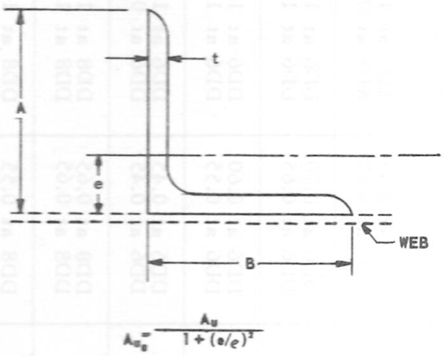
The data in this table has been worked out to give positive margins of safety for ultimate allowable shear flows indicated and applies only to beams with very stiff flanges. (See Sect. 2.2.2.1). It is intended that this table serve only as a general guide in proportioning tension field beams. For each shear flow the necessary web gauge is given and two widely different upright spacings are listed. This should enable the designer to evaluate this effect and interpolate accordingly. From the values given in the table, a good first approximation for A_u seems to be in the range A_u equals 0.6 dt to 0.8 dt.

ULT. ALLOW SHEAR FLOW	WEB THICKNESS	UPRIGHT SPACING	BEAM DEPTH	UPRIGHT (SMALL LEG AT WEB)		RIVET SIZE AND EFFECTIVE SPACING		
q	t	d	h	DIMENSIONS	A_u/dt	Web to Flange	Web to Upright	Upright to Flange
500	.020	4	6	5/8 x 1/2 x .050	.67	AD3 at 0.30	AD3 at 0.60	2 - AD4
500	.020	7	11	1 x 3/4 x .063	.77	AD3 at 0.30	AD3 at 0.60	2 - DD6
625	.025	4	6	5/8 x 5/8 x .063	.75	AD4 at 0.45	AD4 at 0.80	2 - AD4
625	.025	8	12	3/4 x 3/4 x .094	.65	AD4 at 0.40	AD4 at 0.80	2 - DD6
800	.032	4	6	3/4 x 3/4 x .063	.71	AD5 at 0.55	AD5 at 1.05	2 - AD5
800	.032	9	14	1-1/4 x 1 x .094	.70	AD5 at 0.50	AD5 at 0.95	2 - DD8
1050	.040	4	6	1-1/4 x 3/4 x .063	.775	AD5 at 0.45	AD5 at 1.10	2 - AD5
1000	.040	10	15	1-1/4 x 1-1/4 x .125	.745	AD5 at 0.45	AD5 at 0.95	2 - DD8
1350	.051	5	8	1 x 7/8 x .094	.655	DD6 at 0.70	DD6 at 1.30	1 - DD6
1300	.051	11	17	1-3/4 x 1-1/2 x .156	.85	DD6 at 0.65	DD6 at 1.10	2 - ALPPH-T6
1800	.064	5	8	1-1/4 x 1 x .094	.63	DD6 at 0.60	DD6 at 1.30	1 - DD6
1650	.064	12	18	2 x 1-1/2 x .156	.675	DD6 at 0.55	DD6 at 1.00	2 - ALPPH-T6
2300	.081	6	9	1-1/4 x 1-1/4 x .125	.615	DD6 at 0.45	DD6 at 1.15	1 - DD6
2100	.081	13	20	2-1/2 x 2 x .188	.77	DD6 at 0.45	DD6 at 0.90	3 - ALPPH-T6
2900	.102	7	11	1-1/2 x 1-1/4 x .156	.56	DD8 at 0.65	DD8 at 1.50	1 - DD8
2600	.102	14	21	2 x 2 x .250	.645	DD8 at 0.65	DD8 at 1.30	3 - ALPPH-T6
3650	.125	8	12	2 x 1-1/2 x .156	.52	DD8 at 0.55	DD8 at 1.40	1 - DD8
3200	.125	15	23	2-1/2 x 2-1/2 x .250	.625	DD8 at 0.55	DD8 at 1.20	2 - ALPPH-T8
4700	.156	9	14	2 x 1-3/4 x .188	.47	DD8 at 0.40	DD8 at 1.25	1 - DD8
4300	.156	16	24	3 x 2-1/2 x .313	.64	DD8 at 0.40	DD8 at 1.05	2 - ALPPH-T8

TABLE 2.2.2.-2

PROPERTIES OF STANDARD EXTRUDED UPRIGHTS

Reference: AN Standards, Volume 2



AND 10133

ANGLE – EQUAL LEG, EXTRUDED

Dash No.	A	t	A_u	ρ_{yy}	I_{yy}	A_{u_e}
0401	.500	.063	.0582	.147	.0013	.0297
0501	.625	.063	.0749	.186	.0026	.0400
0502		.078	.0908	.185	.0031	.0467
0503		.094	.1068	.181	.0035	.0525
0601	.750	.063	.0908	.227	.0047	.0498
0602		.094	.130	.220	.0063	.0665
0603		.125	.167	.216	.0078	.0802
0701	.875	.063	.107	.265	.0075	.0597
0702		.094	.154	.262	.0106	.0819
0703		.125	.198	.256	.0129	.0990
1001	1.000	.063	.128	.302	.0118	.0735
1002		.094	.183	.296	.0160	.0989
1003		.125	.235	.293	.0202	.121
1201	1.250	.063	.159	.383	.0234	.0932
1202		.094	.230	.379	.0330	.129
1203		.125	.298	.373	.0414	.160
1204		.188	.427	.362	.0561	.210
1401	1.500	.063	.191	.464	.0411	.113
1402		.094	.277	.460	.0585	.158
1403		.125	.360	.454	.0741	.199
1404		.188	.521	.443	.1021	.268
1601	1.750	.094	.330	.536	.0947	.192
1602		.125	.429	.532	.1214	.242
1603		.188	.621	.521	.1683	.329
2001	2.000	.094	.377	.617	.1436	.222
2002		.125	.491	.612	.1838	.281
2003		.188	.715	.601	.2582	.388
2004		.250	.924	.591	.3222	.475
2401	2.500	.125	.616	.774	.3689	.359
2402		.188	.903	.762	.5248	.504
2403		.250	1.17	.751	.6628	.628
3001	3.000	.188	1.09	.924	.9306	.621
3002		.250	1.42	.912	1.1834	.783
3003		.313	1.75	.900	1.4200	.932

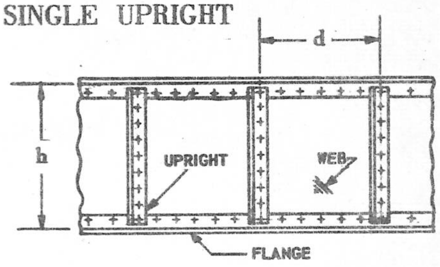
AND 10134

ANGLE – UNEQUAL LEG, EXTRUDED

Dash No.	A	B	t	A_u	ρ_{yy}	I_{yy}	A_{u_e}
0501	.625	.500	.050	.0535	.193	.0020	.0276
0601	.750	.500	.063	.0739	.235	.0041	.0357
0602	.875	.625	.063	.0818	.232	.0044	.0427
0701		.625	.063	.0922	.274	.0069	.0470
0702		.750	.063	.100	.270	.0073	.0539
0703			.125	.184	.258	.0123	.0872
1001	1.000	.625	.063	.100	.316	.0100	.0494
1002			.125	.184	.304	.0170	.0807
1003		.750	.063	.108	.313	.0106	.0565
1004			.125	.200	.302	.0182	.0937
1005		.875	.063	.116	.310	.0111	.0637
1006			.094	.167	.305	.0155	.0872
1201	1.250	.750	.063	.124	.400	.0197	.0616
1202			.094	.179	.393	.0276	.0845
1203			.125	.231	.387	.0346	.104
1204		1.000	.063	.139	.394	.0217	.0756
1205			.094	.202	.389	.0306	.105
1206			.125	.262	.382	.0383	.130
1401	1.500	.750	.094	.204	.476	.0463	.0927
1402			.125	.264	.465	.0584	.114
1403		1.000	.094	.228	.474	.0512	.114
1404			.125	.295	.468	.0647	.142
1405			.156	.361	.463	.0772	.169
1406		1.250	.094	.251	.468	.0550	.135
1407			.125	.327	.462	.0698	.170
1408			.156	.400	.457	.0834	.201
1601	1.750	1.000	.125	.327	.553	.0998	.153
1602		1.250	.125	.358	.549	.1078	.180
1603		1.500	.125	.389	.542	.1144	.208
1604			.156	.478	.537	.1377	.248
2001	2.000	1.000	.125	.358	.637	.1453	.162
2002			.156	.439	.631	.1745	.193
2003		1.250	.125	.392	.643	.1576	.195
2004			.156	.480	.629	.1893	.229
2005			.188	.568	.622	.2197	.263
2006		1.500	.125	.423	.629	.1675	.221
2007			.156	.519	.623	.2016	.263
2008			.188	.615	.618	.2346	.304
2009		1.750	.125	.454	.623	.1760	.249
2010			.156	.558	.617	.2125	.298
2011			.188	.662	.611	.2472	.344
2401	2.500	1.250	.156	.558	.796	.3540	.252
2402			.188	.662	.790	.4137	.292
2403		1.500	.156	.597	.795	.3773	.288
2404			.188	.709	.789	.4416	.334
2405		1.750	.156	.636	.791	.3978	.324
2406			.188	.756	.785	.4660	.376
2407		2.000	.156	.681	.783	.4174	.363
2408			.188	.809	.777	.4889	.421
2409		2.250	.156	.720	.776	.4333	.398
2410			.188	.856	.770	.5077	.462
3001	3.000	1.500	.188	.809	.956	.7387	.367
3002			.250	1.050	.943	.9323	.458
3003		2.000	.188	.903	.950	.8159	.451
3004			.250	1.17	.938	1.034	.564
3005		2.500	.250	1.30	.927	1.115	.676
3006			.313	1.60	.915	1.334	.803

TABLE 2.2.2.–3

SAMPLE CALCULATIONS FOR BEAM WITH SINGLE UPRIGHT



Given: $q = 1300$ lb./in., $h = 10''$, $d = 8''$

Web – .051 75S–T6 Alclad
Beam flanges – 1.25 x 1.25 x .188 75S–T6 Extruded Angle
Upright – 1.00 x 1.00 x .125 75S–T6 Extruded Angle
Rivets – web to flange, DD6 at .7 O.C.
– web to upright, AD5 at .8 O.C.
– flange to upright, 2 – DD6

	EQUATION	REFERENCE	VALUE	ROW	
Shear Flow	q		1300	1	
Web Thickness	t		.051	2	
Beam Depth	h		10	3	
Upright Spacing	d		8	4	
Upright Thickness	t_u		.125	5	
Upright Area	A_u		.235	6	
Upright Radius of Gyration	ρ		.293	7	
Upright Moment of Inertia	I_u		.0202	8	
Upright Effective Area	A_{u_e}		.121	9	
Flange Moment of Inertia	I_f		.0561	10	
Web Shear Stress	f_s	① / ②	25,500	11	
PARAMETERS	d/t	④ / ②	157	12	
	d/h	④ / ③	.80	13	
	ωd	Figure 2.2.2.–3	.46	14	← R
	A_{u_e}/dt	⑨ / ④ ②	.296	15	
	$ht^3 \times 10^3$	③ $②^3$ x 10^3	1.33	16	
	t_u/t	⑤ / ②	2.45	17	
	h/ρ	③ / ⑦	34.2	18	
Diagonal Tension Factor	k	Figure 2.2.2.–4	.42	19	R
Ratio – Upright to Web Stresses	f_u/f_s	Figure 2.2.2.–5	.61	20	↓
Ratio – Upright Stresses	$f_{u_{max}}/f_u$	Figure 2.2.2.–5	1.15	21	
Stress Concentration Factor – Flange Flexibility	C_2	Figure 2.2.2.–6	.22	22	
Stress Concentration Factor – Secondary Bend. Mom.	C_3	Figure 2.2.2.–6	.90	23	
Minimum Mom. of Inertia Required for Upright	I_s	Figure 2.2.2.–7	.0013	24	
Rivet Load – Web to Flange	q_r	Figure 2.2.2.–8	1520	25	
Rivet Load – Web to Upright	q_t	Figure 2.2.2.–9	760	26	
Upright Allowable – Column Yield Stress	F_{co}	Figure 2.2.2.–10	81,500	27	
Upright Allowable – Column	F_{col}	Figure 2.2.2.–10	67,000	28	
Upright Allowable – Forced Crippling	F_o	Figure 2.2.2.–10	19,700	29	↑
Allowable Web Stress	F_s	Figure 2.2.2.–11	25,500	30	R
Upright Stress, Average	f_u	⑪ ⑳	15,550	31	
Upright Stress, Centroidal	$f_{cent.}$	㉛ ⑨ / ⑥	8010	32	
Upright Stress, Maximum	$f_{u_{max}}$	㉛ ㉑	17,900	33	
Rivet Load, Upright to Flange	P_u	㉛ ⑨	1880	34	
Secondary Bending Moment	M_{sb}	1/12 ⑲ ⑪ ② $④^2$ ㉓	2620	35	
Allowable Rivet Load – Web to Flange		Sec. 6.1	1640	36	
Allowable Rivet Load – Web to Upright		Sec. 6.1	885	37	
Allowable Rivet Load – Upright to Flange		Sec. 6.1	2360	38	
Margin of Safety – Web		(㉚ / ⑪) – 1	.00	39	
Margin of Safety – Uprights		*	.10	40	
Margin of Safety – Rivets, Web to Flange		(㊱ / ㉕) – 1	.08	41	
Margin of Safety – Rivets, Web to Upright		(㊲ / ㉖) – 1	.16	42	
Margin of Safety – Rivets, Upright to Flange		(㊳ / ㉞) – 1	.26	43	

* Either $\frac{⑧}{㉔}$ –1, $\frac{㉗}{㉛}$ –1, $\frac{㉘}{㉜}$ –1, or $\frac{㉙}{㉝}$ –1, whichever is least.

Beam flanges are usually affected by webs and forces other than the vertical shear web and hence require separate analysis.

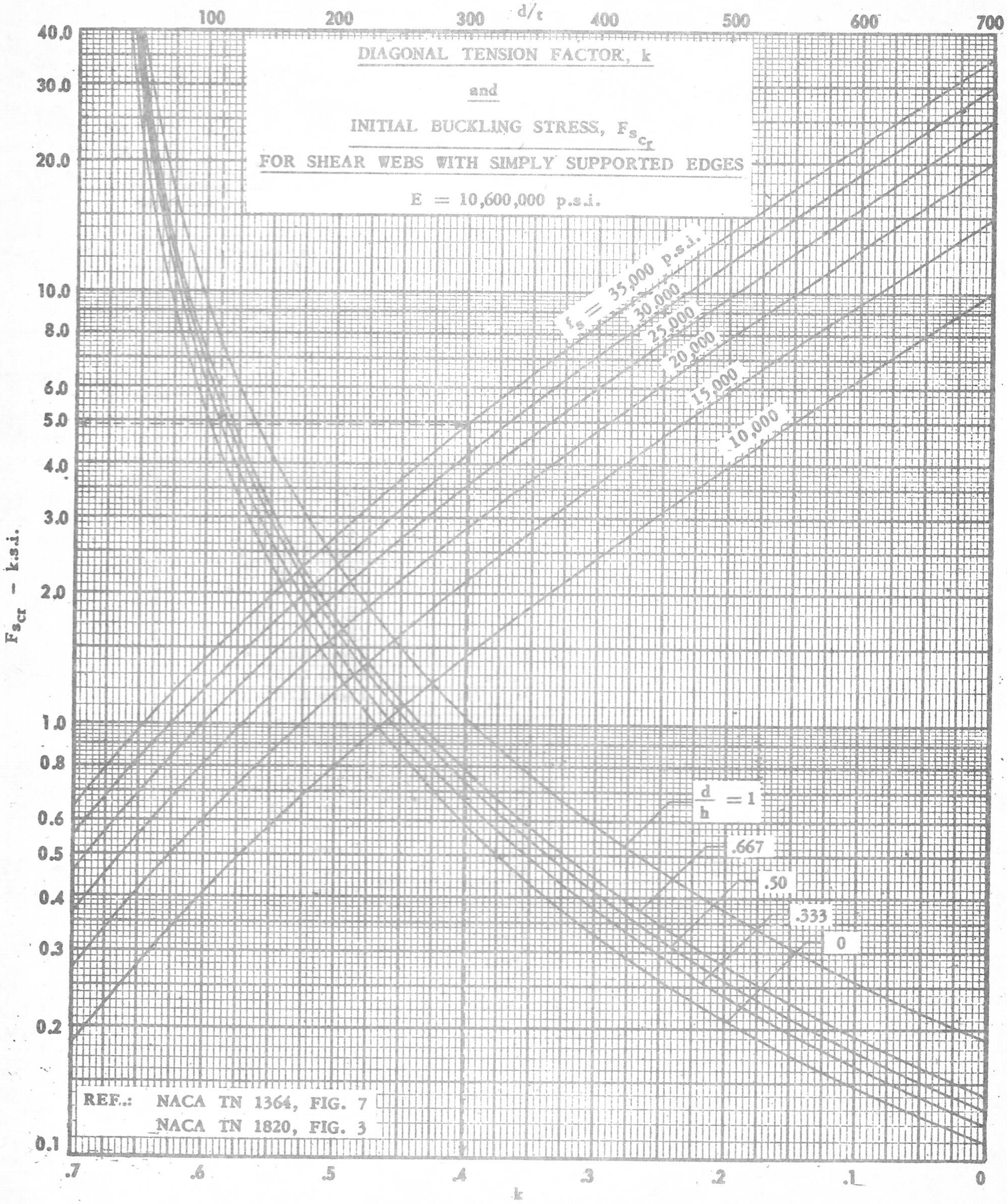
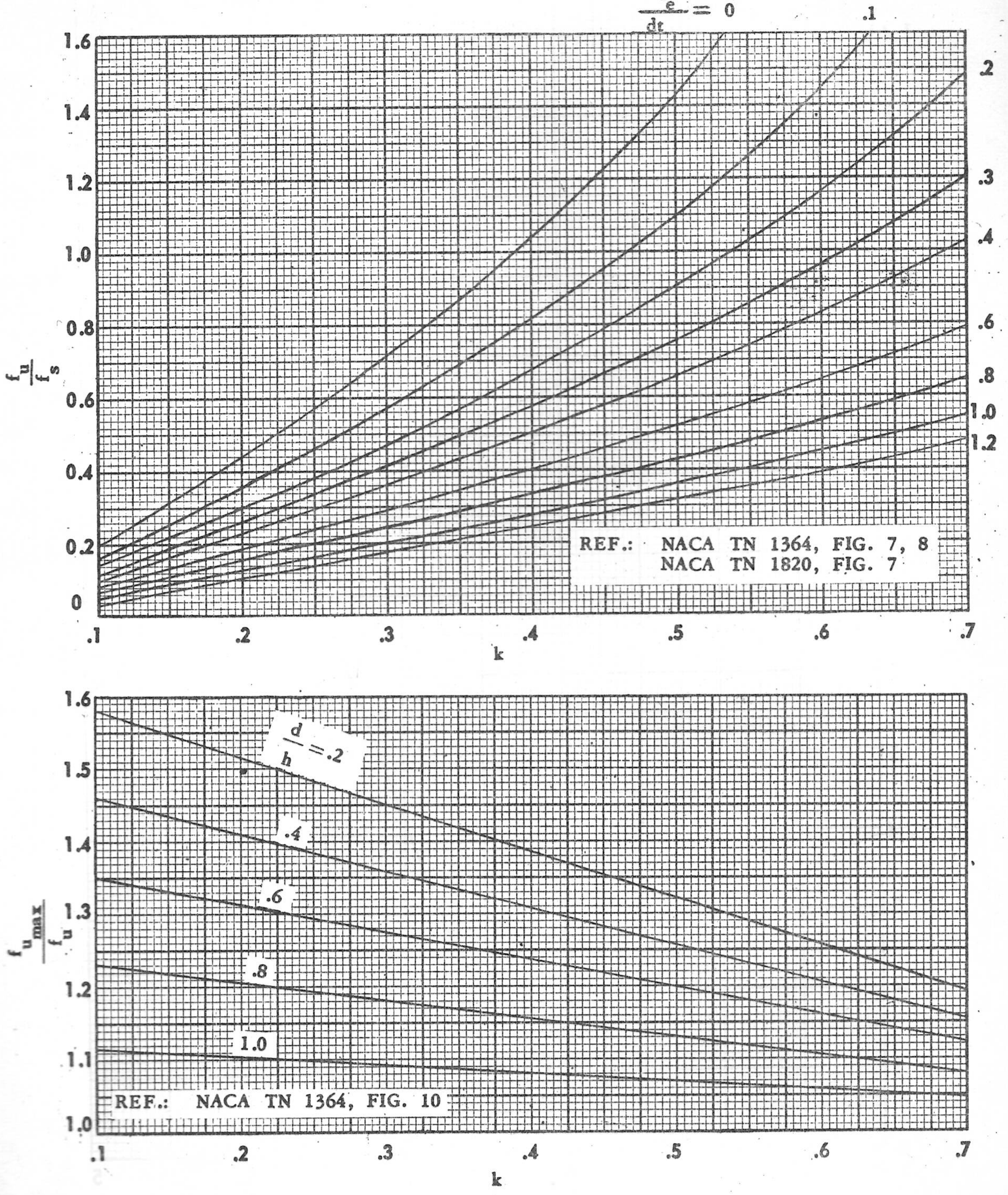

d/t
100
200
300
400
500
600
700
DIAGONAL TENSION FACTOR, k
and
INITIAL BUCKLING STRESS, $F_{s_{cr}}$
FOR SHEAR WEBS WITH SIMPLY SUPPORTED EDGES
E = 10,600,000 p.s.i.
f_s = 35,000 p.s.i.
30,000
25,000
20,000
15,000
10,000
$\frac{d}{h} = 1$
.667
.50
.333
0
40.0
30.0
20.0
10.0
8.0
6.0
5.0
4.0
3.0
2.0
1.0
0.8
0.6
0.5
0.4
0.3
0.2
0.1
$F_{s_{cr}}$ - k.s.i.
.7
.6
.5
.4
.3
.2
.1
0
k
REF.: NACA TN 1364, FIG. 7
NACA TN 1820, FIG. 3

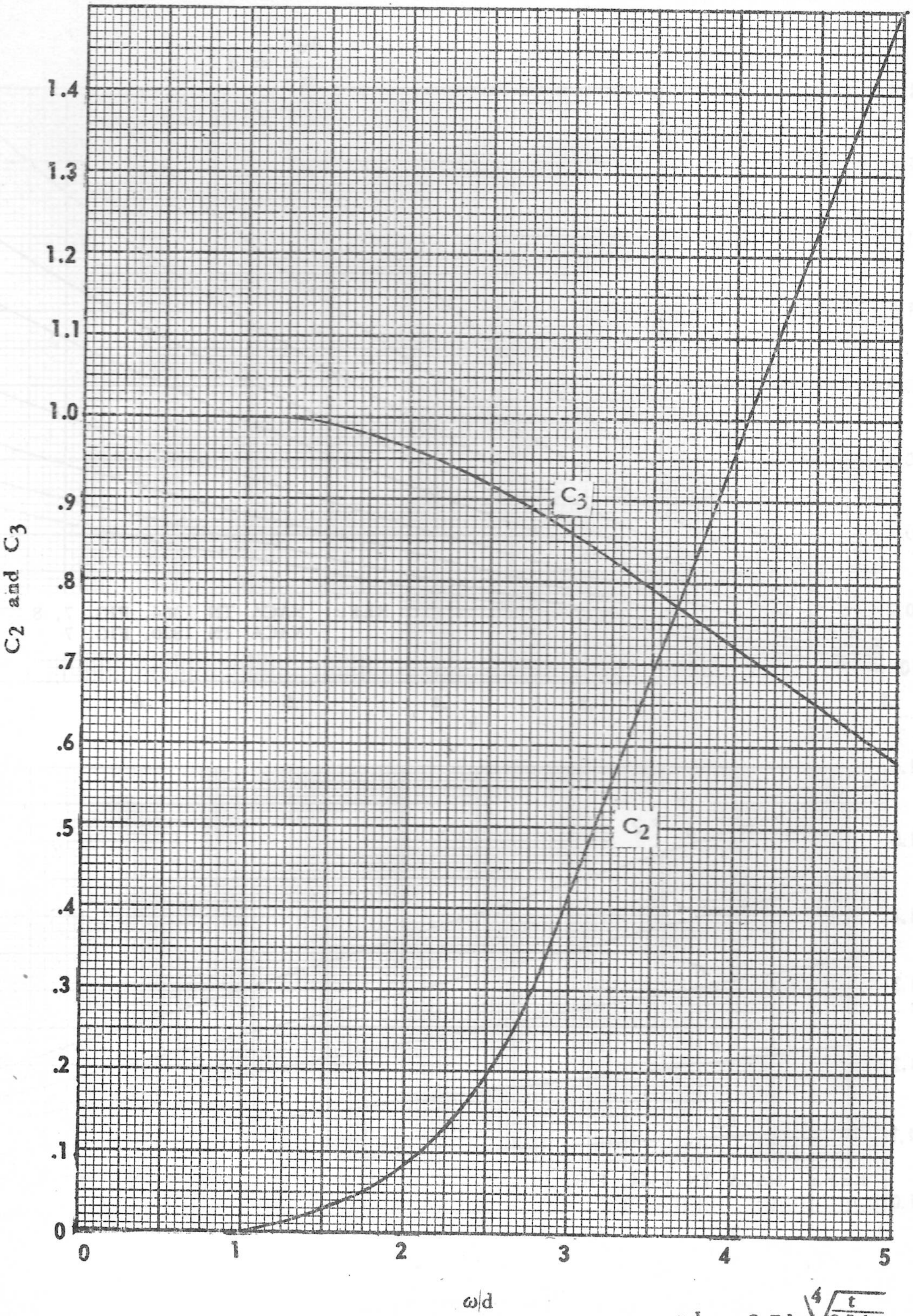
FIGURE 2.2.2-4



REF.: NACA TN 1364, FIG. 7, 8
NACA TN 1820, FIG. 7

REF.: NACA TN 1364, FIG. 10

FIGURE 2.2.2-5
UPRIGHT STRESS RATIOS



$\omega d = 0.7d \sqrt[4]{\frac{t}{2 I_{fh}}}$

FIGURE 2.2.2–6

STRESS CONCENTRATION FACTORS C_2 AND C_3

REF. N.A.C.A. T.N. 1364 FIG. 13

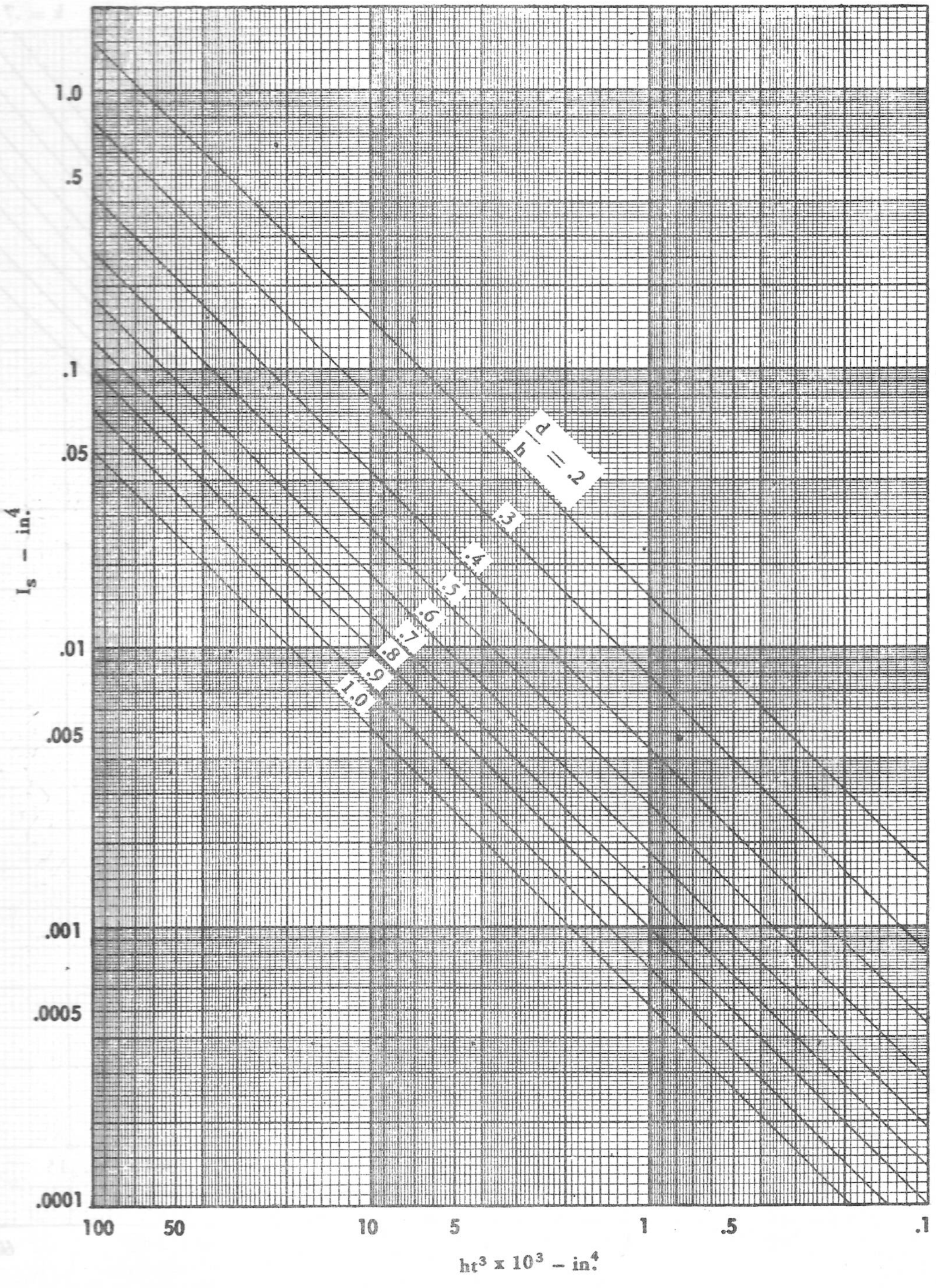


FIGURE 2.2.2-7
MINIMUM MOMENT OF INERTIA REQUIRED
FOR TENSION FIELD BEAM UPRIGHTS

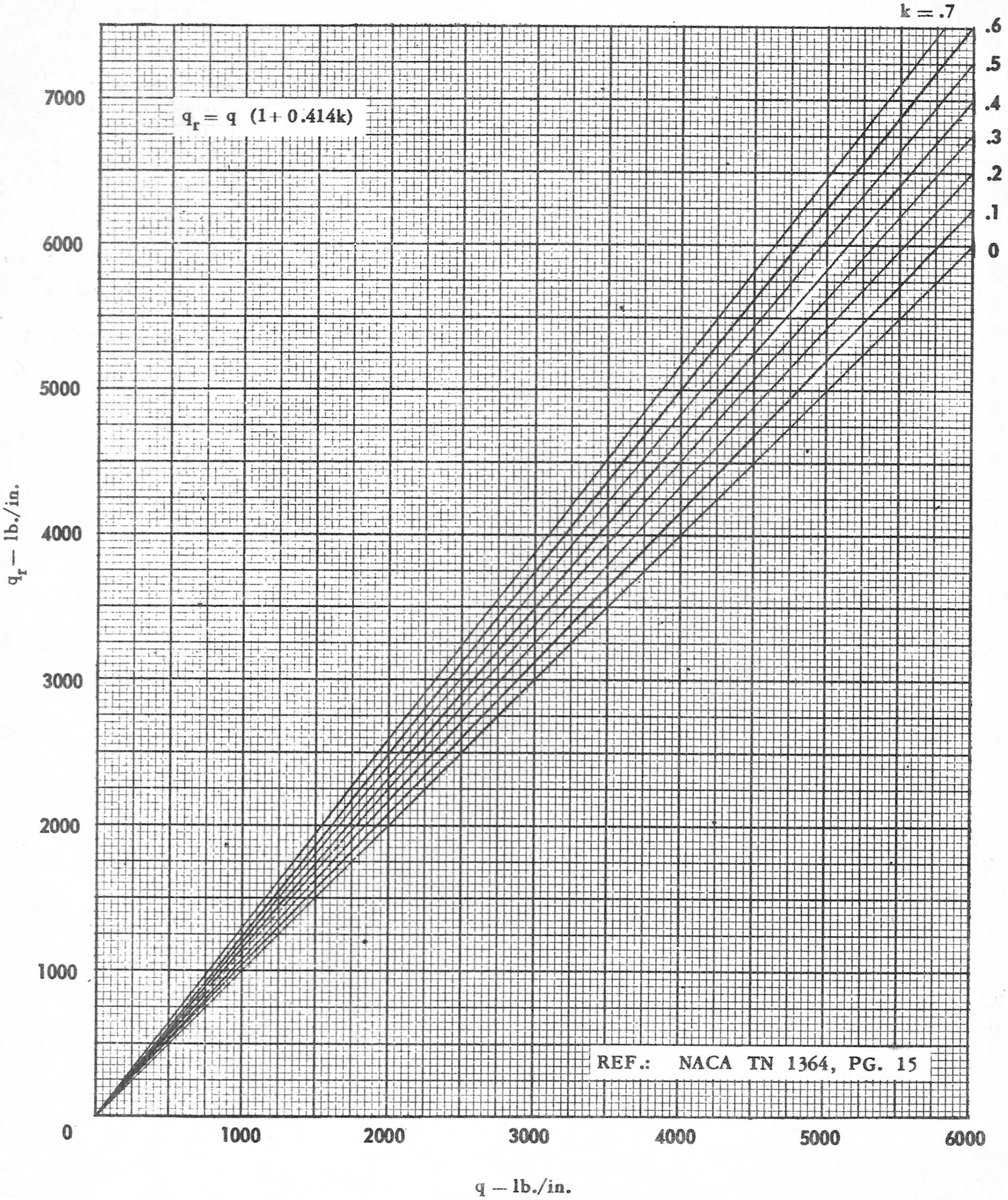
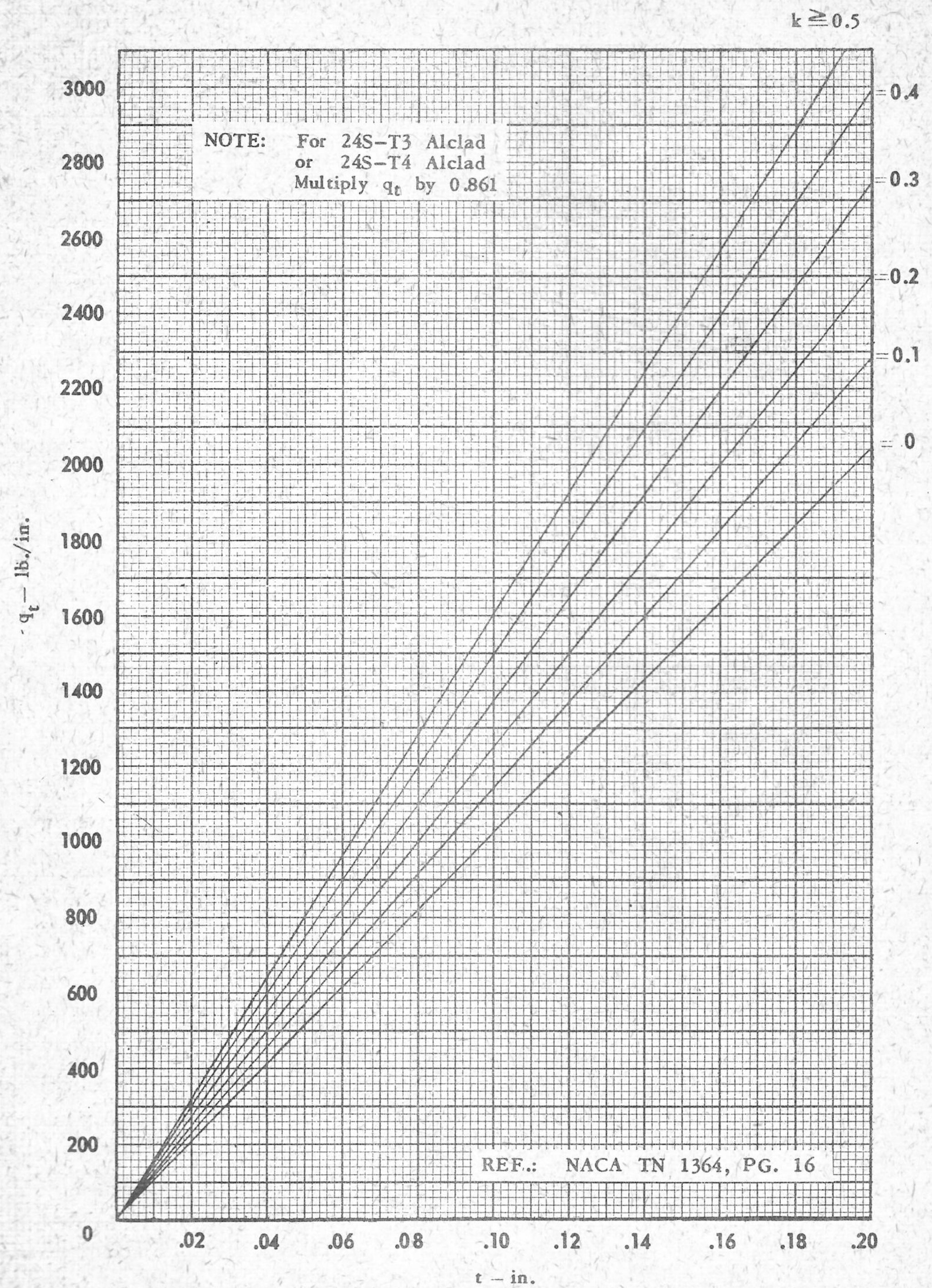


FIGURE 2.2.2-8

RIVET LOADS

WEB-TO-FLANGE AND WEB SPLICES



NOTE: For 24S–T3 Alclad or 24S–T4 Alclad Multiply q_t by 0.861

REF.: NACA TN 1364, PG. 16

FIGURE 2.2.2–9
RIVET TENSION LOADS
WEB TO UPRIGHT
75S–T6 ALCLAD

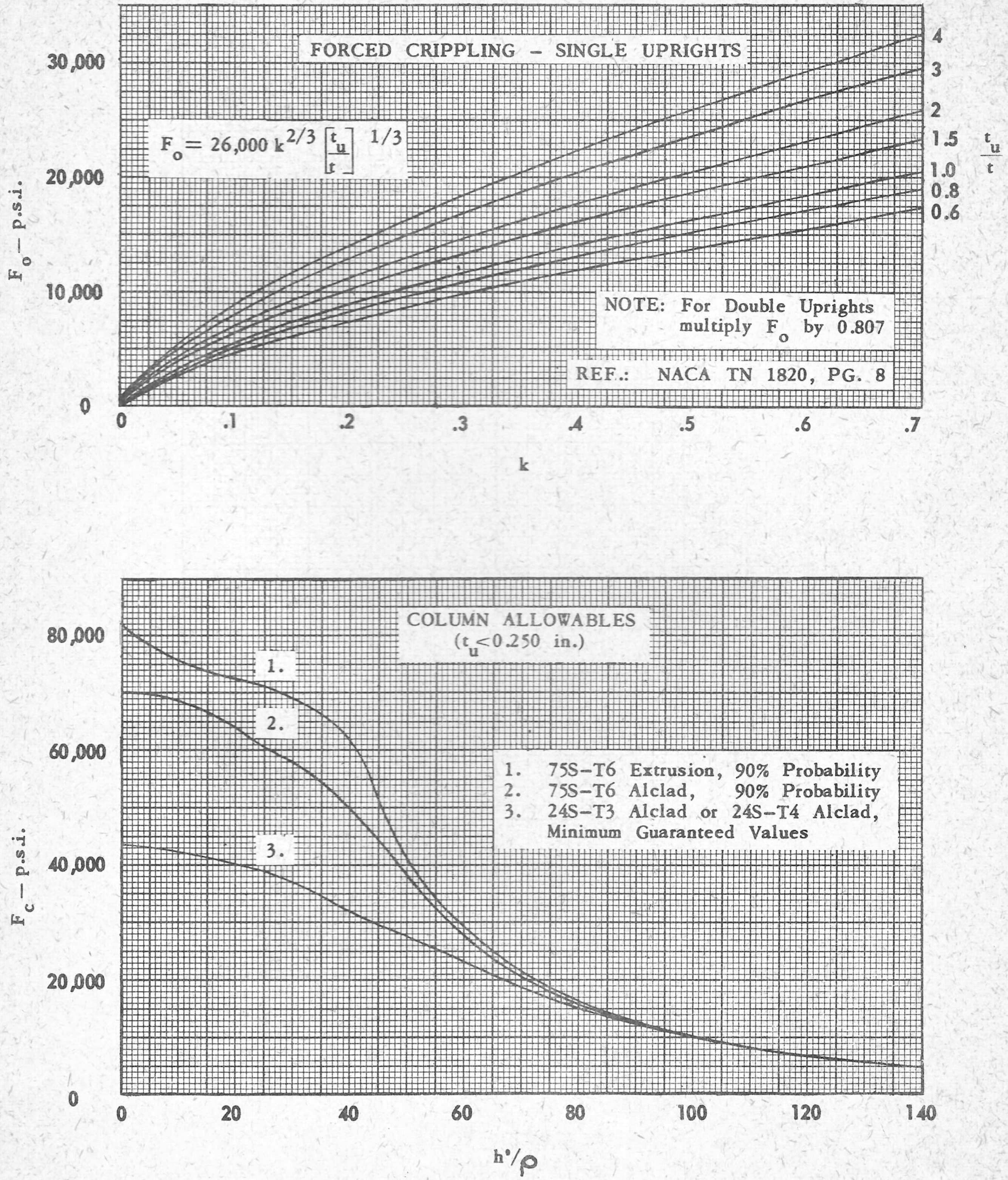


FIGURE 2.2.2–10
UPRIGHT ALLOWABLES

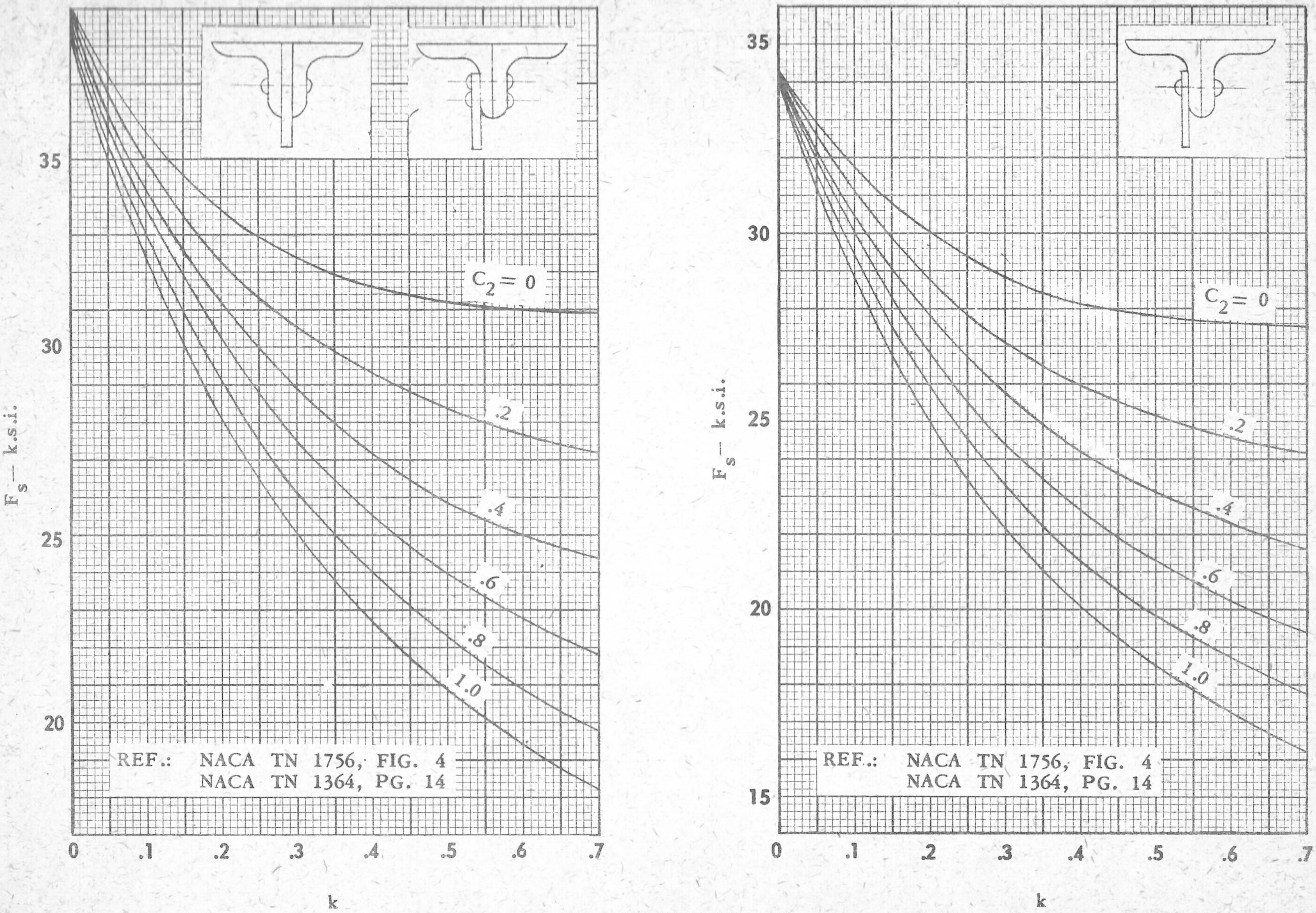
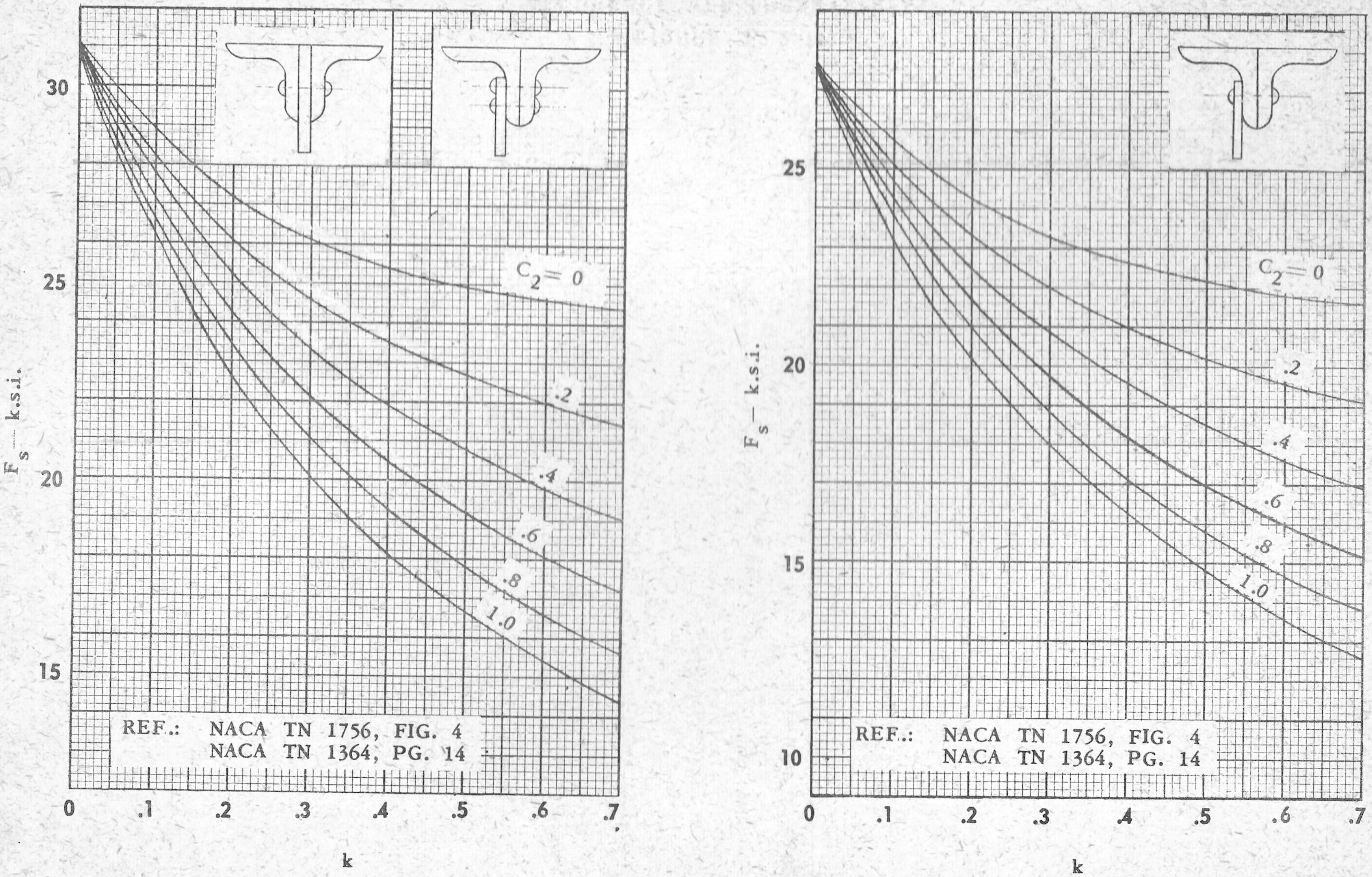


FIGURE 2.2.2-11
ALLOWABLE WEB SHEAR STRESS
75S-T6 ALCLAD



REF.: NACA TN 1756, FIG. 4
NACA TN 1364, PG. 14

REF.: NACA TN 1756, FIG. 4
NACA TN 1364, PG. 14

FIGURE 2.2.2-12
ALLOWABLE WEB SHEAR STRESS
24S-T3 ALCLAD

2.2.3 PLANE TENSION FIELD BEAMS WITH ACCESS HOLES

2.2.3.1 General

Beam Design – This section presents a method of analysis for partial tension field beams with access holes. It consists of a series of empirical corrections which are made to the analysis in Section 2.2.2 to account for the effect of the access hole. The limitations and methods of Section 2.2.2 are used in conjunction with the curves and formulas presented in this section in making the analysis. A sample design procedure is given in Table 2.2.3-1 which combines the methods of Section 2.2.2 and this section, and gives a complete solution.

Effect of a Flange on the Access Hole – The problem of flanging the access hole is entirely different for a shear resistant beam and a partial tension field beam. In a partial tension field beam, the web buckles may cause the flange around the access hole to buckle. Care must therefore be taken to design the flange so that permanent set and cracking will not take place. An integral or lip type flange does tend to develop permanent set and is definitely not recommended for partial tension field beams. The most satisfactory flange is a flat doubler ring reinforcement of about the same gauge as the web which will buckle without permanent set. However, the flange may effect no significant improvement in the static strength of the panel, since yielding of the material around small access holes takes place before failure, and the effect of stress concentration is almost eliminated. The purpose of the flat doubler ring is primarily to protect the hole against notches or cracks which materially increase the stress concentration. This stress concentration can be effectively reduced by the reinforcing rings. Since notches or cracks could be caused by accidental damage or vibration, the desirability of putting ring flanges on access holes in partial tension field beams is a design problem and should be based on the expected service experience of the beam.

2.2.3.2 References

1. Consolidated Report No. SG-673, Plate Girder Webs with Access Holes, October 1, 1941
2. Consolidated Report No. GS-766, A method for the Design of Plate Girder Webs with Access Holes, January 2, 1941
3. Tests of 10-inch 24 S-T Aluminum-Alloy Shear Panels with 1½ Inch Holes. II-Panels Having Holes with Notched Edges. by Paul Kuhn and L.R. Levin. NACA RB No. L4DO1, April 1944.

2.2.3.3 Symbols

Symbols used include those given in Section 2.2.2 and the following, which are used in accounting for the effect of the access hole:

A_e	Effective area of upright, and web, in.^{2}
A'_u	Reduced area of upright, in.^{2}
A_w	Effective area of web, in.^{2}
C_4, C_5	Design reduction factors
D	Diameter of access hole, in.
F'_o	Reduced ultimate allowable compressive stress for forced crippling p.s.i.
F'_s	Reduced allowable web shear stress, p.s.i.
I'_s	Increased minimum moment of inertia required for uprights, in.^{4}
q'_t	Increased rivet tensile load – web to upright, lb./in.

2.2.3.4 Web Design

Use the method given in Section 2.2.2.2 with a reduced allowable web shear stress F'_s, calculated as follows:

$$F'_s = \frac{F_s(A_{e/dt})}{C_5}$$

where F_s = allowable web shear stress found from Figure 2.2.2-11 as a function of k and C_2

C_5 = design reduction factor found from Figure 2.2.3-2 as a function of D/h

$A_e = A_w + A'_u$

$A_u = t\ (d\text{-}D)$

$A'_u = (A_u)(\frac{D}{d})(C_4)$

C_4 = design reduction factor found from Figure

2.2.3-2 as a function of $\frac{t_u/t}{d/h}$

This method gives good correlation with 58 test beams in the following range:

$$.020'' < t < .132''$$

$$.040'' < t_u < .079''$$

$$7.4'' < h < 19.4''$$

$$7.0'' < d < 18.0''$$

$$2.375'' < D < 5.875''$$

This method is *not* recommended for beams outside this range but it can be used as a guide in view of the fact that there is no other information or test results available at the present time.

2.2.3.5 Upright Design

Use the method given in Section 2.2.2.3 with a reduced forced crippling allowable. F'_o, and an increased minimum required moment of inertia, I_s, as follows:

$$F'_o = \frac{F_o}{1 + D/d}$$

$$I'_s = I_s\ (1 + D/d)$$

where (1 + D/d) is a correction factor which takes into account the effect of the access hole. The remainder of the stiffener analysis is not appreciably affected by the access hole and the methods of Section 2.2.2 may be used without correction.

2.2.3.6 Rivet Design

Rivets: Web-to-Flange

Use the method given in Section 2.2.2.4

Rivets: Web-to-Upright

Use the method given in Section 2.2.2.4 with an increased rivet tensile load, q'_t, given by the following equation:

$$q'_t = q_t\ (1 + D/d)$$

Rivets: Upright-to-Flange

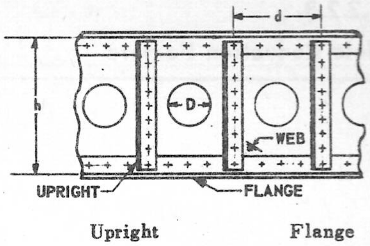
Use the method given in Section 2.2.2.4

2.2.3.7 Flange Design

Use the method given in Section 2.2.2.5

TABLE 2.2.3–1

SAMPLE CALCULATIONS FOR BEAM WITH SINGLE UPRIGHT AND ACCESS HOLE



Given: $q = 525$ lb./ins., $h = 10''$, $d = 8.5''$, $D = 4.5''$

Web – .051" 75S–T6 Alclad	
Beam Flanges 2 – 1¼" x 1¼" x .188"	75S–T6 Extruded Angle
Upright – ¾" x ¾" x .063"	75S–T6 Extruded Angle
Rivets – web to flange	AD5 at 1.00"
– web to upright	DD6 at .85"
– upright to flange,	1–DD6

	EQUATION	REFERENCE	VALUE	ROW	
Shear Flow	q		525	1	
Web Thickness	t		.051	2	
Beam Depth	h		10	3	
Upright Spacing	d		8.5	4	
Access Hole Diameter	D		4.5	5 a	
Upright Thickness	t_u		.063	6	
Upright Area	A_u		.0908	7	
Upright Radius of Gyration	ρ		.227	8	
Upright Moment of Inertia	I_u		.0047	9	
Upright Effective Area	A_{u_e}		.0498	10	
Flange Moment of Inertia	I_f		.1122	11	
Web Shear Stress	f_s	① / ②	10,300	12	
PARAMETERS	dt	④ ②	.433	13	
	d/t	④ / ②	167	14	
	d/h	④ / ③	.85	15	
	ωd	Figure 2.2.2.–3	2.31	16	
	D/h	⑤ / ③	.45	17 a	
	D/d	⑤ / ④	.53	18 a	
	$1 + D/d$	1 + ⑱	1.53	19 a	
	A_w	② (④ – ⑤)	.204	20 a	
	A_{u_e}/dt	⑩ / ⑬	.115	21	
	$ht^3 \times 10^3$	③ ②^3 x 10^3	1.325	22	
	t_u/t	⑥ / ②	1.235	23	
	h/ρ	③ / ⑧	44.0	24	
	$\frac{t_u/t}{d/h}$	㉓ / ⑮	1.45	25 a	
Diagonal Tension Factor	k	Figure 2.2.2.– 4	.26	26	R
Ratio – Upright to Web Stresses	f_u/f_s	Figure 2.2.2.– 5	.47	27	↓
Ratio – Upright Stresses	$f_{u_{max}}/f_u$	Figure 2.2.2.– 5	1.17	28	
Stress Concentration Factor – Flange Flexibility	C_2	Figure 2.2.2.– 6	.17	29	
Stress Concentration Factor – Secondary Bend. Mom.	C_3	Figure 2.2.2.– 6	.93	30	
Minimum Mom. of Inertia Required for Upright	I_s	Figure 2.2.2.– 7	.00115	31	
Rivet Load – Web to Flange	q_r	Figure 2.2.2.– 8	580	32	
Rivet Load – Web to Upright	q_t	Figure 2.2.2.– 9	670	33	
Upright Allowable – Column Yield Stress	F_{co}	Figure 2.2.2.–10	81,500	34	
Upright Allowable – Column	F_{col}	Figure 2.2.2.–10	54,000	35	
Upright Allowable – Forced Crippling	F_o	Figure 2.2.2.–10	11,200	36	
Allowable Web Stress	F_s	Figure 2.2.2.–11	28,000	37	
Design Reduction Factor Due to Access Hole	C_4	Figure 2.2.3.– 2	.465	38 a	↑
Design Reduction Factor Due to Access Hole	C_5	Figure 2.2.3.– 2	1.377	39 a	R
Reduced Area of Upright	A_u	⑦ ⑱ ㊳	.0224	40 a	
Effective Area of Upright and Web	A_e	⑳ + ㊵	.226	41 a	
Parameter	A_e/dt	㊶ / ⑬	.522	42 a	
Increased Upright Min. Mom. of Inertia Required	I_s	㉛ ⑲	.00176	43 a	
Reduced Upright Allowable – Forced Crippling	F_o'	㊱ / ⑲	7320	44 a	
Reduced Allowable Web Shear Stress	F_s	㊲ ㊷ / ㊴	10,600	45 a	
Upright Stress, Average	f_u	⑫ ㉗	4840	46	
Upright Stress, Centroidal	$f_{cent.}$	㊻ ⑩ / ⑦	2660	47	
Upright Stress, Maximum	$f_{u_{max}}$	㊻ ㉘	5660	48	
Rivet Load, Upright to Flange	P_u	㊻ ⑩	241	49	
Secondary Bending Moment	M_{sb}	1/12 ㉖ ⑫ ② ④^{2}㉚	765	50	
Increased Rivet Load – Web to Upright	q_t	㉝ ⑲	1025	51 a	
Allowable Rivet Load – Web to Flange		6.1	594	52	
Allowable Rivet Load – Web to Upright		6.1	1080	53	
Allowable Rivet Load – Upright to Flange		6.1	1148	54	
Margin of Safety – Web		㊺ / ⑫ - 1	.03	55	
Margin of Safety – Upright		*	.29	56	
Margin of Safety Rivets, Web to Flange		(52) / ㉜ - 1	.02	57	
Margin of Safety Rivets, Web to Upright		(53) / (51) - 1	.05	58	
Margin of Safety Rivets, Upright to Flange		(54) / ㊾ - 1	3.76	59	

* Either ⑨/㊸ -1, ㉞/㊻ -1, ㉟/㊼ -1, or ㊹/㊽ -1, whichever is least.

[a] **The above steps are those added to account for the effect of the access hole. The rest of the table is the same as Table 2.2.2–3.** ← R

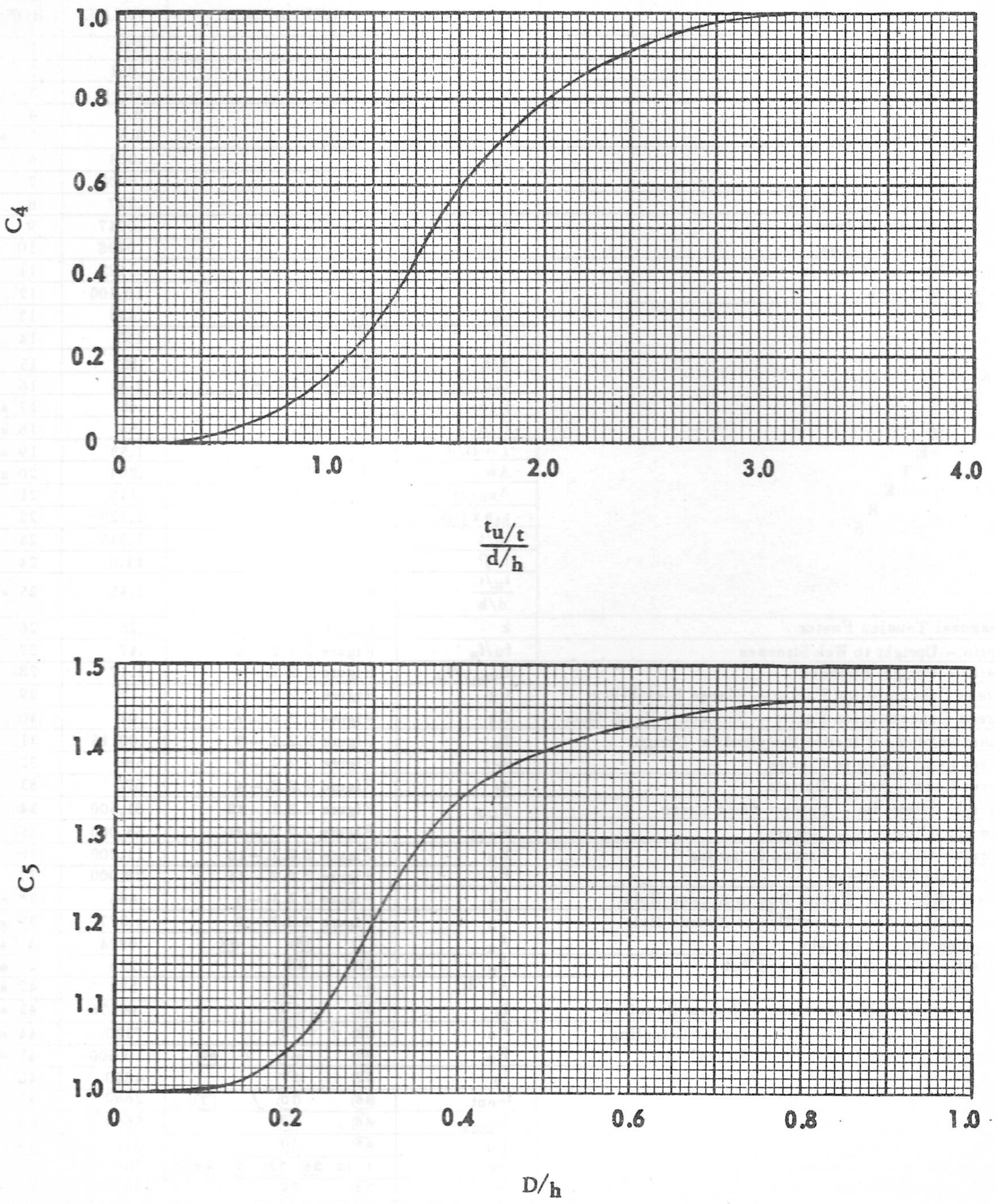


FIGURE 2.2.3-2
DESIGN REDUCTION FACTORS
DUE TO ACCESS HOLE

2.2.4 CURVED WEB TENSION FIELD BEAMS

2.2.4.1 General

The following section applies strictly to complete cylinders or to beams with single curvature subjected to uniform shear stresses. However, it may be used as an approximation for cylinders or beams having non-uniform shear stress distribution. The method is an adaptation of that presented in section 2.2.2 for tension field beams with plane webs. Information is included for both 24 S-T and 75 S-T aluminum alloys.

References:

1. "Diagonal Tension in Curved Webs", by Paul Kuhn and George E. Griffith, NACA TN 1481, 1947.
2. "A Summary of Diagonal Tension Part I" by Kuhn, Peterson and Levin, NACA, TN 2661, June 1951.
3. "Critical Combinations of Shear and Direct Axial Stress for Curved Rectangular Panels" by Scheldcrout and Stein, NACA TN 1928, August 1949.
4. Airplane Structural Analysis and Design, by Sechler and Dunn, 1942.
5. Experimental Investigation of Stiffened Cylinders Subjected to Torsion and Compression, NACA, T.N. 2188, 1950. by James P. Peterson.

Symbols:

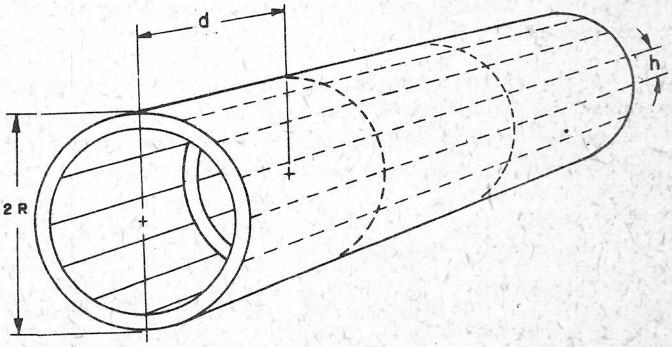
A	Cross sectional area, in^2
E	Young's modulus, psi.
P	Load, lbs.
Q	Beam shear, lbs.
R	Radius of curvature, in.
T	Applied torque, in.-lbs.
Z	Curvature parameter $\left(\frac{.954h^2}{Rt}\right.$ or $\left.\frac{.954d^2}{Rt}\right)$ (Use whichever is smaller of h or d)
d	Spacing of rings, in.
ϵ	Normal strain, in./in.
h	Spacing of stiffeners, in.
k	Diagonal - tension factor
η	Ratio of tangent modulus at the critical stress to the elastic modulus
q	Shear flow, lbs./in.
t	Thickness, in. (Without subscript signifies thickness of web)
α	Angle of diagonal tension, deg.
μ	Poissons ratio
σ	Normal stress, psi.
τ	Shear stress, psi.
ρ	Radius of gyration of cross section, in.
Δ	Correction factor for allowable shear stress

Subscripts:

all.	Allowable
cr	Critical
e	Effective
max.	Maximum
DT	Diagonal Tension
PDT	Pure Diagonal Tension
R	Rivet
RG	Ring
S	Shear
ST	Stringer

Special Combinations:

K_s	Critical shear-stress coefficient, established by geometry of the type of edge support.
σ_o	"Basic" allowable compressive stress for forced crippling of stiffeners (valid for stresses below proportional limit of material), psi.
τ_{all}^*	"Basic" allowable shear stress, psi.



2.2.4.2 Initial Buckling Stress of Web

The stress at which the web starts to buckle is given by the formula

$$\frac{\tau_{cr}}{\eta_s} = K_s E \left(\frac{t}{h}\right)^2 \quad \text{if } h < d \qquad (1\,a)$$

$$= K_s E \left(\frac{t}{d}\right)^2 \quad \text{if } d < h \qquad (1\,b)$$

where K_s is obtained from Fig. 2.2.4-1a or 2.2.4-1b

In no case should the stringer or ring gauge be less than one gauge lighter than the web gage.

2.2.4.3 Determination of Diagonal Tension Field Factor, k

The factor, k, specifies the degree to which the diagonal tension is developed; when k = O, there is no diagonal tension action; when k = 1.O, the diagonal tension is fully developed.

When the parameter, $\frac{300\ td}{Rh}$, has been calculated, the value of k may be read from Fig. 2.2.4.-2

2.2.4.4 Determination of Stresses in Stringers, Rings, and Web.

The stress in a stringer is given by:

$$\sigma_{ST} = \frac{k\,\tau \cot \alpha}{\frac{A_{ST}}{ht} + 0.5(1\text{-}k)} \qquad (2)$$

and the stress in a ring by

$$\sigma_{RG} = \frac{k\,\tau \tan \alpha}{\frac{A_{RG}}{dt} + 0.5\,(1\text{-}k)} \qquad (3)$$

For floating rings (not attached to skin but its stringers only) the factor .5(1-k) representing effective skin in formula (3) is omitted.

The above stresses are compressive stresses and must be added to other stresses not arising from diagonal-tension action where such stresses exist. The shear stress in the web is found as follows:

$$\tau = \frac{T}{2\,\pi\,R^2\,t} \qquad \text{(complete cylinder)}$$

$$\tau = q/t \qquad \text{(curved web beam)}$$

The angle α between a generatrix at the cylinder and the direction of the diagonal tension is given by the formula:

$$\tan^2 \alpha = \frac{\epsilon - \epsilon_{ST}}{\epsilon - \epsilon_{RG} + \frac{1}{24}\left(\frac{h}{R}\right)^2} \quad \text{if } d > h \qquad (4a)$$

$$\tan^2 \alpha = \frac{\epsilon - \epsilon_{ST}}{\epsilon - \epsilon_{RG} + \frac{1}{8}\,\frac{d}{R}^2 \tan^2 \alpha} \quad \text{if } h > d \quad (4b)$$

where ϵ is the strain in the sheet along the direction of diagonal tension, $\epsilon_{ST} = \frac{\sigma_{ST}}{E}$ is the strain in the stringer, and $\epsilon_{RG} = \frac{\sigma_{RG}}{E}$ is the strain in the ring.

The strain ϵ in the sheet is given by the formula:

$$\epsilon = \frac{\tau}{E}\left[\frac{2K}{\sin 2\,\alpha} + (1\text{–}k)\,(1+\mu)\,\sin 2\,\alpha\right] \qquad (5)$$

which is evaluated in Fig. 2.2.4.-3

From the above it may be seen that formulas (2) through (5) are interdependent and must be solved by successive approximation. The process is rapidly convergent and normal accuracy is usually obtained within three cycles.

A first approximation to the angle α may be made by means of the formula

$$\alpha \approx \alpha_{PDT}\left(\frac{\alpha}{\alpha_{PDT}}\right) \qquad (6)$$

with α_{PDT} taken from Fig. 2.2.4.-4 and the ratio $\frac{\alpha}{\alpha_{PDT}}$ from Fig. 2.2.4.-5.

This first approximation will give results that are probably close enough for most cases of curved tension field webs if a M.S. $\geqq$ .20 is held for the stringers and rings and a M.S. $\geqq$ .15 is held for the web. If more accuracy is desirable or necessary the proper value of α may be determined by successive approximation as given above.

2.2.4.5 Analysis of Web

The ultimate strength of the web can be estimated by means of the empirical formula

$$\tau_{all} = \tau^{*}_{all}\ (0.65 + \Delta) \qquad (7)$$

where τ^{*}_{all} is the "basic" allowable shear stress taken from Fig. 2.2.4.-6 or 2.2.4.-7 and Δ is obtained from the graph in Fig. 2.2.4.-8.

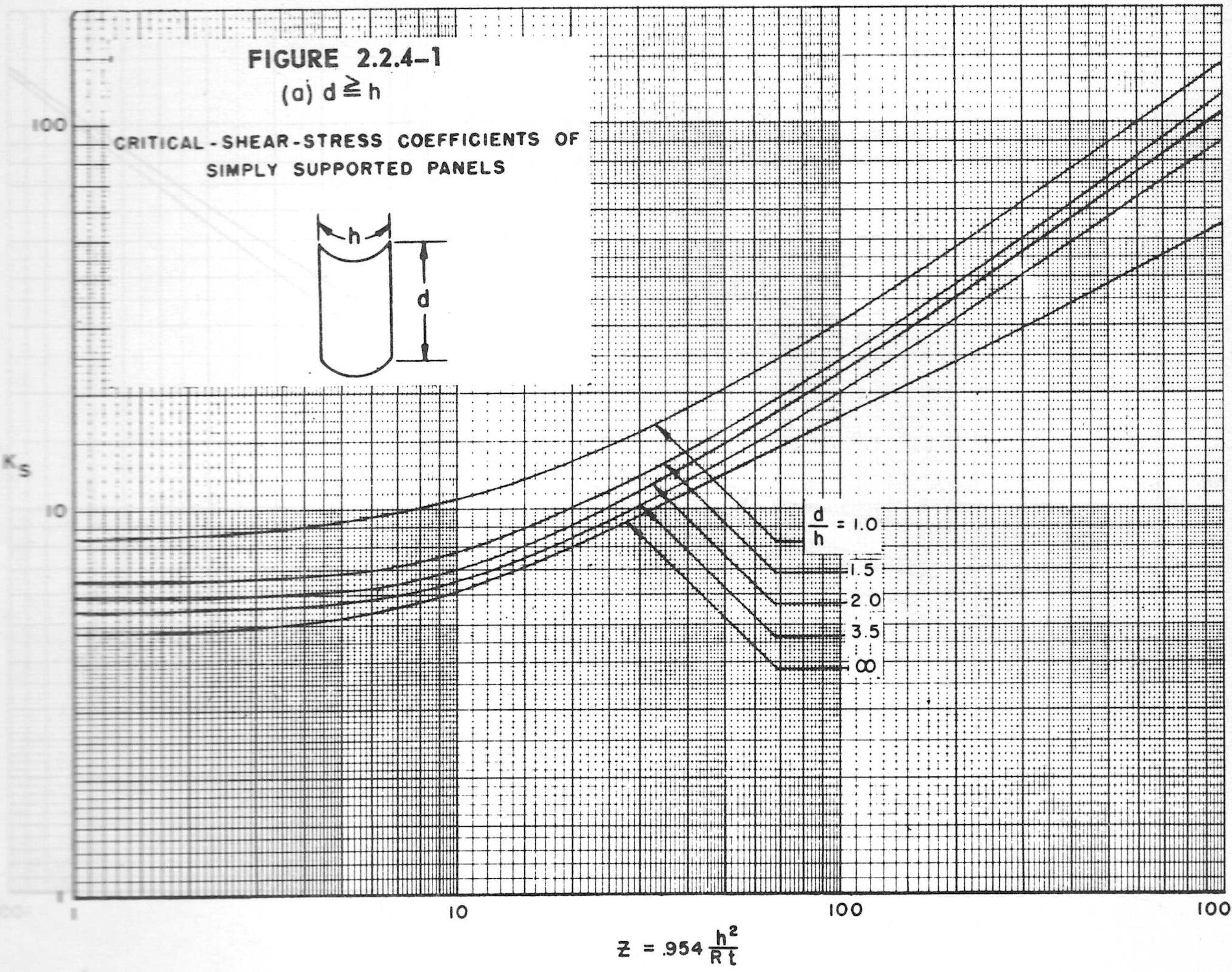


FIGURE 2.2.4-1
(a) d ≧ h
CRITICAL-SHEAR-STRESS COEFFICIENTS OF SIMPLY SUPPORTED PANELS

2.2.4.6 Analysis of Stringers

The stringer stress given by formula (2) is the average stress over the length of the stringer. The maximum stress at the midpoint of a panel is given by the formula

$$\sigma_{ST_{max.}} = \sigma_{ST} \left(\frac{\sigma_{ST_{max.}}}{\sigma_{ST}} \right) \qquad (8)$$

where the ratio $\frac{\sigma_{ST_{max.}}}{\sigma_{ST}}$ is obtained from Fig. 2.2.4.-9 using h/d as the other parameter.

Because of the polygonal shape acquired by the cross section of the cylinder as diagonal tension develops each tension diagonal experiences a change in direction as it crosses the stringer. The magnitude of this inward radial pressure per running inch is

$$p = \frac{k \tau t h}{R} \tan \alpha \qquad (9)$$

If this pressure were distributed uniformly along the length of the stringer, the primary peak bending moment (at the junction with the ring) would be given by the formula

$$M_{ST} = k \tau \frac{t h d^2}{12R} \tan \alpha \qquad (10)$$

A secondary peak moment would exist half-way between rings and its magnitude would be one-half the primary peak.

Since the rings are actually circular (or curved) a portion of the tension diagonal near each end of a

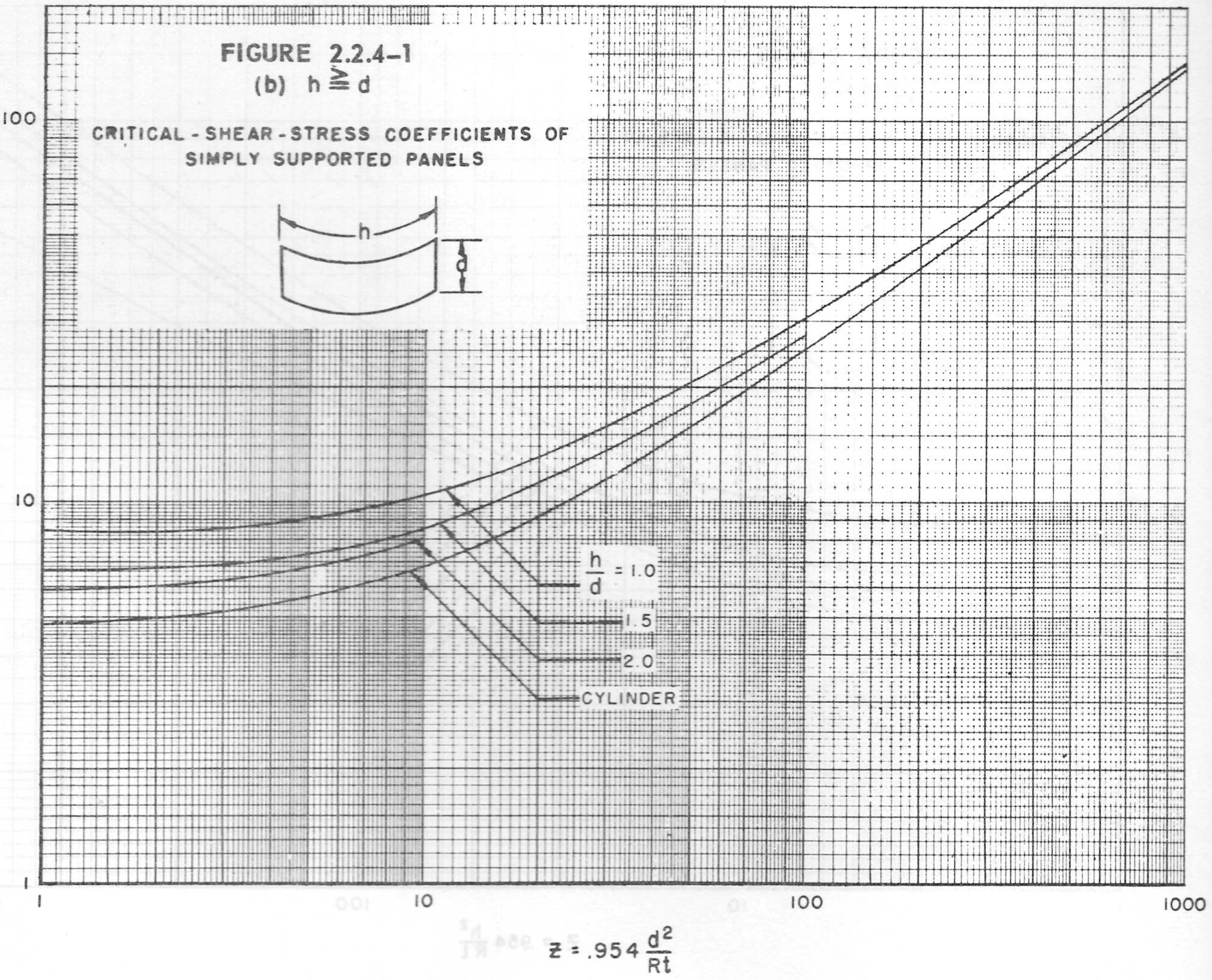


FIGURE 2.2.4-1
(b) h ≧ d
CRITICAL-SHEAR-STRESS COEFFICIENTS OF SIMPLY SUPPORTED PANELS

panel will be forced to remain in the original cylindrical surface and experience little change in direction. Tests indicate that the secondary peak moment (halfway between rings) agrees roughly with the calculated value

$$M_{ST} = k\,r\,\frac{t\,h\,d^2}{24\,R}\quad\tan\alpha \qquad (11)$$

Strength tests on cylinders have led to the conclusion that the maximum moment at the ring is no larger than the calculated secondary moment above.

(1) A check should be made against column failure normal to the skin (using the local crippling stress for the stringer as allowable for $\frac{L}{\rho} = 0$) with some reduction to allow for effect of skin buckles unless t_{ST}/t is larger than 3. Some allowance should be made for beam column effect.

(2) The stiffened cylinder should be checked against general instability by **Fig. 2.2.4-10**

(3) The stringers should be checked for forced crippling in the same manner as for plain webs. See Sec. 2.2.2.3.

2.2.4.7 Analysis of Rings

The ring stress given by formula (3) is the average stress in the ring. The maximum stress exists at the mid-point of the panel and is given by the formula

$$\sigma_{RGmax.} = \sigma_{RG}\;\frac{\sigma_{RG\,max.}}{\sigma_{RG}} \qquad (12)$$

where the ratio $\frac{\sigma_{RG\,max.}}{\sigma_{RG}}$ is obtained from Figure 2.2.4.-9 using $^{d}/h$ as the other parameter.

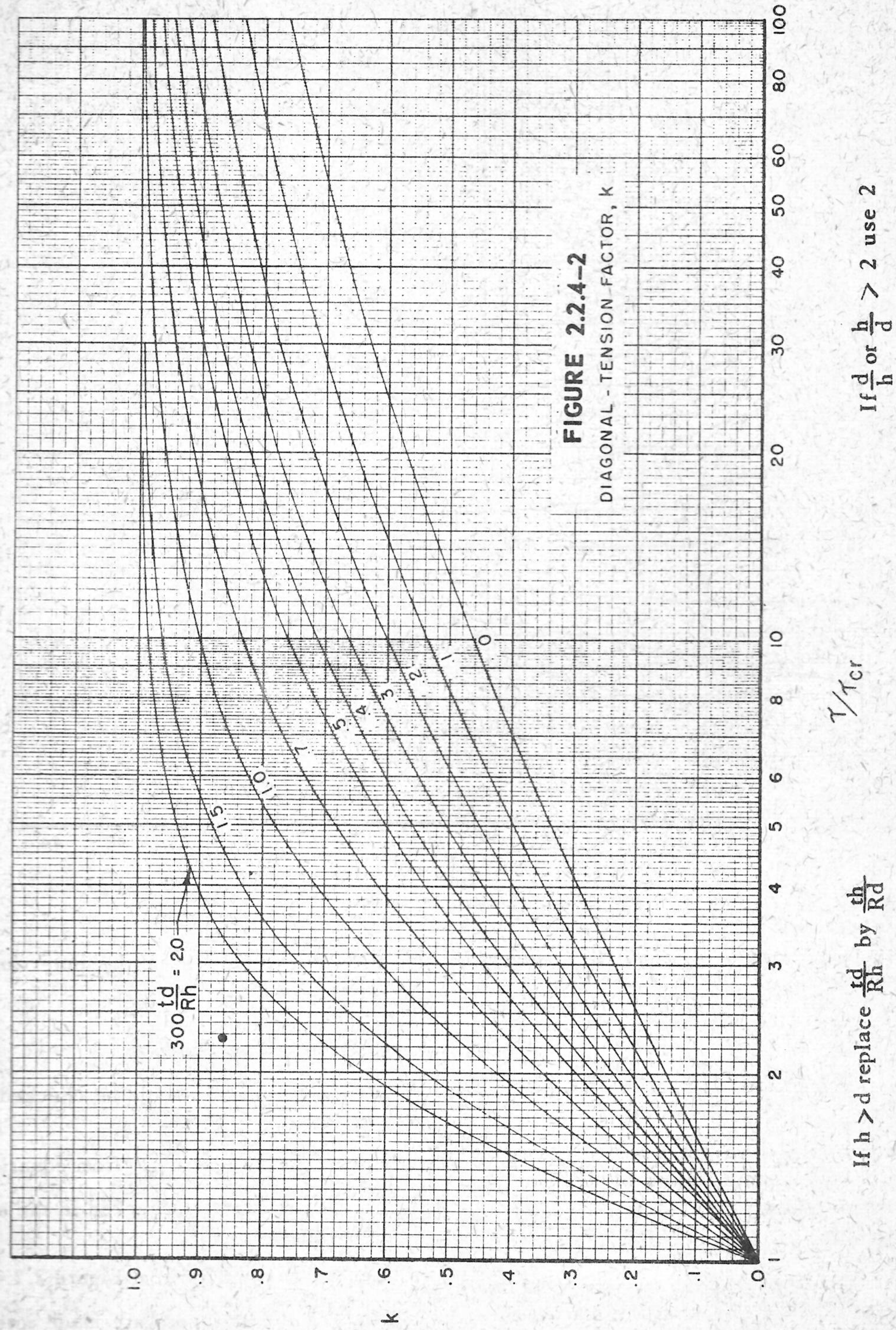


FIGURE 2.2.4-2

DIAGONAL-TENSION-FACTOR, k

If h > d replace $\frac{td}{Rh}$ by $\frac{th}{Rd}$

If $\frac{d}{h}$ or $\frac{h}{d} > 2$ use 2

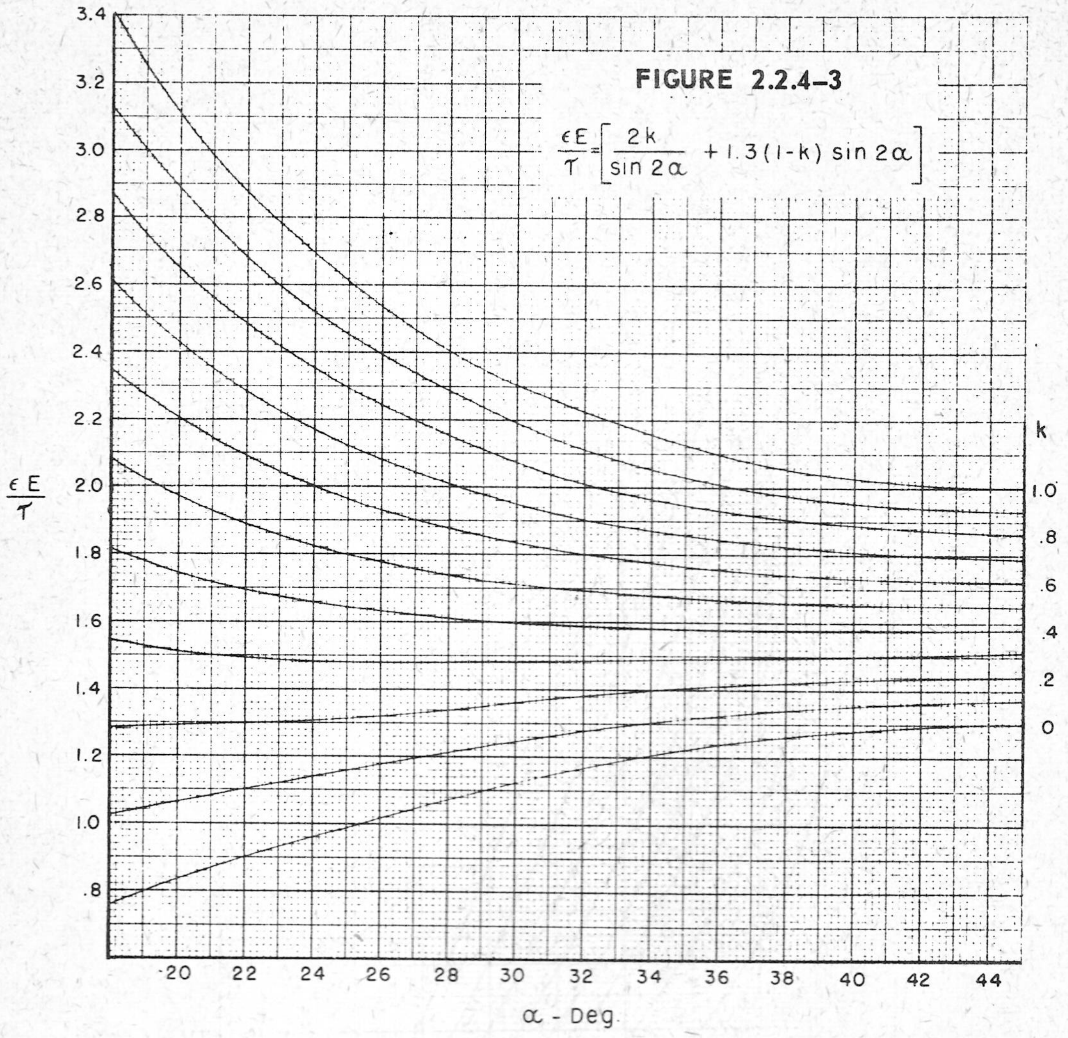


FIGURE 2.2.4–3

$$\frac{\epsilon E}{\tau} = \left[\frac{2k}{\sin 2\alpha} + 1.3(1-k)\sin 2\alpha\right]$$

Floating Rings (not attached to skin but to stringers only)

The primary bending moment in a floating ring (at the junction with a stringer)is

$$M_{RG} = k\,\tau\,\frac{t h^2 d}{12 R}\tan\alpha \qquad (13)$$

the secondary maximum half-way between stringers is half as large. This type of ring should be designed to carry the loads shown below:

(1) Hoop compression σ_{RG} from equation (3)

(2) Bending moment due to M_{RG} shown above at the junction with stringer. A check at the station midway between stringers (where the moment is only half as large, but of opposite sign) may be necessary if the cross section of the ring is such that allowable stresses in the outer and inner fibers differ greatly.

Rings Attached to Skin

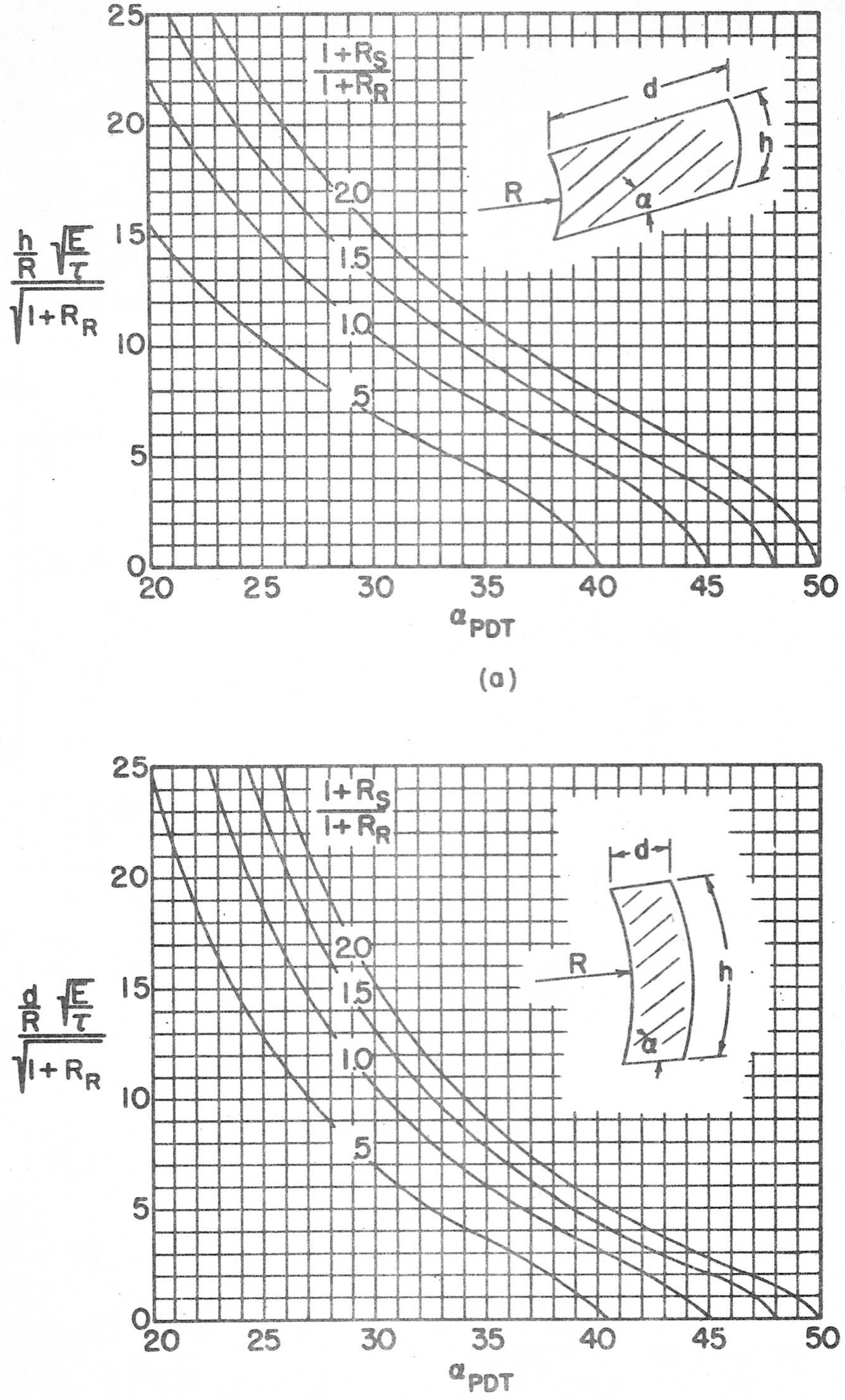
This type of ring should be checked for the following stresses.

(1) Hoop compression σ_{RG} from equation (3)

(2) Forced crippling is checked in the same manner as for stringers

(3) General instability from Figure 2.2.4.-10

Unless the stringers are made intercostal (which leads to loss of efficiency in bending strength of the cylinder and therefore is seldom done) the ring must be notched to permit the stringers to pass through. At



(a)

(b)

FIGURE 2.2.4-4 – Angle of pure diagonal tension

$$R_R = \frac{dt}{A_{RG}} \; ; \; R_S = \frac{ht}{A_{ST}}$$

the notch the ring stress is increased because the cross section is reduced; this effect is aggravated also by the suddenness of reduction. The free edge of the notch should be checked against local crippling failure. If the stringer is connected to the ring by a clip-angle of sufficient length riveted to the web of the ring, the section at the notch is increased, and the edge of the notch can be stiffened so much that there is no danger of this type of failure. The section at the notch should be checked for axial load on the net section plus the bending moment caused by the displacement of the centroid of the net section from that of the gross section. See Section 4.4.

2.2.4.8. Analysis of Rivets

For a complete cylinder only the rivets in a web splice need be checked. The load per inch for rivets in a splice may be taken as

$$P_R = q(1 + 0.414k) \qquad (14a)$$

For a curved web beam the rivets need be checked at two points only.

Web-to-flange rivets

The load per inch run is, according to eq. 15 of NACA TN 1364:

$$P_R = q(1 + 0.414k) \qquad (14b)$$

Upright-to-flange rivets

The upright-to-flange rivets must carry the load existing in the upright into the flange.

$$P_{RG} = A_{RG}\,\sigma_{RG} \qquad (15)$$

2.2.4.9 Analysis of Flanges

Flanges on a curved web exist only where the member is not a complete cylinder. Where they exist the method given in Section 2.2.2–5 should be followed.

2.2.4.10 Sample Calculation

A cylinder having the following dimensions and subjected to a pure torque of 934,000 in. lbs. will be analysed as a sample problem to show the use of the method. The tabular form used in this problem is to be used as a guide only and no standard form will be adopted unless experience shows the desirability of doing so.

Material – 24 S-T Alc.

Skin thickness, $t = .040$ in.

Radius of curvature, $R = 15.0$ in.

Spacing of rings, $d = 15.0$ in.

Spacing of stringers, $h = 7.85$ in.

Thickness of stringers, $t_{ST} = .081$ in.

Thickness of rings, $t_{RG} = .102$ in.

Area of stringer (1.0 x 2.0 x .081 Channel, 24 S-T3 Alc.), $A_{ST} = .287$ in.^{2}

Area of ring (1.0 x 2.0 x .102 Channel, 24 S-T4 Alc.), $A_{RG} = .365$ in.^{2}

$$q = \frac{T}{2\,A} = \frac{934{,}000}{2 \times \pi\,(15.00)^2} = 660 \text{ lbs./in.}$$

Computations of stringer as a column

$C = 3.0$ (assumed)

$I = .165$ in.^{4}

$$\rho_{ST} = \sqrt{\frac{I}{A_{ST}}} = \sqrt{\frac{.165}{.287}} = .758 \text{ in.}$$

$$L'/\rho = \frac{15.00}{.758\sqrt{3.0}} = 11.42$$

$$\sigma_{col.} = 48{,}800 \text{ lb./in.}^2$$ See Sec. 3.1

Computations of general instability

$$\rho_{RG} = 1.919\sqrt{\frac{\frac{1.919}{.960} + 6}{12\left(\frac{1.919}{.960} + 2\right)}} = .784$$

See Table 10.5, p. 310. Niles & Newell, Vol. 1, Second Edition

$$\sqrt{\frac{\rho_{ST} \times \rho_{RG}}{hd}}\left(\frac{\sqrt{\rho_{ST} \times \rho_{RG}}}{R}\right)^{3/4} =$$

$$\sqrt{\frac{.758 \times .784}{15.0 \times 7.85}}\left(\frac{\sqrt{.758 \times 784}}{15.0}\right)^{3/4}$$

$$= .00756$$

Fig. 65 of NACA TN 1197 shows that the general instability failure stress would be very high for this case. Therefore, no further investigation of this phenomena will be made.

NO.	ITEM	EQUATION	VALUE
1	Shear Flow, q, lb./in.	$\frac{T}{2A}$ or $\frac{Q}{h}$	660
2	Skin Thickness, t, in.		.040
3	Shear Stress, τ, psi.	① / ②	16,500
4	Radius of curvature, R, in.		15.00
5	Spacing of Rings or Upright, d, in.		15.00
6	Spacing of Stringers h, in.		7.85
7	Thickness of Stringers t_{ST}, in.		.081
8	Thickness of Ring, t_{RG}		.102
9	Area of Stringer Cross Section, A_{ST}, in.^{2}		.287
10	Area of Ring Cross Section, A_{RG}, in.^{2}		.365
11	Curvative Parameter, Z	$\frac{.954 \times \textcircled{5}^2}{\textcircled{4} \times \textcircled{2}}$ or $\frac{.954 \times \textcircled{6}^2}{\textcircled{4} \times \textcircled{2}}$ whichever is least	98.22
12	K_s	See Fig.1, NACA TN 1481 or Fig. 2.2.4–1	22.7
13	Critical Shear Stress, τ_{cr}, psi.	See Eq. (1a) or (1b)	6000
14	τ / τ_{cr}	③ / ⑬	2.75
15	300 td/Rh	$\frac{300 \times \textcircled{2} \times \textcircled{5}}{\textcircled{4} \times \textcircled{6}}$	1.53
16	Diagonal Tension Factor, k	See Fig. 2, NACA TN 1481 or Fig. 2.2.4–2	.71
17	$\frac{h}{R} \sqrt{E/\tau}$	$\frac{\textcircled{6}}{\textcircled{4}} \sqrt{\frac{\textcircled{E}}{\textcircled{3}}}$	14.5
18	$\sqrt{1 + 0.5 \left(\frac{ht}{A_{ST}} + \frac{dt}{A_{RG}} \right)}$	$\sqrt{1 + 0.5 \quad \frac{\textcircled{6} \times \textcircled{2}}{\textcircled{9}} + \frac{\textcircled{5} \times \textcircled{2}}{\textcircled{10}}}$	1.54
19		⑰ / ⑱	9.42
20	α_{PDT}	See Fig. 5, NACA TN 1481 or Fig. 2.2.4–4	31.6°
21	α / α_{PDT}	See Fig. 6, NACA TN 1481 or Fig. 2.2.4–5	.929
22	α	⑳ x ㉑	29° 20'
23	cot α		1.7796
24	tan α		.56194
25	$\frac{A_{ST}}{ht} + 0.5 (1 - K)$	$\frac{\textcircled{9}}{\textcircled{6} \times \textcircled{2}} + 0.5(1 - \textcircled{16})$	1.059
26	$\frac{A_{RG}}{dt} + 0.5 (1 - K)$	$\frac{\textcircled{10}}{\textcircled{5} \times \textcircled{2}} + 0.5(1 - \textcircled{16})$	.754
27	σ_{ST}	$-\frac{\textcircled{16} \times \textcircled{3} \times \textcircled{23}}{\textcircled{25}}$	-19,700
28	ϵ_{ST}	㉗ / E	-.00193
29	σ_{RG}	$-\frac{\textcircled{16} \times \textcircled{3} \times \textcircled{24}}{\textcircled{26}}$	-8740
30	ϵ_{RG}	㉙ / E	-.00086

NO.	ITEM	EQUATION	VALUE
31	$\epsilon E/\tau$	See Fig. 4, NACA TN 1481 or Fig. 2.2.4–3	1.98
32	ϵ	$㉛ \times \frac{③}{E}$	.00320
33	$\tan \alpha_1$	$\sqrt{\frac{㉜ - ㉘}{㉜ - ㉚ + \frac{1}{24}\left(\frac{⑥}{④}\right)^2}}$	.576
34	α_1	arc tan ㉝	29° 57'
35	$\cot \alpha_1$		1.7361
36	σ_{ST}	$-\frac{⑯ \times ③ \times ㉟}{㉕}$	- 20,200
37	ϵ_{ST}	㊱ / E	-.00198
38	σ_{RG}	$-\frac{⑯ \times ③ \times ㉝}{㉖}$	- 8510
39	ϵ_{RG}	㊳ / E	- .00084
40	$\epsilon E/\tau$	See Fig. 4, NACA TN 1481 or Fig. 2.2.4–3	1.96
41	ϵ	$㊵ \times \frac{③}{E}$	.00317
42	$\tan \alpha_2$	$\sqrt{\frac{㊶ - ㊲}{㊶ - ㊴ + \frac{1}{24}\left(\frac{⑥}{④}\right)^2}}$	.578
43	α_2		30° 2'
44	$\cot \alpha_2$		1.7301
45	σ_{ST}	$-\frac{⑯ \times ③ \times ㊹}{㉕}$	- 20,250
46	ϵ_{ST}	㊺ / E	-.00199
47	σ_{RG}	$-\frac{⑯ \times ③ \times ㊷}{㉖}$	- 8490
48	ϵ_{RG}	㊼ / E	-.00083
49	$\epsilon E/\tau$	See Fig. 4, NACA TN 1481 or Fig. 2.2.4–3	1.95
50	ϵ	$㊾ \times \frac{③}{E}$	.00315
51	$\tan \alpha_3$	$\sqrt{\frac{㊿ - ㊻}{㊿ - ㊽ + \frac{1}{24}\left(\frac{⑥}{④}\right)^2}}$	.57791
52	α_3		30° 1.4'
53	$\cot \alpha_3$		1.7302
54	σ_{ST}	$-\frac{⑯ \times ③ \times (53)}{㉕}$	20,250

NO.	ITEM	EQUATION	VALUE
55	σ_{RG}	$-\frac{\text{⑯} \times \text{③} \times \text{(51)}}{\text{㉖}}$	8490
56	τ_{all} *	See Fig. 7, NACA TN 1481 or Fig. 2.2.4–6 or 7	18,940
57	Δ	See Fig. 8, NACA TN 1481 or Fig. 2.2.4–8	.235
58	τ_{all}	(56) (.65 + (57))	16,780
59	M.S. of Web	$\frac{\text{(58)}}{\text{③}} - 1$	.01
60	$\sigma_{ST_{max.}} / \sigma_{ST}$	See Fig. 3, NACA TN 1481 or Fig. 2.2.4–9	1.126
61	$\sigma_{ST_{max.}}$	(54) x (60)	22,800
62	col.	See Sample Calculation	48,800
63	M. S. (Column)	$\frac{\text{(62)}}{\text{(61)}} - 1$	1.14
64	$\frac{t_u}{t}$	$\frac{\text{⑦}}{\text{②}}$	2
65	σ_O	See Figure 2.2.2 - 7	27,500
66	M. S. (Forced Crippling)	$\frac{\text{(65)}}{\text{(61)}} - 1$	.21
67	$\sigma_{max.}$	See Section 3.1.1	38,500
68	M. S. (Natural Crippling)	$\frac{\text{(67)}}{\text{(61)}} - 1$	.75
69	$\sigma_{RG_{max.}} = \sigma_{RG}$ (Assumed)	(55)	8,490
70	$\frac{t_u}{t}$	$\frac{\text{⑧}}{\text{②}}$	2.6
71	O	See Figure 2.2.2-7	29,500
72	M. S. (Forced Crippling)	$\frac{\text{(71)}}{\text{(69)}} - 1$	2.48

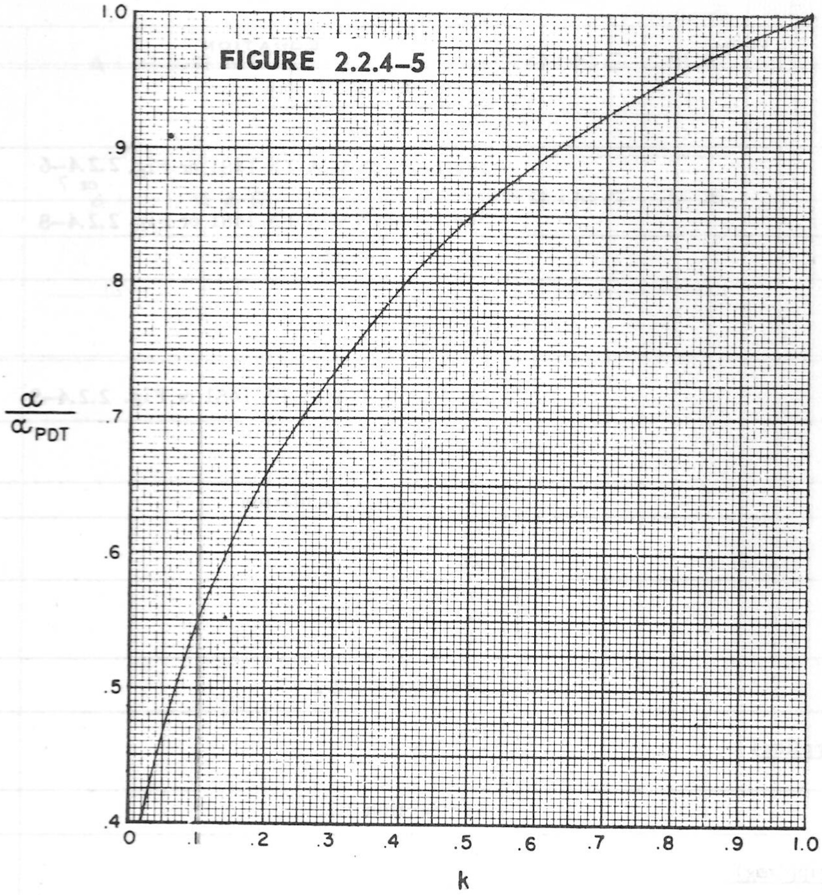


FIGURE 2.2.4-5

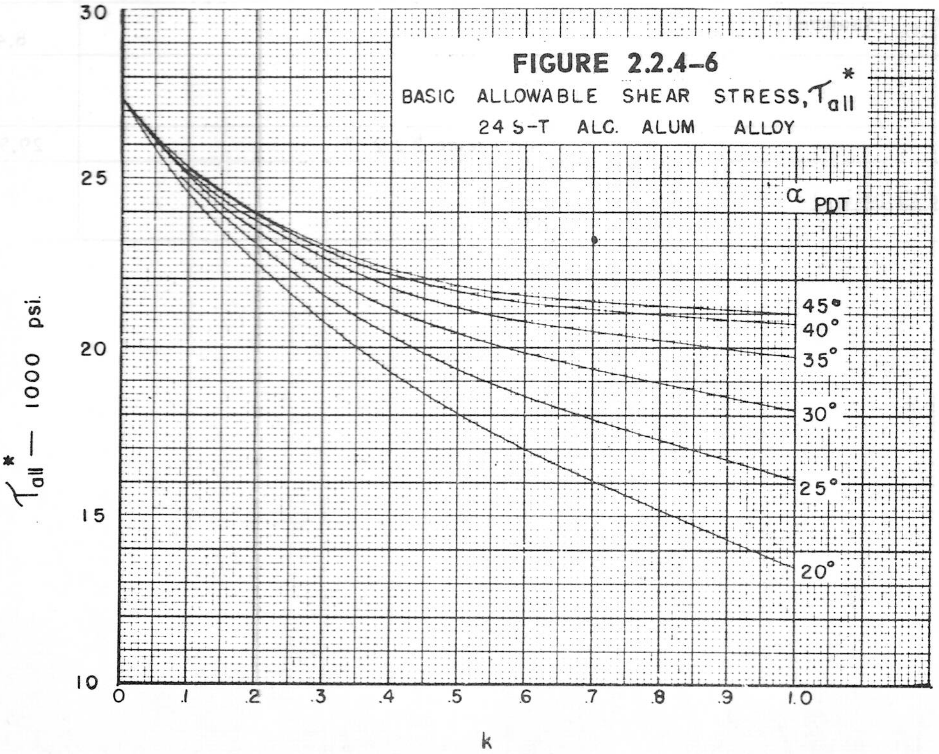


FIGURE 2.2.4-6

BASIC ALLOWABLE SHEAR STRESS, τ_{all}^{*}

24 S-T ALC. ALUM ALLOY

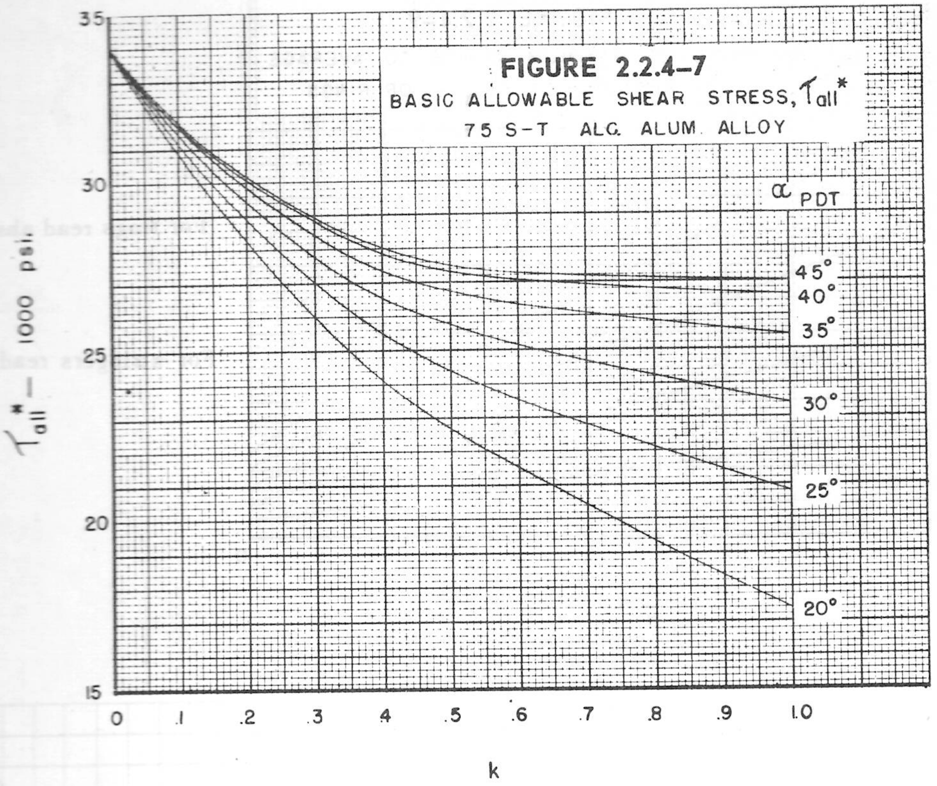


FIGURE 2.2.4–7

BASIC ALLOWABLE SHEAR STRESS, τ_{all}^{*}

75 S-T ALC. ALUM. ALLOY

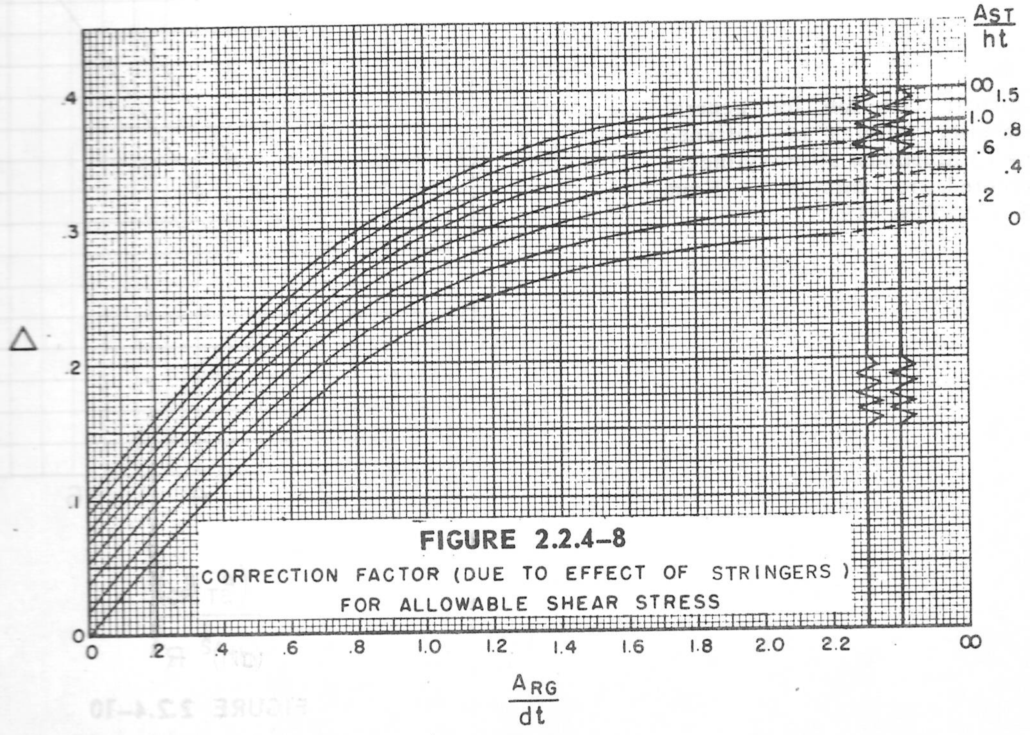
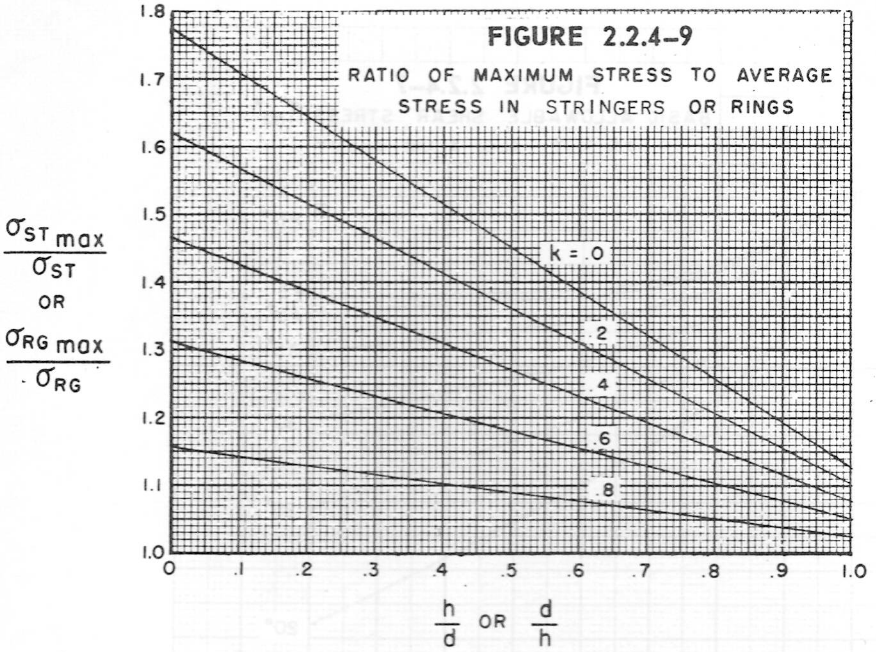


FIGURE 2.2.4–8

CORRECTION FACTOR (DUE TO EFFECT OF STRINGERS)
FOR ALLOWABLE SHEAR STRESS



For rings read abscissa $\frac{d}{h}$

For stringers read abscissa $\frac{h}{d}$

Empirical criterion for general instability failures of stiffened 24S–T3 aluminum alloy cylinders subjected to torsion. From N.A.C.A T.N. 1197.

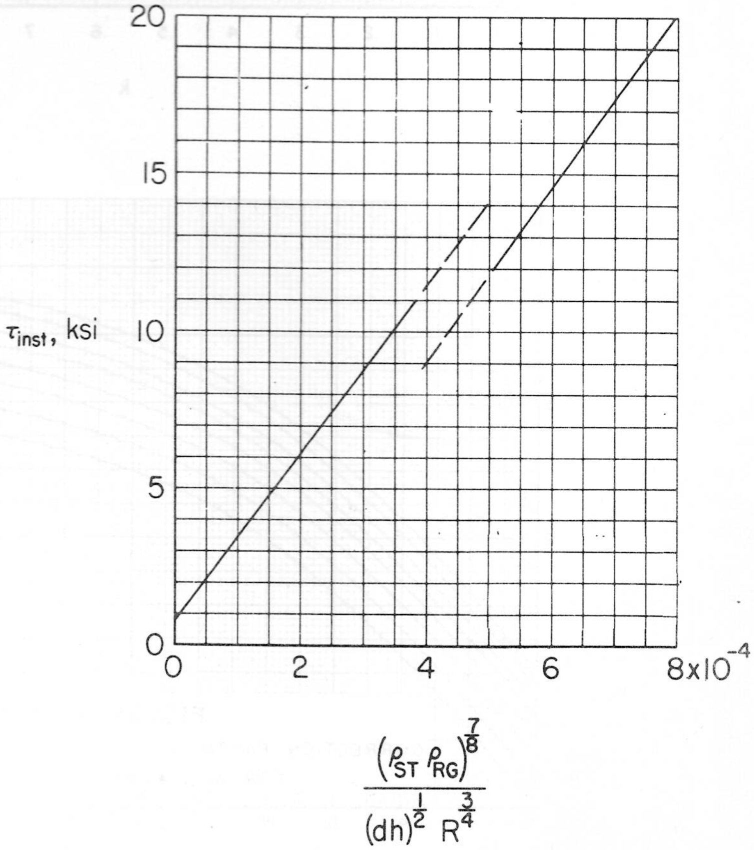
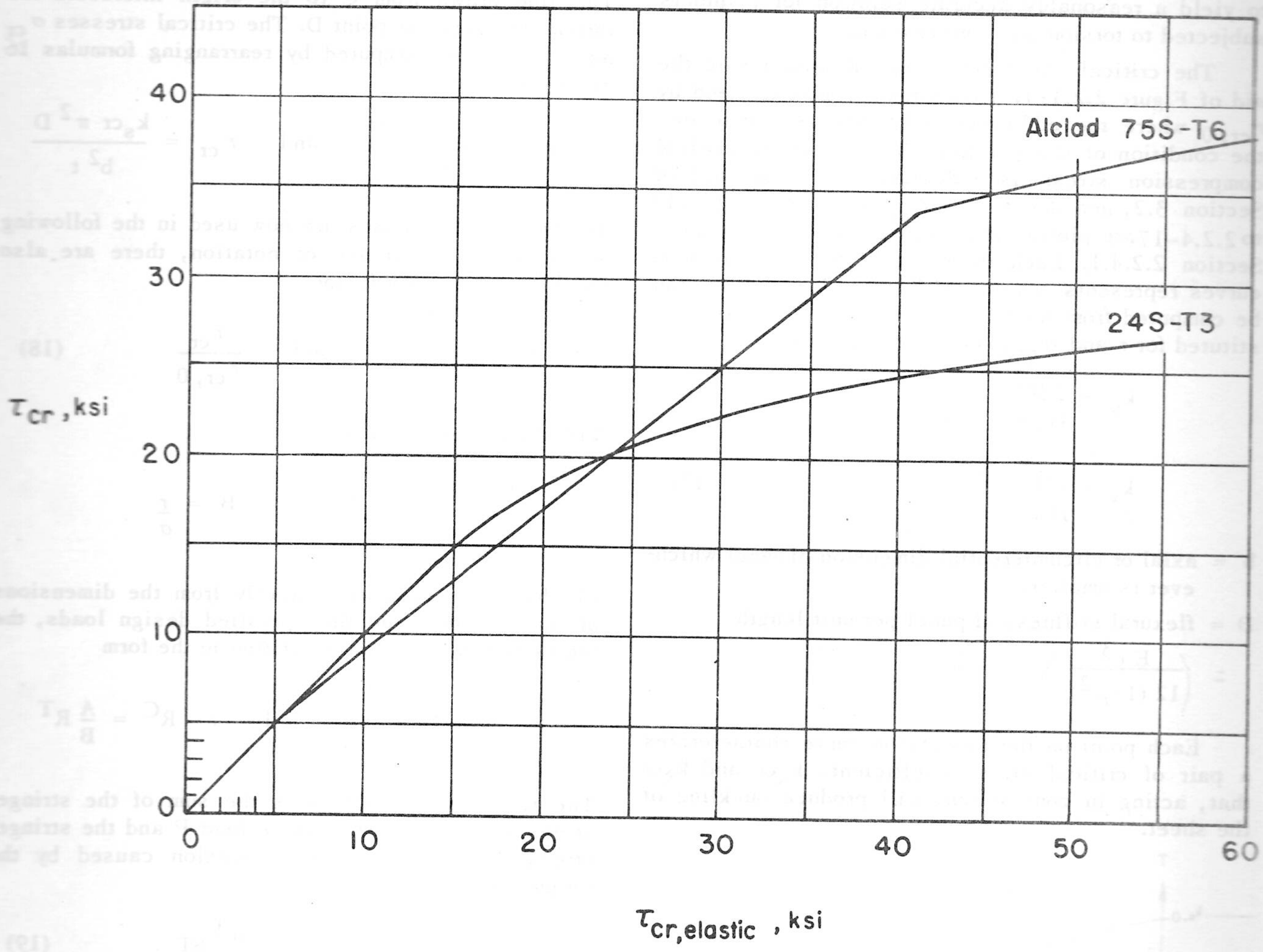


FIGURE 2.2.4–10



(c) Plasticity correction.

FIGURE 2.2.4–11

2.2.4.11 Combined Loading

The following method of analysis has been found to yield a reasonably accurate solution for cylinders subjected to torsion and compression.

The critical shear stress is calculated with the aid of Figure 2.2.4-1. This stress is now denoted by $\tau_{cr,\,0}$ where the additional subscript zero indicates the condition of shear acting alone. Next the critical compression stress is calculated with the aid of Section 3.2, and denoted by $\sigma_{cr,\,0}$. Figures 2.2.4–14 to 2.2.4–17are plotted from data given in Reference 3 Section 2.2.4.1. Each intercept of these interaction curves represents a value of $k_{s,0}$ or $k_{x,0}$ which can be computed from formulas 16 and 17if $\tau_{cr,0}$ is substituted for τ and $\sigma_{cr,0}$ is substituted for σ

$$k_s = \frac{\tau\, tb^2}{D\, \pi^2} \qquad (16)$$

$$k_x = \frac{tb^2}{D\, \pi^2} \qquad (17)$$

b = axial or circumferential dimension of panel whichever is smaller.

D = flexural stiffness of panel per unit length

$$= \left(\frac{E\, t^3}{12\, (1 - \mu^2)} \right)$$

Each point on the interaction curve characterizes a pair of critical stress coefficients $k_s cr$ and $k_x cr$ that, acting in conjunction, will produce buckling of the sheet.

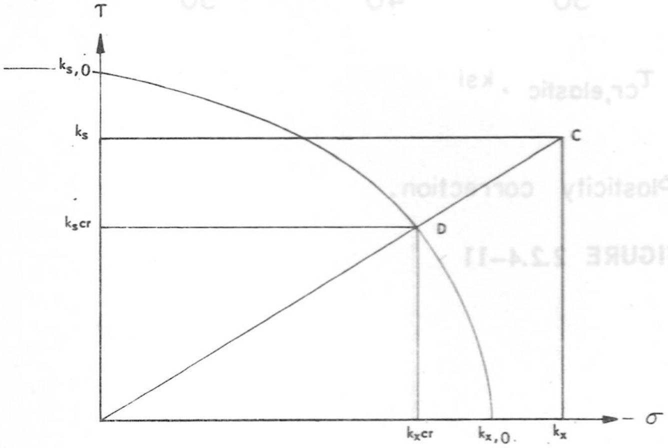


FIGURE 2.2.4–12

Let σ denote the compressive stress that would exist in the cylinder if the sheet did not buckle (i.e., remained fully effective) under the design compression load P. Similarly let τ denote the shear that would exist if the sheet did not buckle under the action of the design torque T. If these values of σ and τ are used in formulas 16 and 17 the magnitude of k_x and k_s can be found, thus locating point C in Figure 2.2.4–12 The line drawn from C to the origin intersects the interaction curve at point D. The critical stresses σ_{cr} and τ_{cr} can be computed by rearranging formulas 16 and 17 to read

$$\sigma_{cr} = \frac{k_x cr\, \pi^2\, D}{b^2\, t} \qquad \text{and} \qquad \tau_{cr} = \frac{k_s cr\, \pi^2\, D}{b^2\, t}$$

These critical stresses are now used in the following steps. For convenience of notation, there are also used the interaction factors

$$R^C = \frac{\sigma_{cr}}{\sigma_{cr,0}} \quad ; \quad R^T = \frac{\tau_{cr}}{\tau_{cr,\,0}} \qquad (18)$$

With the aid of the ratios

$$A = \frac{\tau_{cr,\,0}}{\sigma_{cr,0}} \quad ; \quad B = \frac{\tau}{\sigma}$$

which can be computed directly from the dimensions of the structure and the specified design loads, the interaction factors can be written in the form

$$R^T = \frac{-A}{2B} + \sqrt{\frac{A^2}{4B^2} + 1} \quad ; \quad R^C = \frac{A}{B}\, R^T$$

The total stringer stress is the sum of the stringer stress due to the compressive load P and the stringer stress due to the diagonal tension caused by the torque, or

$$\sigma_{ST} = \sigma^C{}_{ST} + \sigma^T{}_{ST} \qquad (19)$$

The stress $\sigma^C{}_{ST}$ is computed by the formula

$$\sigma^C{}_{ST} = \frac{P}{n\,(A_{ST} + ht_\eta{}^C)} \qquad (20)$$

The load P must be taken as negative because it is compressive; n is the number of stringers, A_{ST} is the area of one stringer, and η^C is the effective-width factor. This factor is taken as the Kármán-Sechler expression for effective width (reference 4), multiplied by the ratio R^C in order to make allowance for the presence of the torque loading; thus

$$\eta^C = R^C\, 0.89 \sqrt{\frac{\sigma_{cr}}{\sigma^C{}_{ST}}} \qquad (21)$$

If expression (21) is substituted into equation (20), a quadratic equation is obtained which yields

$$\sigma^{C}_{ST} = \frac{P}{nA_{ST}} - 2D^2 + 2D\sqrt{D^2 - \frac{P}{nA_{ST}}} \qquad (22)$$

where

$$D = 0.445\ \frac{ht}{A_{ST}}\ R^{C}\sqrt{-\sigma_{cr}} \qquad (23)$$

The stress σ^{T}_{ST} is computed by formula (2), modified by the ratio R^T in order to allow for the presence of the compressive load P; the modified formula is

$$\sigma^{T}_{ST} = - \frac{k\,\tau \cot \alpha}{\frac{A_{ST}}{ht} + 0.5(1-k)\,R^{T}} \qquad (24)$$

The interaction factors R^C and R^T, by definition, describe the interaction between compression and torque at the instant of buckling. Their use in formulas (21) and (24) to describe the interaction on the effective width is fundamentally arbitrary. However, in the usual design range, the effect of moderate errors in estimating the effective width is unimportant; any reasonable method for estimating the effect of interaction on effective width is therefore acceptable for the time being.

The stress in a ring is computed, by the unmodified formula (3).

The diagonal strain in the sheet is computed by equation (5), on the implied assumption that it is not modified significantly by the compressive force carried by the sheet. The angle α is computed by formula (4a) or (4b), the strain ϵ_{ST} being computed from the total compressive stress σ_{ST} given by expression (25). The diagonal-tension factor k is obtained from Figure 2.2.4-2 by using τ_{cr} (not $\tau_{cr,0}$).

$$\sigma_{ST} = \sigma_{ST}{}^{C} + \sigma_{ST}{}^{T} \qquad (25)$$

The stress computation for the case of combined loading thus differs from that for the case of pure torque loading in the following items:

(i) The critical stress is reduced by interaction.

(ii) The stringer stress due to the load P must be added; this calculation involves an interaction factor.

(iii) The calculation of the stringer stress due to the torque involves an interaction factor.

Concerning item (i), there is ample theoretical and experimental evidence to justify the belief that the calculation is sufficiently accurate for design purposes. The factors used in items (ii) and (iii) are arbitrary, but they have only a very minor effect except for low loading ratios. Consequently, the accuracy with which the stresses can be computed under combined loading might be expected to be about the same as for pure torque loading, as long as the ratio τ/τ_{cr} is greater than 2.

The question of allowable stresses for failure is more problematical. The allowable value of skin shear stress is probably not changed significantly by added compression, but there is no experimental evidence on this score. As far as true column failure of the stringers is concerned, it would be immaterial whether the compressive stress in the stringer arises directly from the axial load P, or indirectly (through diagonal-tension action) from the torque; in other words, column failure would be assumed to take place when the total stringer stress given by expression (19) reaches the column allowable value. The condition of true column failure would only exist, however, if the cross section of the stringer were completely immune to forced deformations induced by skin buckles. As mentioned previously, the problem of interaction between forced deformation and column failure is probably more serious in curved than in plane webs, and fragmentary data indicate that no practical stringer section may be completely free from interaction effects.

Since it appears that there will be some interaction in most cases, the investigation of reference 5 was carried out in the region where the interaction is clearly large; namely, on stringers designed to fail by forced crippling in the case of pure-torque loading. Five cylinders of identical construction were built; one was tested in pure compression, one in pure torsion, and the other three in combined compression and torsion. The results were fitted by the interaction formula

$$\left(\frac{T}{T_o}\right)^{1.5} + \frac{P}{P_o} = 1.00 \qquad (26)$$

where T and P are the torque and the compressive load that cause stringer failure when acting simultaneously. T_o is the torque causing stringer failure when acting alone, and P_o is the compressive load causing stringer failure when acting alone. When this formula is used, it is not necessary to compute the stringer stress by the method described previously for combined loading; a stringer-stress computation is made only for the case of a pure torque to calculate T_o. Ideally, the load P_o would also be calculated, but at present it would be safer to obtain this load by a compression test on one bay of the complete cylinder, or on a sector of this bay large enough to contain at least five stringers.

The discussions and formulas for curved diagonal tension have been given on the assumption that the structure considered is a circular cylinder. Evidently, more general types of structure may be analyzed by the same formulas by the usual device of analyzing small regions or individual panels. The questions of detail procedure that will arise must be answered by individual judgment, because more general methods are not available at present. The results will obviously be more uncertain, for instance, if there are large changes in shear flow from one panel to the next. It should be borne in mind that in such cases problems in stress distribution exist even when the skin is not buckled into a diagonal-tension field; the existence of these problems is often overlooked because elementary theories are normally used to compute the shear flows.

As numerical examples of strength analyses of curved diagonal-tension webs, two cylinders will be analyzed that were tested in the investigation of reference 5. The cylinders were of nominally identical construction and differed only in loading conditions. They had 12 stringers of Z-section and rings also of Z-section. The rings were notched to let the stringers pass through them. Clip angles were used to connect the stringers to the rings and at the same time to reinforce the edge of the notch. The analysis will be made for the test loads that produced failure. The third example illustrates the calculation of the angle of twist for the cylinder used in the first example.

Example 1. Pure torsion

The example chosen is cylinder 1 of reference 5. The material is 24S - T3 aluminum alloy.

Basic data:

$R = 15.0$ in. $\quad t = 0.0253$ in. $\quad d = 15.0$ in.

$E = 10.6 \times 10^3$ksi $\quad \mu = 0.32 \quad h = 7.87$ in.

$\pi R^2 (1 - \frac{1}{6} \phi^2) = 675 \qquad G = 4.0 \times 10^3$ ksi

Stringers: Z-section $\frac{3}{4} \times 1 \times \frac{3}{4} \times 0.040$;

$A_{ST} = 0.0925$ in.^{2}

Rings: Z-section $\frac{3}{4} \times 2 \times \frac{3}{4} \times 0.081$;

$A_{RG} = 0.251$ in.^{2}

Nominal shear stress

$$\tau = \frac{T}{2\pi R^2 (1 - \frac{1}{6}\phi^2)} \qquad (27)$$

where ϕ is the angle subtended by two stringers. The quantity $(1 - \frac{1}{6}\phi^2)$ in the denomination is neglected if its value is small.

$$\tau = \frac{388}{2 \times 675 \times 0.0253} = 11.36 \text{ ksi}$$

Buckling stress

$$Z = \frac{7.87^2}{15.0 \times 0.0253} \sqrt{1 - 0.32^2} = 155$$

From figure 2.2.4 - 1 $k_s = 35$

$$\tau_{cr} = 35 \frac{\pi^2 \times 10.6 \times 10^3 \times 7.87^2}{12 \times 15^2 \times 155^2} = 3.50 \text{ ksi}$$

Loading ratio

$$\frac{\tau}{\tau_{cr}} = \frac{11.36}{3.50} = 3.24$$

Diagonal - tension factor

$$300 \frac{td}{Rh} = 300 \frac{0.0253 \times 15.0}{15.0 \times 7.87} = 0.965$$

From Figure 2.2.4 - 2 $k = 0.63$

First approximation for angle of diagonal tension

$$R_R = \frac{dt}{A_{RG}} = \frac{15.0 \times 0.0253}{0.251} = 1.513$$

$$R_S = \frac{ht}{A_{ST}} = \frac{7.87 \times 0.0253}{0.0925} = 2.155$$

$$\frac{1 + R_S}{1 + R_R} = 1.256$$

$$\frac{\frac{h}{R}\sqrt{\frac{E}{\tau}}}{\sqrt{1 + R_R}} = \frac{\frac{7.87}{15.0}\sqrt{\frac{10.6 \text{ x } 10^3}{11.36}}}{\sqrt{1 + 1.513}} = 10.1$$

From figure 2.2.4-4: $\alpha_{PDT} = 32.3^{o}$

From figure 2.2.4-5: $\frac{\alpha}{\alpha_{PDT}} = 0.90$

$$\alpha = 32.3^{o} \text{ x } 0.90 = 29.0^{o}$$

Stress and strain formulas

From formulas (2) and (3):

$$\sigma_{ST} = -\frac{0.63 \text{ x } 11.36}{0.465 + 0.5(1 - 0.63)} \cot \alpha = -11.03 \cot \alpha \text{ ksi}$$

$$\epsilon_{ST} = -1.04 \text{ x } 10^{-3} \cot \alpha$$

$$\sigma_{RG} = -\frac{0.63 \text{ x } 11.36 \tan \alpha}{0.660 + 0.5(1 - 0.63)} = -8.46 \tan \alpha \text{ ksi}$$

$$\epsilon_{RG} = -0.800 \text{ x } 10^{-3} \tan \alpha$$

$$\frac{1}{24}\left(\frac{h}{R}\right)^2 = \frac{1}{24}\left(\frac{7.87}{15.0}\right)^2 = 11.48 \text{ x } 10^{-3}$$

$$\frac{\tau}{E} = 1.07 \text{ x } 10^{-3}$$

First cycle

$\alpha = 29^{o}$ $\quad$ $\tan \alpha = 0.554$ $\quad$ $\cot \alpha = 1.805$

From figure 2.2.4-3:

$$\frac{\epsilon E}{\tau} = 1.90; \ \epsilon = 1.90 \text{ x } 1.07 \text{ x } 10^{-3} = 2.035 \text{ x } 10^{-3}$$

$$\epsilon_{ST} = -1.04 \text{ x } 10^{-3} \text{ x } 1.805 = -1.875 \text{ x } 10^{-3}$$

$$\epsilon_{RG} = -0.800 \text{ x } 10^{-3} \text{ x } 0.554 = -0.444 \text{ x } 10^{-3}$$

According to formula (4a):

$$\tan^2 \alpha = \frac{2.035 + 1.875}{2.035 + 0.440 + 11.48} = 0.280$$

$$\tan \alpha = 0.529$$

Second cycle

The final value of α is closer to the computed value of the preceding cycle than to the initially assumed value; therefore, take as the next approximation

$$\tan \alpha = 0.529 + \frac{1}{4}(0.554 - 0.529) = 0.535$$

$\cot \alpha = 1.87$ $\quad$ $\alpha = 28^{o} \, 10'$

This slenderness ratio is so low that there is obviously a large margin against column failure at the computed value of stringer stress,

Stringers, forced-crippling failure:

From figure 2.2.4-9:

$$\frac{\sigma_{max.}}{\sigma} = 1.16$$

$$\sigma_{ST_{max}} = 20.6 \text{ x } 1.16 = -23.85 \text{ ksi}$$

$$\frac{t_{ST}}{t} = \frac{0.0404}{0{:}0253} = 1.60$$

From figure 2.2.2-10 $\sigma_o = -22.3$ ksi

Note: The "design allowable" value of the stringer stress (−22.3 ksi) is 7 percent greater than the calculated value of −23.85 ksi. Therefore, the calculation would have predicted a torque 7 percent lower than the actual failing torque, that is, the calculation is 7 percent conservative. The "best possible estimate" of the allowable stress (based on the middle of the scatter band instead of the lower edge) would be 25 percent higher than the "design allowable" value; a strength prediction based on this value thus would have been 18 percent unconservative.

Example 2 Combined loading

The example chosen to demonstrate the analysis of a cylinder under combined torsion and compression is cylinder 5 of reference 5. In order to simplify the demonstration by making use of partial results obtained in example 1, it will be assumed that the dimensions given for example 1 apply; actually, some of the dimensions differed by as much as 2 percent.

Basic data

Dimensions as in example 1.

T = 303 inch-kips $\quad$ P = −13.5 kips

Compression area:

12 stringers = 12 x 0.0925 = 1.11 in.^{2}

Sheet (100%) = π x 30 x 0.0253 = 2.38 in.^{2}

Total = 3.49 in.^{2}

Basic stresses

$$\sigma_{cr,0} = 3.50 \text{ ksi (see example 1)}$$

$$\sigma_{cr,0} = -5.95 \text{ ksi}$$

The latter value is computed according to the recommendations of section 3.2.

At the design loads, the nominal stresses are

$$\sigma = \frac{-13.5}{3.49} = -3.87 \text{ ksi}$$

$$\tau = \frac{303}{34.2} = 8.86 \text{ ksi}$$

Interaction factors

From formulas 18:

$$A = \frac{3.50}{-5.95} = 0.588 \qquad B = \frac{8.86}{-3.87} = -2.29$$

$$R^T = 0.878 \qquad R^C = 0.228$$

$$\tau_{cr} = 0.878 \times 3.50 = 3.07 \text{ ksi}$$

$$\sigma_{cr} = -0.228 \times 5.95 = 1.356 \text{ ksi}$$

Compressive stress due to axial load

From formulas (23) and (22), respectively:

$$D = 0.445 \times 2.155 \times 0.228 \times \sqrt{1.356} = 0.254$$

$$\sigma^C_{ST} = -10.55 \text{ ksi} \qquad \epsilon^C_{ST} = -0.996 \times 10^{-3}$$

Diagonal-tension factor

$$\frac{\tau}{\tau_{cr}} = \frac{8.86}{3.07} = 2.88 \qquad k = 0.59$$

Stress and strain formulae

From formula (24):

$$\sigma^T_{ST} = -\frac{0.59 \times 8.86 \cot \alpha}{0.465 + 0.5(1 - 0.59) \times 0.878}$$

$$= -8.10 \cot \alpha \text{ ksi}$$

$$\epsilon^T_{ST} = -0.764 \times 10^{-3} \cot \alpha$$

From formula (3):

$$\sigma^T_{RG} = -\frac{0.59 \times 8.86 \tan \alpha}{0.660 + 0.5(1 - 0.59)} = -6.05 \tan \alpha \text{ ksi}$$

$$\epsilon_{RG} = -0.570 \times 10^{-3} \tan \alpha$$

Computation cycle

Only the last cycle will be shown here. This computation is essentially the same as for a case of pure torsion (example 1), except that the stringer strain due to axial load (ϵ^C_{ST}) is added to the strain due to the torque (ϵ^T_{ST}).

The first approximation to the angle α may be obtained by disregarding the compression, that is to say, in the same manner as in example 1. An analyst with some experience may improve this approximation by adding a correction for the effect of the axial load (compression load will steepen the angle).

Assume $\alpha = 28^\circ 30'$; $\tan \alpha \doteq 0.543$; $\cot \alpha = 1.84$

From figure 2.2.4-3:

$$\frac{\epsilon E}{\tau} = 1.86 \; ;$$

$$\epsilon = 1.86 \times \frac{8.86}{10.6 \times 10^3} = 1.552 \times 10^{-3}$$

From the strain formulas:

$$\epsilon^T_{ST} = -0.764 \times 10^{-3} \times 1.84 = -1.405 \times 10^{-3}$$

$$\epsilon_{RG} = -0.570 \times 10^{-3} \times 0.543 = -0.310 \times 10^{-3}$$

$$\tan^2 \alpha = \frac{1.552 + 1.405 + 0.996}{1.552 + 0.310 + 11.48} = 0.296$$

$$\tan \alpha = 0.544$$

This result agrees with the assumed value within the accuracy of calculation and thus constitutes the final value.

By the stress formulas:

$$\sigma^T_{ST} = \frac{-8.10}{0.544} = -14.90 \text{ ksi}$$

Therefore the total stringer stress is

$$\sigma_{ST} = -14.90 - 10.55 = -25.45 \text{ ksi}$$

The value measured (on a cylinder with slightly different actual dimensions) was –25 ksi.

Failure

Since the torque is much less than in example 1 (pure-torque case), there is a wide margin against web rupture.

The margin against stringer failure is evaluated by formula (24). According to test (reference 5), the cylinder failed under pure compression at $P_o = 42.0$ kips. Under pure torque, the test gave $T_o = 388$ inch-kips: (the calculated value of T_o (example 1) is 7 percent lower). Thus, with the "design loads" T = 303 inch-kips and P = 13.5 kips

$$\left(\frac{T}{T_o}\right)^{1.5} + \frac{P}{P_o} = \left(\frac{303}{388}\right)^{1.5} + \frac{13.5}{42.0} = 1.01$$

Note: Because the "design loads" T and P used in this example were actually test failing loads and because the interaction curve was based on a series of tests on cylinders of these dimensions, the calculated value of 1.01 indicates that the analytical expression chosen for the interaction curve fits this particular test very well.

Example 3. Angle of twist

In this example, the angle of twist will be calculated for the cylinder of example 1 at the failing torque.

According to example 1:

$\tan a = 0.535$; $\cot a = 1.87$; $\sin 2a = 0.832$; $k = 0.63$

$$\frac{E}{G_{DT}} = \frac{4}{\sin^2 2a} + \frac{\tan^2 \alpha}{\frac{A_{ue}}{dt} + 0.5(1-k)} + \frac{\cot^2 a}{\frac{2A_F}{ht} + 0.5(1-k)}$$

$$\frac{E}{G_{DT}} = \frac{4}{0.832^2} + \frac{0.535^2}{0.660 + 0.5(1\text{-}0.63)} + \frac{1.872}{0.465 + 0.5(1\text{-}0.63)}$$

$$\frac{E}{G_{DT}} = 5.77 + 0.34 + 5.39 = 11.50$$

$$G_{DT} = \frac{E}{11.50} = \frac{10.6 \times 10^3}{11.50} = 0.922 \times 10^3 \text{ ksi}$$

$$\frac{1}{G_{IDT}} = \frac{1-k}{G} + \frac{k}{G_{DT}} \qquad (27)$$

$$\frac{1}{G_{IDT}} = \frac{0.37}{4 \times 10^3} + \frac{0.63}{0.922 \times 10^3} =$$

$$0.0925 \times 10^{-3} + 0.683 \times 10^{-3} = 0.775 \times 10^{-3}$$

$$G_{IDT} = 1.29 \times 10^3 \text{ ksi}$$

The torsion constant for the polygon section is

$$J = 2 \times \pi \times 15^3 \times 0.0253 \left(1 - \frac{7}{24} \times 0.524^2\right) = 492$$

For a length of 60 inches, the angle of twist is

$$\frac{TL}{G_{IDT}J} = \frac{388 \times 60}{1.29 \times 10^3 \times 492} = 0.0366 \text{ radian}$$

Interaction Curves

This section provides a solution for curved rectangular panels and cylinders under a combination of shear and direct axial stress. Ref. N.A.C.A. T.N. 1928.

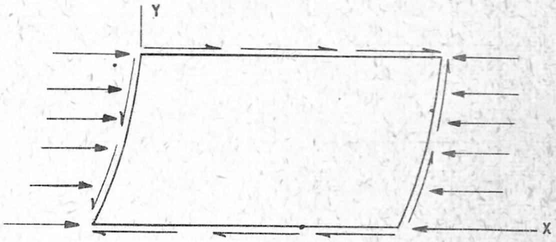


FIGURE 2.2.4–13

The combination of shear and axial stress that causes curve panels to buckle may be obtained from the equations

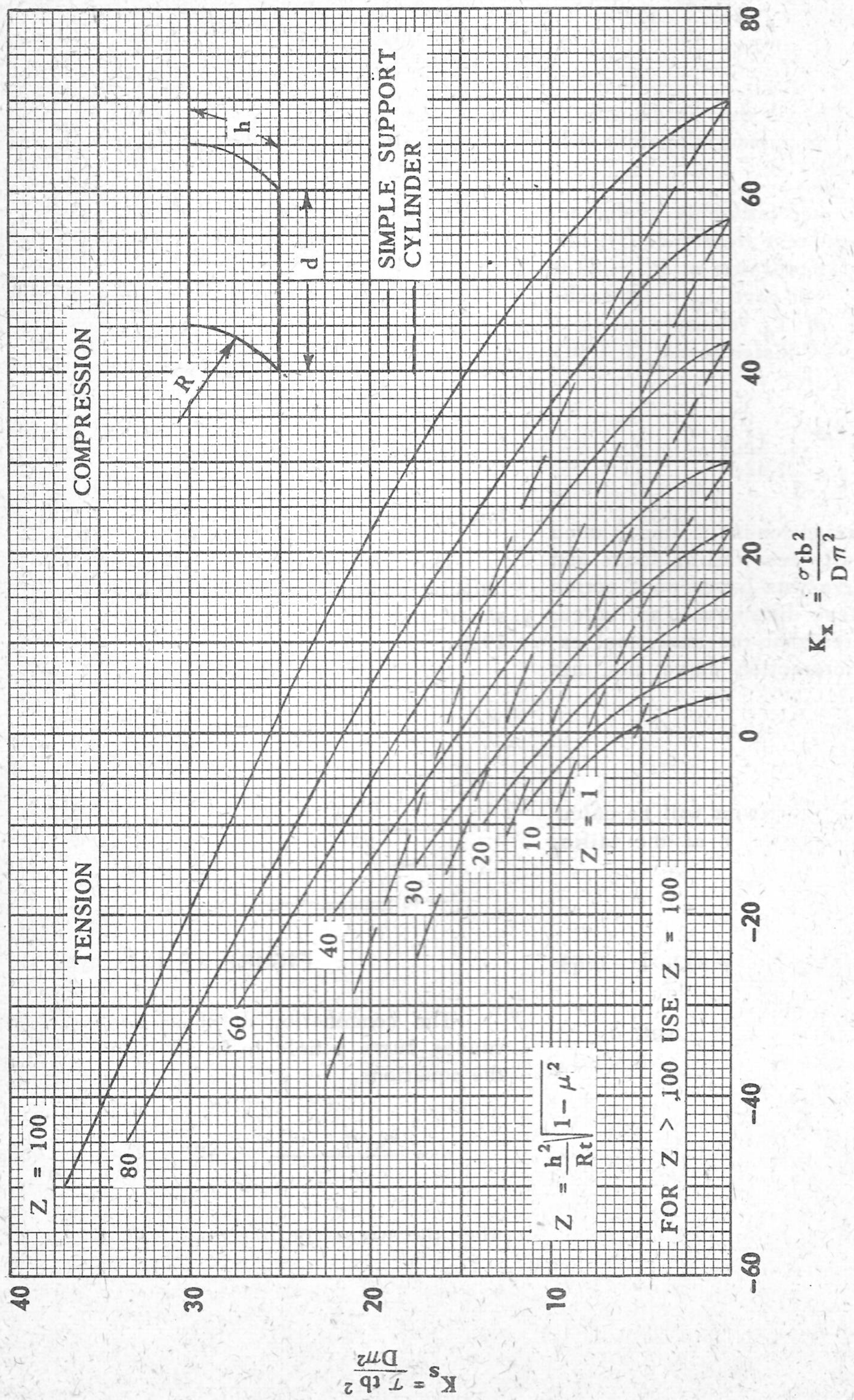
$$\tau = \frac{k_s \pi^2 D}{b^2 t} \quad \text{and} \quad \sigma_x = \frac{k_x \pi^2 D}{b^2 t}$$

when the stress coefficients k_s and k_x are known.

k_s and k_x are determined by use of Figures 2.2.4–14 to 2.2.4–17 b = axial or circumferential dimension of panel, whichever is smaller.

$$D = \frac{E\,t^3}{12\,(1 - \mu^2)}$$

The theoretical combinations of stress coefficients for curved rectangular panels having different ratios of circumferential to axial dimension are given in figures 2.2.4–14 to 2.2.4–17.

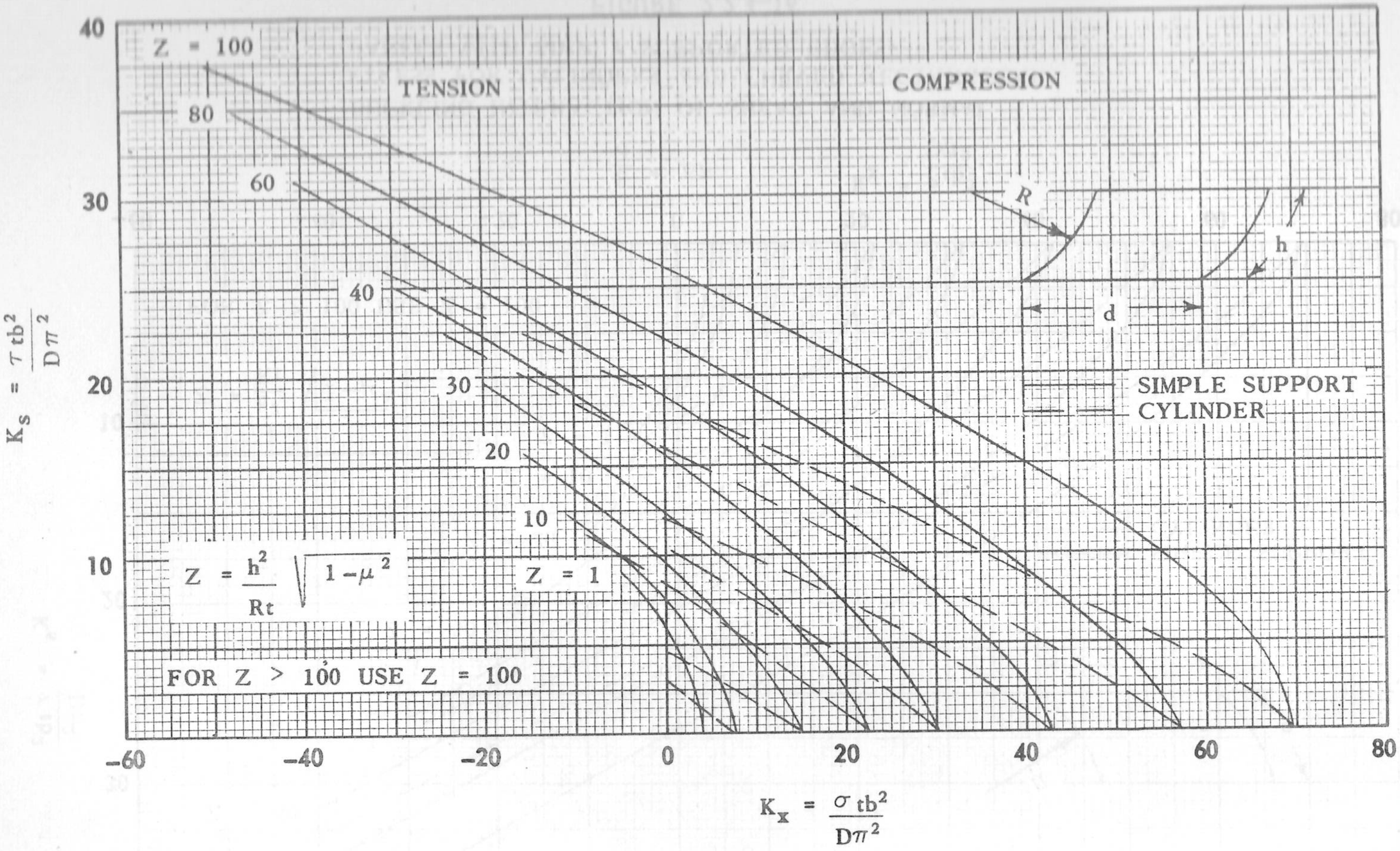


$\frac{h}{d} = .33$

BUCKLING INTERACTION OF SHEAR AND NORMAL STRESS FOR CYLINDERS AND CURVED RECTANGULAR PANELS WITH SIMPLY SUPPORTED EDGES

FIGURE 2.2.4–14

REF.: NACA TN 1928

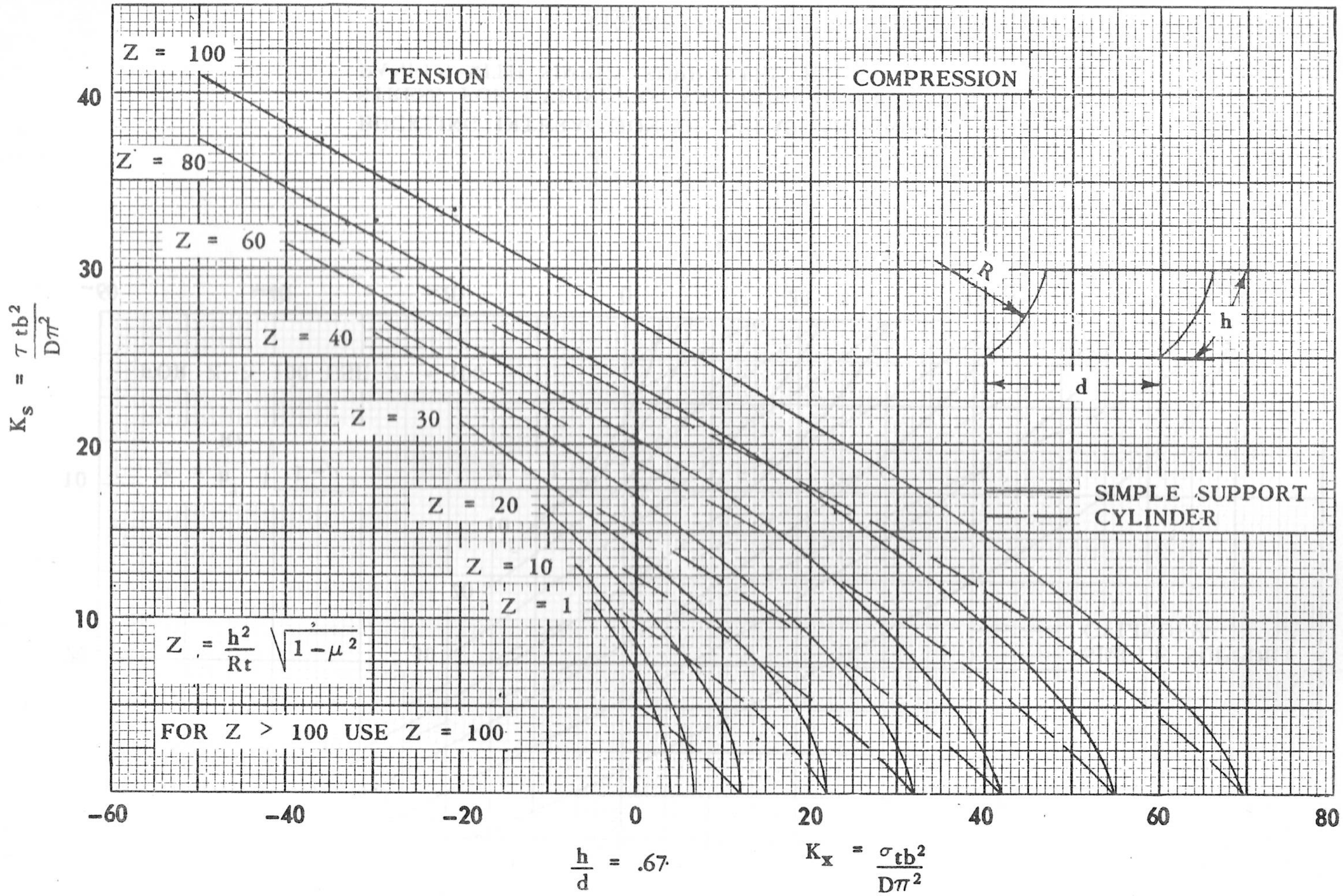


$\frac{h}{d} = .5$

BUCKLING INTERACTION OF SHEAR AND NORMAL STRESS FOR CYLINDERS AND CURVED RECTANGULAR PANELS WITH SIMPLY SUPPORTED EDGES

FIGURE 2.2.4-15

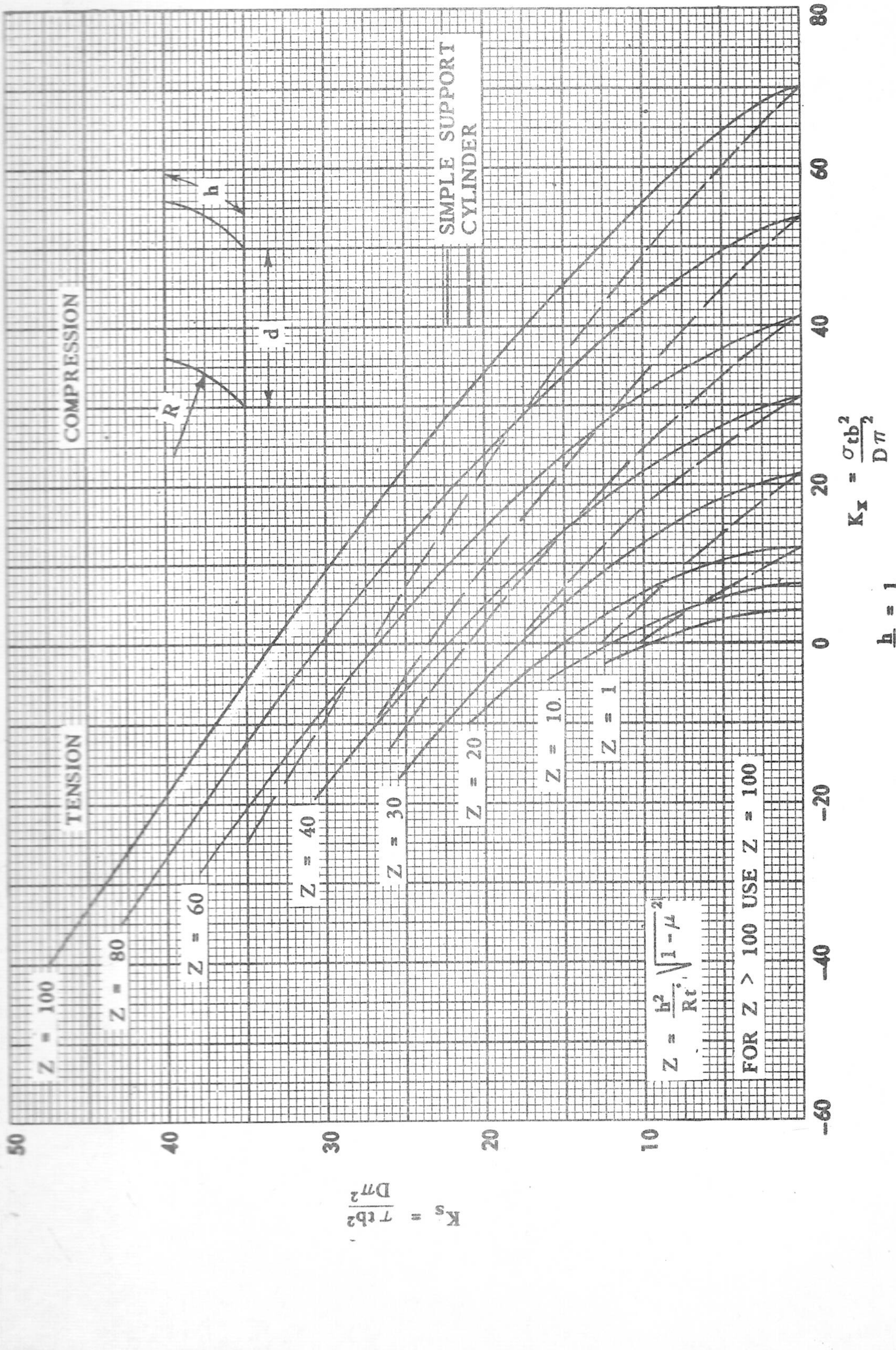
REF.: NACA TN 1928



BUCKLING INTERACTION OF SHEAR AND NORMAL STRESS FOR CYLINDERS AND CURVED RECTANGULAR PANELS WITH SIMPLY SUPPORTED EDGES.

FIGURE 2.2.4–16

REF.: NACA TN 1928



$\frac{h}{d} = 1$

BUCKLING INTERACTION OF SHEAR AND NORMAL STRESS FOR CYLINDERS AND CURVED RECTANGULAR PANELS WITH SIMPLY SUPPORTED EDGES

FIGURE 2.2.4–17

REF.: NACA TN 1928

BEAMS IN GENERAL

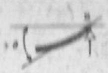
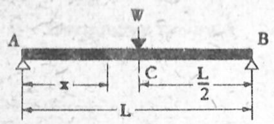
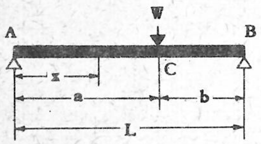
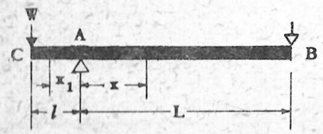
TABLE OF CONTENTS

3 Beams In General

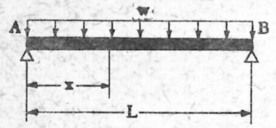
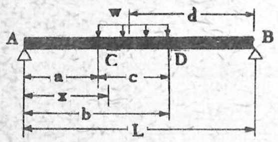
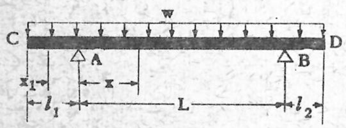
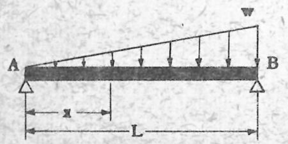
Use the Reference Table below for determining the method of solution of many common beam problems.

REFERENCE TABLE

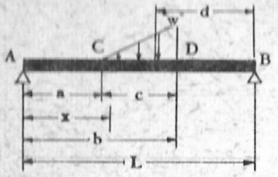
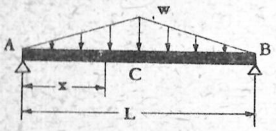
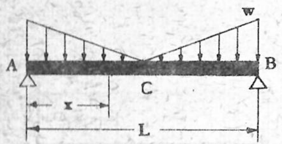
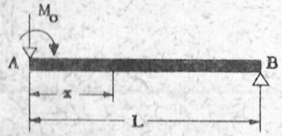
BEAM TYPE	REMARKS
Simply Supported	I Constant } See Section 3.1 for common types of loading. I Varying } Reactions, shear and B.M. are unaffected by varying I.
Cantilever	I Constant } See Section 3.2 for common types of loading. I Varying } Shear and B.M. are unaffected by varying I
Both Ends Fixed (Fixed Beams)	I Constant See Section 3.3.1 for standard loading cases. 3.3.2 for irregular loading cases. 3.3.3 for effect of axial load. 3.3.4 for effect of sinking support. I Varying See Section 3.3.5 for fixed end moments. 3.3.6 for stiffness and carry-over. 3.3.7 for effect of sinking support.
Propped Cantilever	I Constant See Section 3.4.1 for cases with prop at end. For irregular loading and/or prop at intermediate position, see Section 3.4.2. I Varying See Section 3.4.3.
Continuous Beam	I Constant throughout See Section 3.5.1, 3.5.2 and 3.5.3. I Constant per span See Section 3.5.4, 3.5.5 and 3.5.6. I Varying per span See Section 3.5.7.
R→ Curved Beams	For circumferential stresses see 3.6

SIMPLY SUPPORTED BEAM	REACTIONS R_A AND R_B (LB) [POSITIVE UPWARD]	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD] AND SLOPE θ (RADIANS) POSITIVE THUS 
(1) CENTRE LOAD 	$R_A = R_B = \frac{W}{2}$	(A to C) $M = \frac{Wx}{2}$ (C to B) $M = \frac{W}{2}(L - x)$ Max. M $= \frac{WL}{4}$ at C	(A to C) $y = -\frac{Wx}{48\,EI}(3L^2 - 4x^2)$ Max. $y = -\frac{WL^3}{48\,EI}$ at C $\theta = -\frac{WL^2}{16\,EI}$ at A
(2) INTERMEDIATE LOAD  a > b	$R_A = \frac{Wb}{L}$ $R_B = \frac{Wa}{L}$	(A to C) $M = \frac{Wb}{L}x$ (B to C) $M = \frac{Wa}{L}(L - x)$ Max. $M = \frac{Wab}{L}$ at C	(A to C) $y = -\frac{Wbx}{6\,EIL}[a(L+b) - x^2]$ (B to C) $y = -\frac{Wa(L-x)}{6\,EIL}[b(L+a) - (L-x)^2]$ Max. $y = -\frac{Wb(L^2 - b^2)^{3/2}}{9\sqrt{3}\,EIL}$ at $x = \sqrt{\frac{L^2 - b^2}{3}}$ Approx. y at centre $(x = L/2)$ $= -\frac{Wb}{48\,EI}[3L^2 - 4b^2]$ $\theta = -\frac{Wab}{6EIL}(L+b)$ at A $\theta = \frac{Wab}{6EIL}(L+a)$ at B
(3) OVERHANGING LOAD 	$R_A = \frac{W}{L}(L + l)$ $R_B = -\frac{Wl}{L}$	(C to A) $M = -W(l - x_1)$ (A to B) $M = -\frac{Wl}{L}(L - x)$ Max. $M = -Wl$ at A	(C to A) $y = -\frac{Wl}{6EI}[3lx_1 - x_1^2 + 2Ll]$ (A to B) $y = +\frac{Wlx}{6EIL}(L - x)(2L - x)$ Max. y upward $= \frac{WlL^2}{15.59\,EI}$ at x = 0.42265 L Max. y downward $= -\frac{Wl^2}{3\,EI}(L + l)$ at C

3.1 Simply Supported Beams: Tabular Summary

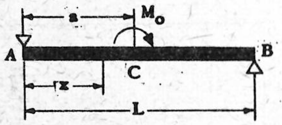
SIMPLY SUPPORTED BEAM	REACTIONS R_A AND R_B (LB) [POSITIVE UPWARD]	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD] AND SLOPE θ (RADIANS) POSITIVE THUS 
(4) UNIFORM LOADING 	$R_A = R_B = \frac{W}{2}$ where $W = wL$	$M = \frac{1}{2} W \left(x - \frac{x^2}{L}\right)$ Max. $M = \frac{WL}{8}$ at $x = \frac{L}{2}$	$y = -\frac{1}{24}\frac{Wx}{EIL}\left[L^3 - 2Lx^2 + x^3\right]$ Max. $y = -\frac{5}{384}\frac{WL^3}{EI}$ at $x = \frac{L}{2}$ $\theta = -\frac{1}{24}\frac{WL^2}{EI}$ at A
(5) PARTIAL UNIFORM LOADING 	$R_A = \frac{Wd}{L}$ $R_B = \frac{W}{L}\left(a + \frac{c}{2}\right)$ where $W = wc$ and $d = L - \left(a + \frac{c}{2}\right)$	(A to C) $M = R_A x$ (C to D) $M = R_A x - \frac{W(x-a)^2}{2c}$ (D to B) $M = R_A x - W\left[x - \frac{1}{2}(a+b)\right]$ Max. $M = \frac{Wd}{L}\left(a + \frac{cd}{2L}\right)$ at $x = \left(a + \frac{cd}{L}\right)$	(A to C) $y = \frac{Wx}{48EIL}\left[8d(x^2 - L^2 + d^2) + c^2(2L - 2b + c)\right]$ (C to D) $y = \frac{W}{48EIL}\left[8dx(x^2 - L^2 + d^2) + c^2x(2L - 2b + c) - \frac{2L}{c}(x-a)^4\right]$ (D to B) $y = \frac{Wx}{48EIL}\left[8d(x^2 - L^2 + d^2) + c^2(c - 2b)\right]$ $\theta = \frac{W}{48EIL}\left[8d(d^2 - L^2) + c^2(2L + c - 2b)\right]$ at A $\theta = \frac{W}{48EIL}\left[8d(L-d)(2L-d) + c^2(c-2b)\right]$ at B
(6) OVERHUNG AT BOTH ENDS UNIFORM LOAD 	$R_A = \frac{W}{2L}\left[L + l_1 - l_2\right]$ $R_B = \frac{W}{2L}\left[L + l_2 - l_1\right]$ where $W = w(L + l_1 + l_2)$	(C to A) $M = -\frac{w}{2}(l_1 - x_1)^2$ (A to B) $M = -\frac{w}{2}(l_1 + x)^2 + R_A x$	At C $y = -\frac{Wl_1}{24EIL'}, \left[2L(l_2^2 + 2l_1^2) + 3l_1^2 - L^3\right]$ where $L^1 = (L + l_1 + l_2)$ At D $y = -\frac{Wl_2}{24EIL'}, \left[2L(l_1^2 + 2l_2^2) + 3l_2^2 - L^3\right]$
(7) TRIANGULAR LOADING 	$R_A = \frac{1}{3}W$ $R_B = \frac{2}{3}W$ where $W = \frac{1}{2}wL$	$M = \frac{W}{3}\left(x - \frac{x^3}{L^2}\right)$ Max. $M = 0.128\ WL$ at $x = \frac{L\sqrt{3}}{3} = 0.5774L$	$y = -\frac{1}{180}\frac{Wx}{EIL^2}(3x^4 - 10L^2x^2 + 7L^4)$ Max. $y = -0.01304\frac{WL^3}{EI}$ at $x = 0.519L$ $\theta = -\frac{7}{180}\frac{WL^2}{EI}$ at A $\theta = +\frac{8}{180}\frac{WL^2}{EI}$ at B

3.1 Simply Supported Beams: Tabular Summary (Cont'd)

SIMPLY SUPPORTED BEAM	REACTIONS R_A AND R_B (LB) [POSITIVE UPWARD]	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD] AND SLOPE θ (RADIANS) POSITIVE THUS 
(8) PARTIAL TRIANGULAR LOADING 	$R_A = \frac{Wd}{L}$ $R_B = \frac{W}{L}(L-d)$ where $W = \frac{1}{2}wc$ $d = L - (a + \frac{2}{3}c)$	(A to C) $M = R_A x$ (C to D) $M = R_A x - \frac{W(x-a)^3}{3c^2}$ (D to B) $M = R_A x - \frac{W}{3}(3x - a - 2b)$ Max. $M = \frac{Wd}{L}(a + \frac{2}{3}c\sqrt{\frac{d}{L}})$ at $x = a + c\sqrt{\frac{d}{L}}$	(A to C) $y = \frac{Wx}{6EIL}[d(x^2 - L^2 - d^2 + \frac{c^2}{6}) + \frac{c^3}{135}]$ (C to D) $y = \frac{Wx}{6EIL}[d(x^2 - L^2 + d^2 + \frac{c^2}{6}) + \frac{c^3}{135} - \frac{L}{10x}\frac{(x-a)^5}{c^2}]$ (D to B) $y = \frac{W(L-x)}{6EIL}[c^2(\frac{17}{270}c - \frac{b}{6}) - d(x^2 + Lx + d^2)]$ $\theta = \frac{W}{6EIL}[\frac{c^2}{6}(L - b + \frac{17}{270}c) - d(L^2 - d^2)]$ at A $\theta = \frac{W}{6EIL}[d(L-d)(2L-d) + \frac{c^2}{6}(\frac{17}{45} - b)]$ at B
(9) TRIANGULAR LOADING 	$R_A = R_B = \frac{W}{2}$ where $W = \frac{1}{2}wL$	(A to C) $M = \frac{W}{6}(3x - \frac{4x^3}{L^2})$ (C to B) $M = \frac{W}{6}[3(L-x) - 4\frac{(L-x)^3}{L^2}]$ Max. $M = \frac{WL}{6}$ at C	(A to C) $y = -\frac{Wx}{30EIL^2}(\frac{5}{4}L^2 - x^2)^2$ Max. $y = -\frac{1}{60}\frac{WL^3}{EI}$ at C $\theta = -\frac{5}{96}\frac{WL^2}{EI}$ at A
(10) TRIANGULAR LOADING 	$R_A = R_B = \frac{W}{2}$ where $W = \frac{1}{2}wL$	(A to C) $M = \frac{1}{2}W(x - \frac{2x^2}{L} + \frac{4}{3}\frac{x^3}{L^2})$ (C to B) $M = \frac{1}{2}W[(L-x) - 2\frac{(L-x)^2}{L} + \frac{4}{3}\frac{(L-x)^3}{L^2}]$ Max. $M = \frac{WL}{12}$ at C	(A to C) $y = \frac{1}{12}\frac{W}{EI}(x^3 - \frac{x^4}{L} + \frac{2}{5}\frac{x^5}{L^2} - \frac{3}{8}L^2 x)$ Max. $y = -\frac{3}{320}\frac{WL^3}{EI}$ at C $\theta = -\frac{1}{32}\frac{WL^2}{EI}$ at A
(11) END COUPLE 	$R_A = -\frac{M_o}{L}$ $R_B = +\frac{M_o}{L}$	$M = M_o + R_A x$ Max. $M = M_o$ at A	$y = \frac{1}{6}\frac{M_o}{EI}(3x^2 - \frac{x^3}{L} - 2Lx)$ Max. $y = -0.0642\frac{M_oL^2}{EI}$ at $x = 0.422L$ $\theta = -\frac{1}{3}\frac{M_oL}{EI}$ at A

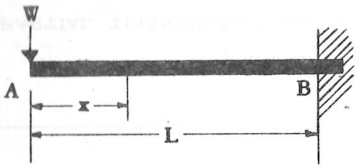
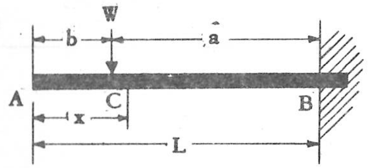
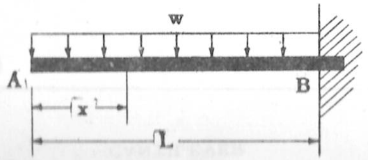
For numerical example, see Report Gen/1090/801.4

3.1 Simply Supported Beams: Tabular Summary (Cont'd)

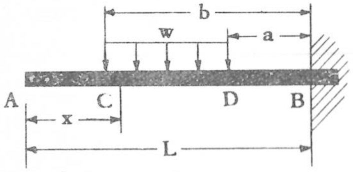
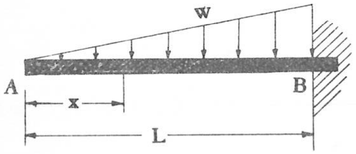
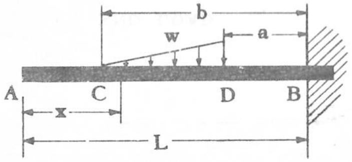
SIMPLY SUPPORTED BEAM	REACTIONS R_A AND R_B (LB) [POSITIVE UPWARD]	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD] AND SLOPE θ (RADIANS) POSITIVE THUS 
(12) INTERMEDIATE COUPLE 	$R_A = -\frac{M_o}{L}$ $R_B = +\frac{M_o}{L}$	(A to C) $M = R_A x$ (C to B) $M = R_A x + M_o$ Max. $-M = R_A a$ just to the left of C Max. $+M = R_A a + M_o$ just to the right of C	(A to C) $y = \frac{1}{6}\frac{M_o}{EI}\left[\left(6a - \frac{3a^2}{L} - 2L\right)x - \frac{x^3}{L}\right]$ (C to B) $y = \frac{1}{6}\frac{M_o}{EI}\left[3a^2 + 3x^2 - \frac{x^3}{L} - \left(2L + \frac{3a^2}{L}\right)x\right]$ $\theta = -\frac{1}{6}\frac{M_o}{EI}\left(2L - 6a + \frac{3a^2}{L}\right)$ at A $\theta = +\frac{1}{6}\frac{M_o}{EI}\left(L - \frac{3a^2}{L}\right)$ at B $\theta = \frac{M_o}{EI}\left(a - \frac{a^2}{L} - \frac{L}{3}\right)$ at C

For numerical example, see Report Gen/1090/801.5

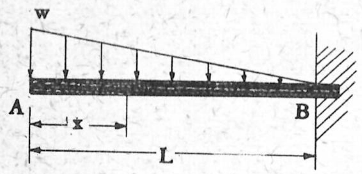
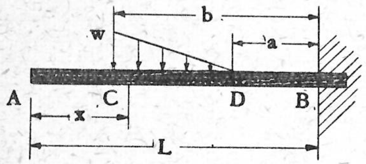
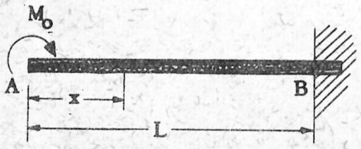
3.1 Simply Supported Beams: Tabular Summary (Cont'd)

CANTILEVER	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD] AND SLOPE θ (RADIANS)
(1) END LOAD 	$M = -Wx$ $\text{Max. } M = -WL \text{ at } B$	$y = -\frac{1}{6}\frac{W}{EI}(x^3 - 3L^2x + 2L^3)$ $\text{Max. } y = -\frac{1}{3}\frac{WL^3}{EI} \text{ at } A$ $\theta = \frac{1}{2}\frac{WL^2}{EI} \text{ at } A$
(2) INTERMEDIATE LOAD 	(A to C) $M = 0$ (C to B) $M = -W(x-b)$ $\text{Max. } M = -Wa \text{ at } B$	(A to C) $y = -\frac{1}{6}\frac{W}{EI}(3a^2L - a^3 - 3a^2x)$ (C to B) $y = -\frac{1}{6}\frac{W}{EI}[(x-b)^3 - 3a^2(x-b) + 2a^3]$ $\text{Max. } y = -\frac{1}{6}\frac{W}{EI}(3a^2L - a^3)$ $\theta = \frac{1}{2}\frac{Wa^2}{EI} \text{ from A to C}$
(3) UNIFORM LOADING 	$M = -\frac{W}{2L}x^2$ where $W = wL$	$y = -\frac{1}{24}\frac{W}{EIL}(x^4 - 4L^3x + 3L^4)$ $\text{Max. } y = -\frac{WL^3}{8EI}$ $\theta = \frac{1}{6}\frac{WL^2}{EI} \text{ at } A$

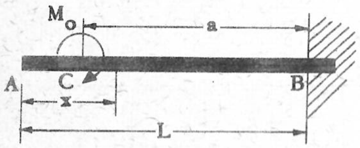
3.2 Cantilevers: Tabular Summary

CANTILEVER	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD] AND SLOPE θ (RADIANS)
(4) PARTIAL UNIFORM LOADING 	(A to C) $M = 0$ (C to D) $M = -\frac{1}{2}\frac{W}{(b-a)}(x-L+b)^2$ (D to B) $M = -\frac{1}{2}W(2x-2L+a+b)$ Max. $M = -\frac{1}{2}W(a+b)$ at B where $W = w(b-a)$	For y and θ, see Roark, p. 98, Case 4.
(5) TRIANGULAR LOADING 	$M = -\frac{1}{3}\frac{W}{L^2}x^3$ Max. $M = -\frac{WL}{3}$ at B where $W = \frac{1}{2}wL$	$y = -\frac{1}{60}\frac{W}{EIL^2}(x^5 - 5L^4x + 4L^5)$ Max. $y = -\frac{1}{15}\frac{WL^3}{EI}$ at A $\theta = \frac{1}{12}\frac{WL^2}{EI}$ at A
(6) PARTIAL TRIANGULAR LOADING 	(A to C) $M = 0$ (C to D) $M = -\frac{1}{3}\frac{W(x-L+b)^3}{(b-a)^2}$ (D to B) $M = -\frac{1}{3}W(3x-3L+b+2a)$ Max. $M = -\frac{1}{3}W(b+2a)$ at B where $W = \frac{1}{2}w(b-a)$	For y and θ see Roark, p. 99, Case 6.

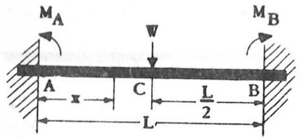
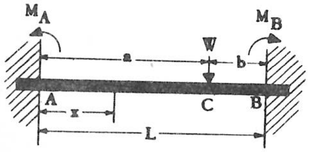
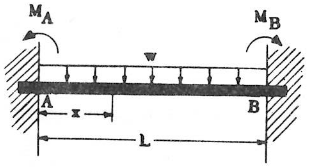
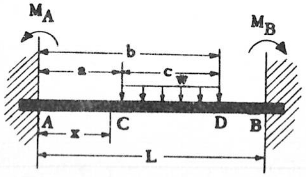
3.2 Cantilevers: Tabular Summary (Cont'd)

CANTILEVER	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD] AND SLOPE θ (RADIANS)
(7) TRIANGULAR LOADING 	$M = -\frac{1}{3}\frac{W}{L^2}(3Lx^2 - x^3)$ $\text{Max. } M = -\frac{2}{3} WL$ at B where $W = \frac{1}{2} wL$	$y = -\frac{1}{60}\frac{W}{EIL^2}(5Lx^4 + 11L^5 - x^5 - 15L^4x)$ $\text{Max. } y = -\frac{11}{60}\frac{WL^3}{EI}$ at A $\theta = \frac{1}{4}\frac{WL^2}{EI}$ at A
(8) PARTIAL TRIANGULAR LOADING 	(A to C) $M = 0$ (C to D) $M = -\frac{1}{3} W\left[\frac{3(x - L + b)^2}{(b - a)} - \frac{(x - L + b)^3}{(b - a)^2}\right]$ (D to B) $M = -\frac{1}{3} W(3x - 3L + 2b + a)$ $\text{Max. } M = -\frac{1}{3} W(2b + a)$ at B where $W = \frac{1}{2} w(b - a)$	For y and θ see Roark, p. 99, Case 8.
(9) END COUPLE 	$M = M_o$ $\text{Max. } M = M_o$ (throughout)	$y = \frac{1}{2}\frac{M_o}{EI}(L^2 - 2Lx + x^2)$ $\text{Max. } y = \frac{1}{2}\frac{M_o L^2}{EI}$ at A $\theta = -\frac{M_o L}{EI}$ at A

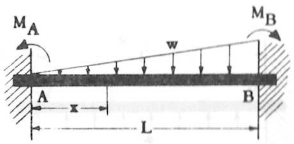
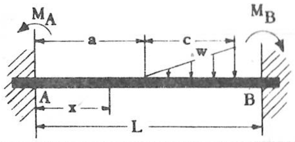
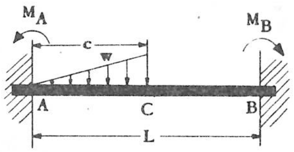
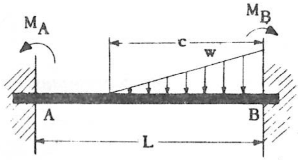
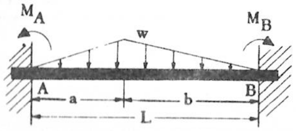
3.2 Cantilevers: Tabular Summary (Cont'd)

CANTILEVER	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD] AND SLOPE θ (RADIANS)
(10) INTERMEDIATE COUPLE  	(A to C) $M = 0$ (C to B) $M = M_o$ Max. $M = M_o$ (C to B)	(A to C) $y = \frac{M_o a}{EI}(L - \frac{1}{2}a - x)$ (C to B) $y = \frac{1}{2}\frac{M_o}{EI}(L - x)^2$ Max. $y = \frac{M_o a}{EI}(L - \frac{1}{2}a)$ at A $\theta = -\frac{M_o a}{EI}$ (A to C)

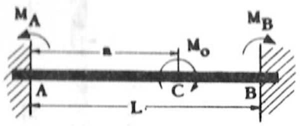
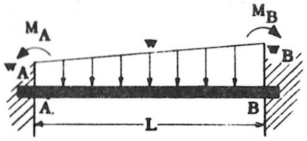
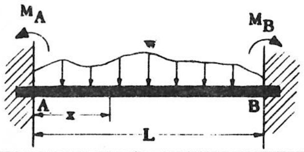
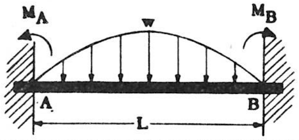
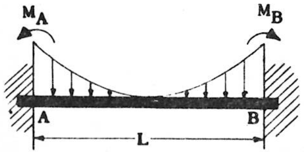
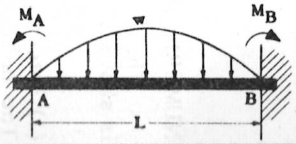
3.2 Cantilevers: Tabular Summary (Cont'd)

FIXED BEAM	REACTIONS R_A AND R_B (LB.) [POSITIVE UPWARD]	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD]
(1) CENTRE LOAD 	$R_A = R_B = \frac{W}{2}$	$M_A = -\frac{WL}{8}$ $M_B = -\frac{WL}{8}$ (A to C) $M = \frac{1}{8} W(4x - L)$ (C to B) $M = \frac{1}{8} W(3L - 4x)$ Max. + $M = \frac{1}{8} WL$ at C	(A to C) $y = -\frac{1}{48}\frac{W}{EI}(3Lx^2 - 4x^3)$ Max. $y = -\frac{1}{192}\frac{WL^3}{EI}$ at C
(2) INTERMEDIATE LOAD 	$R_A = \frac{Wb^2}{L^3}(3a + b)$ $R_B = \frac{Wa^2}{L^3}(3b + a)$	$M_A = -\frac{Wab}{L^2} b$ $M_B = -\frac{Wab}{L^2} a$ (A to C) $M = -\frac{Wab^2}{L^2} + R_A x$ (C to B) $M = -\frac{Wab^2}{L^2} + R_A x - W(x - a)$ Max. + M (at C) $= -\frac{Wab^2}{L^2} + R_A a$	(A to C) $y = \frac{1}{6}\frac{Wb^2x^2}{EIL^3}(3ax + bx - 3aL)$ (C to B) $y = \frac{1}{6}\frac{Wa^2(L - x)^2}{EIL^3}[(3b + a)(L - x) - 3bL]$ Max. $y = -\frac{2}{3}\frac{W}{EI}\frac{a^3b^2}{(3a + b)^2}$ at $x = \frac{2aL}{(3a + b)}$ for $a > b$
(3) UNIFORM LOADING 	$R_A = R_B = \frac{W}{2}$ where $W = wL$	$M_A = -\frac{WL}{12}$ $M_B = -\frac{WL}{12}$ $M = \frac{1}{2} W(x - \frac{x^2}{L} - \frac{L}{6})$ Max. + $M = \frac{1}{24} WL$ at $x = \frac{L}{2}$	$y = \frac{1}{24}\frac{Wx^2}{EIL}(2Lx - L^2 - x^2)$ Max. $y = -\frac{1}{384}\frac{WL^3}{EI}$ at $x = \frac{L}{2}$
(4) PARTIAL UNIFORM LOADING 	See Roark, p. 106, Case 34.		

3.3.1 Fixed Beams of Constant I

FIXED BEAM	REACTIONS R_A AND R_B (LB.) [POSITIVE UPWARD]	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD]
(5) TRIANGULAR LOADING 	$R_A = \frac{3}{10} W$ $R_B = \frac{7}{10} W$ $W = \frac{1}{2} wL$	$M_A = -\frac{WL}{15}$ $M_B = -\frac{WL}{10}$ $M = W(\frac{3}{10} x - \frac{L}{15} - \frac{1}{3}\frac{x^3}{L^2})$ Max. + M = 0.043 WL at x = 0.548 L	$y = \frac{1}{60}\frac{W}{EI}(3x^3 - 2Lx^2 - \frac{x^5}{2})$ Max. $y = -0.002617 \frac{WL^3}{EI}$ at x = 0.525 L
(6) PARTIAL TRIANGULAR LOADING 	See Roark, p. 107, Case 36		
(7) PARTIAL TRIANGULAR LOADING 	$W = \frac{1}{2} wC$ $R_B = W \frac{c^2}{L^2} (\frac{3}{2} - \frac{4}{5}\frac{c}{L})$ $R_A = W - R_B$	$M_A = -\frac{Wc}{15L^2}(10L^2 - 15cL + 6c^2)$ $M_B = -\frac{Wc^2}{10L^2}(5L - 4c)$	A special case of (6)
(8) PARTIAL TRIANGULAR LOADING 	$R_A = \frac{Wc^2}{10L^3}(5L - 2c)$ $R_B = W - R_A$ $W = \frac{1}{2} wc$	$M_A = -\frac{Wc^2}{30L^2}(5L - 3c)$ $M_B = -\frac{Wc}{30L^2}(10L^2 - 10cL + 3c^2)$	A special case of (6)
(9) TRIANGULAR LOADING 	$W = \frac{1}{2} wL$	$M_A = -\frac{W}{30L^2}[2a^2(a + 4b) + 3b^2(4a + b)]$ $M_B = -\frac{W}{30L^2}[3a^2(a + 4b) + 2b^2(4a + b)]$ For derivation, see Report GEN/1090/801.2	

3.3.1 Fixed Beams of Constant I (Cont'd)

FIXED BEAM	REACTIONS R_A AND R_B (LB.) [POSITIVE UPWARD]	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD]
(10) INTERMEDIATE COUPLE 	$R_A = - 6 \frac{M_o}{L^3} (aL - a^2)$ $R_B = 6 \frac{M_o}{L^3} (aL - a^2)$	$M_A = - \frac{M_o}{L^2} (4La - 3a^2 - L^2)$ $M_B = - \frac{M_o}{L^2} (2La - 3a^2)$	See Roark, p. 107, Case 37.
(11) TRAPEZOIDAL LOADING 	$W = \frac{W_A + W_B}{2} L$	$M_A = - \frac{L^2}{60} [5w_A + 2(w_B - w_A)]$ $M_B = - \frac{L^2}{60} [5w_A + 3(w_B - w_A)]$	
(12) LOADING $w = f(x)$ 		$M_A = - \frac{1}{L^2} \int_o^L wx(L - x)^2 \, dx$ $M_B = - \frac{1}{L^2} \int_o^L wx^2(L - x) \, dx$	
(13) PARABOLIC LOADING 	$W = \frac{2}{3} wL$	$M_A = - \frac{WL}{10}$ $M_B = - \frac{WL}{10}$ For derivation, see Report GEN/1090/801.3	
(14) INVERTED PARABOLIC LOADING 	$W = \frac{1}{3} wL$	$M_A = - \frac{WL}{20}$ $M_B = - \frac{WL}{20}$	
(15) SINUSOIDAL LOADING 	$W = \frac{2}{\pi} wL$	$M_A = - \frac{WL}{\pi^2}$ $M_B = - \frac{WL}{\pi^2}$ For derivation, see Report GEN/1090/801.1	

3.3.1 Fixed Beams of Constant I (Cont'd)

3.3.2 Fixed Beam of Constant I: Fixed end moments for irregular loading.

When it is not possible to determine the fixed end moments by means of one or more of the standard cases of 3.3.1, use the following method employing slope deflection formulas. (See Report GEN/1090 /803 for derivation).

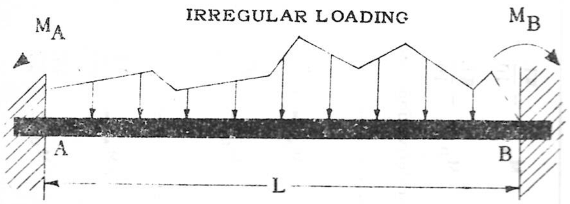
PROCEDURE:

By graphical integration (or otherwise) find the area of the free bending moment diagram and the spanwise position of its centroid. The fixed end moments, acting as shown in Fig. 3.3.2–1, are then:–

$$M_A = -\frac{2A}{L^2}(\bar{x}_A - 2\bar{x}_B)$$

$$M_B = \frac{2A}{L^2}(2\bar{x}_A - \bar{x}_B)$$

where L = beam span,
A = area under free B.M. curve,
$\bar{x}_A$ = distance of its centroid from A
and $\bar{x}_B$ = distance of its centroid from B



There is zero slope at A and B due to building in.

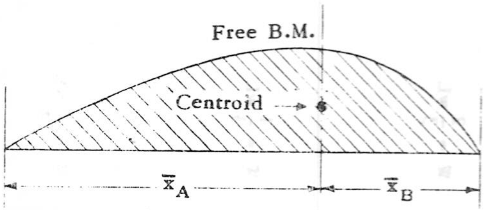


Figure 3.3.2–1

NUMERICAL EXAMPLE:

The loading on a fixed beam of span L = 20 in is shown in Curve I of Fig. 3.3.2–2.

(1) Integrate Curve I graphically to give Curve II. The ordinate at B of this curve represents the total load on the beam, viz 30.62 lb.

(2) Integrate Curve II graphically to find the moment of the external loads about B. Moment about B = 330.5 lb. in.

Free reaction at A, $R_A{}^1 = \frac{330.5}{20} = 16.53$ lb.

(3) Free B.M. on beam at any station can now be obtained and the free B.M. curve drawn (Curve III).

(4) Integrate Curve III to give Curve IV, the ordinate of which at B represents the area under the free B.M. curve, viz 1224. Integration of Curve IV gives the moment of the free B.M. about B, viz 12530 and hence:

$$\bar{x}_B = \frac{12530}{1224} = \underline{10.22 \text{ in.}}$$

$$\bar{x}_A = \underline{9.78 \text{ in.}}$$

(5) $L = 20$ $L^2 = 400$ $\bar{x}_A = 9.78$ $\bar{x}_B = 10.22$

$$\frac{2A}{L^2} = \frac{2 \times 1224}{400} = 6.12$$

$$M_A = -6.12(9.78 - 20.44) = +6.12 \times 10.66 = \underline{65.3 \text{ lb.in.}}$$

$$M_B = 6.12(19.56 - 10.22) = 6.12 \times 9.34 = \underline{57.2 \text{ lb.in.}}$$

The resultant B.M. is shown shaded in Fig. 3.3.2–3.

Check: For the same total load, uniformly distributed:

$$M_A = M_B = \frac{WL}{12} = 30.62 \times \frac{20}{12} = \underline{51.0 \text{ lb. in.}}$$

3.3.3 Fixed Beam of Constant I: Effect of axial load.

(a) The end moments on a fixed beam of constant I due to the combined effect of axial load and lateral load are given under Beam Columns (6.5) for common loading cases.

For moment distribution calculations, however, stiffness and carry-over factors, as well as the fixed-end moments, are required. These are given in Niles and Newell, Vol. II, in Tables 14:8 and 14:9 on pp. 122 – 126, for certain standard loading cases.

NUMERICAL EXAMPLE 1

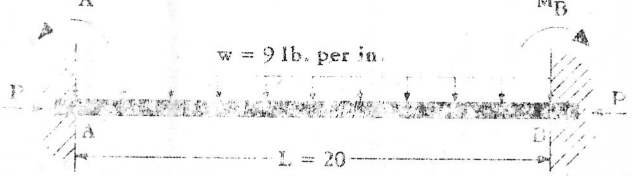


Figure 3.3.3–1

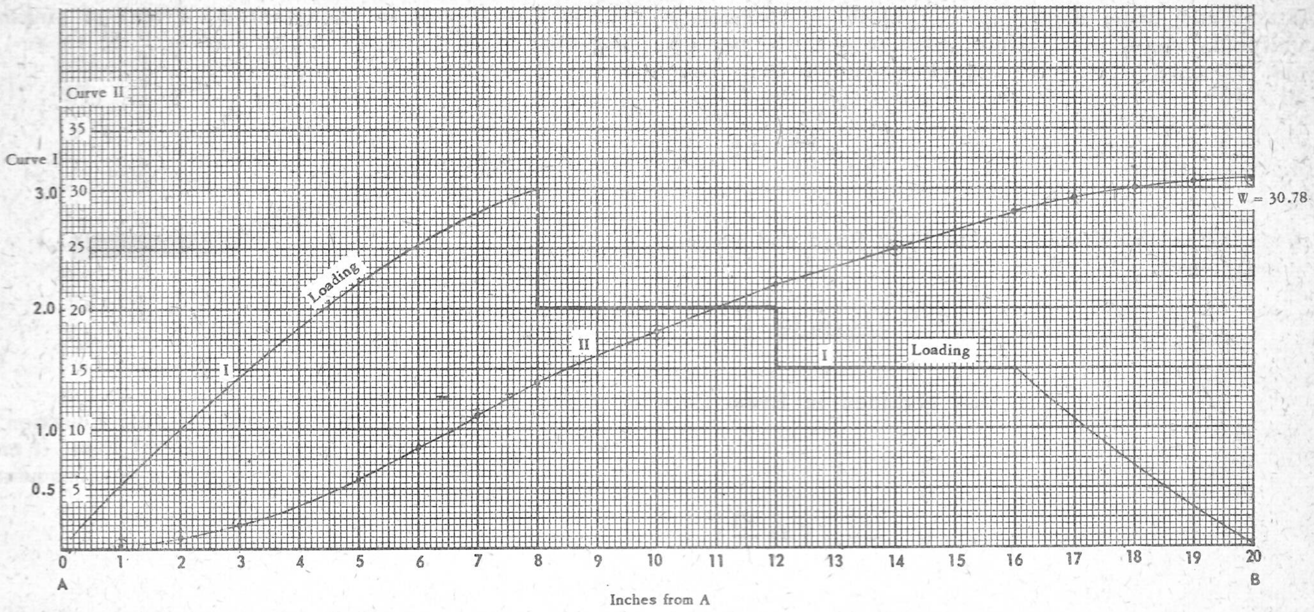


Figure 3.3.2–2

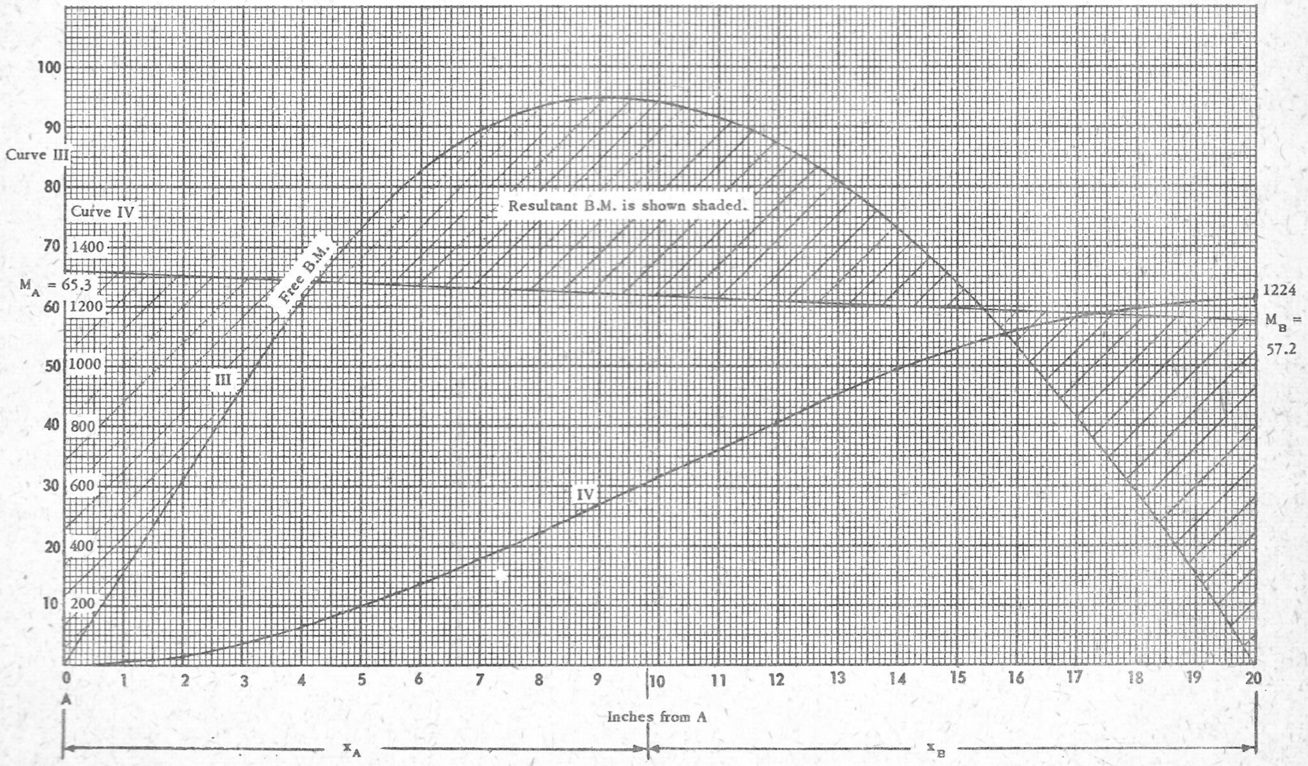


Figure 3.3.2–3

Fixed end beam shown in Fig. 3.3.3–1 carrying a uniformly distributed lateral load w = 9 lb. per in. and a compressive end load P = 10,000 lb. Take $EI = 10^6$

$$j^2 = \frac{EI}{P} = 100 \qquad j = 10 \qquad L/j = \frac{20}{10} = 2.0$$

From Niles and Newell, Table 14:8, fixed end moment coefficient C = 11.176

$$M_A = \frac{WL}{C} = \frac{9 \times 20 \times 20}{11.176} = \underline{322 \text{ lb. in.}}$$

This value checks with that obtained in Beam Columns, (6.5).

Carry-over factor = .6263
Stiffness coefficient = .8590
Absolute stiffness $= .8590 \times \frac{4EI}{L} = 3.44\,\frac{EI}{L}$

This is because the absolute stiffness of a non-tapered beam with fixed ends, without end load, is $\frac{4EI}{L}$, as shown in Report GEN/1090/804.

3.3.4 Fixed Beam of Constant I: Effect of sinking support.

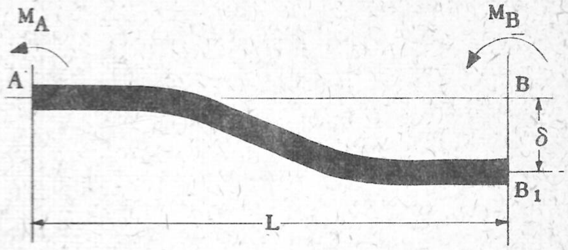


Figure 3.3.4–1

It is shown in Report GEN/1090/803, Equation 6 that, for a fixed beam carrying no lateral load, if end B sinks an amount δ relative to A, the end moments induced are:–

$$M_A = M_B = -\frac{6EI}{L}\cdot\frac{\delta}{L} \qquad \text{(See Fig. 3.3.4–1)}$$

This moment must be superimposed on the moment due to lateral load and incorporated in the moment distribution calculations for a continuous beam, (see 3.5.5).

3.3.5 Fixed Beam of Varying I: End moment determination.

A method is given in Report GEN/1090/805 using the Column Analogy.

Briefly, this method entails the replacement of the original beam by an analogous column having a width equal to $\frac{1}{EI}$ (or $\frac{1}{I}$ if more convenient), a depth equal to the span L and a length l which is taken to be very small.

However, in the case of linear tapered beams the following design charts can be used for the particular loadings given.

END MOMENTS FOR LINEAR TAPERED BEAMS

Figs. 3.3.5–1 to 5 give design charts for linearly tapered beams under common loading conditions, from which the fixed end moments can be found.

$$W = wL$$

$$M_A = \frac{WL}{K_A}$$

$$M_B = \frac{WL}{K_B}$$

When $d_A = d_B$,

$$d_B/d_A = 1.0$$

and $M_A = M_B = \frac{WL}{12}$

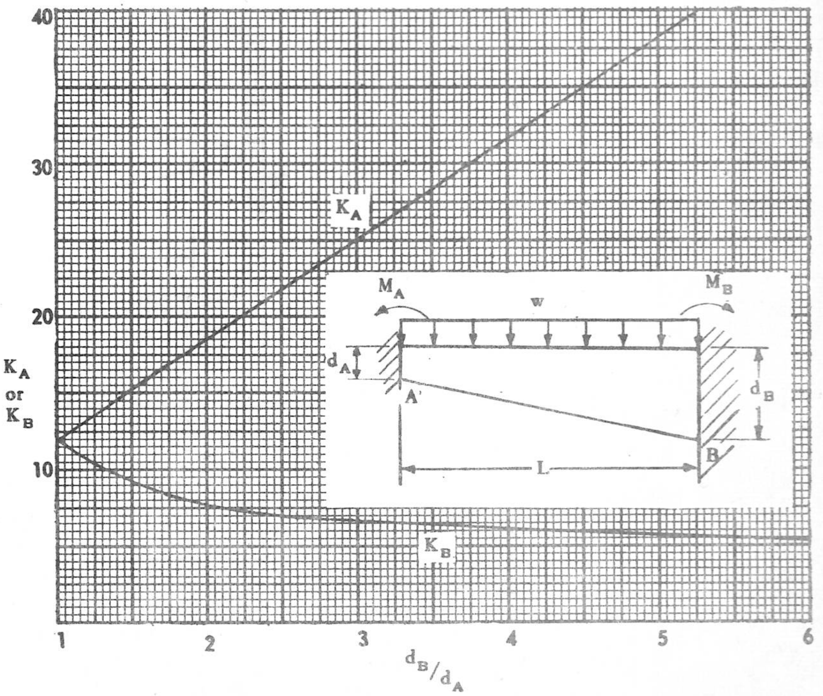


Figure 3.3.5–1 Uniform Load

End Moments for Linear Tapered Beams (Cont'd)

W = total load on beam

$M_A = \frac{WL}{K_A}$

$M_B = \frac{WL}{K_B}$

When $d_A = d_B$, $d_B/d_A = 1.0$

and $M_A = \frac{WL}{15}$

$M_B = \frac{WL}{10}$

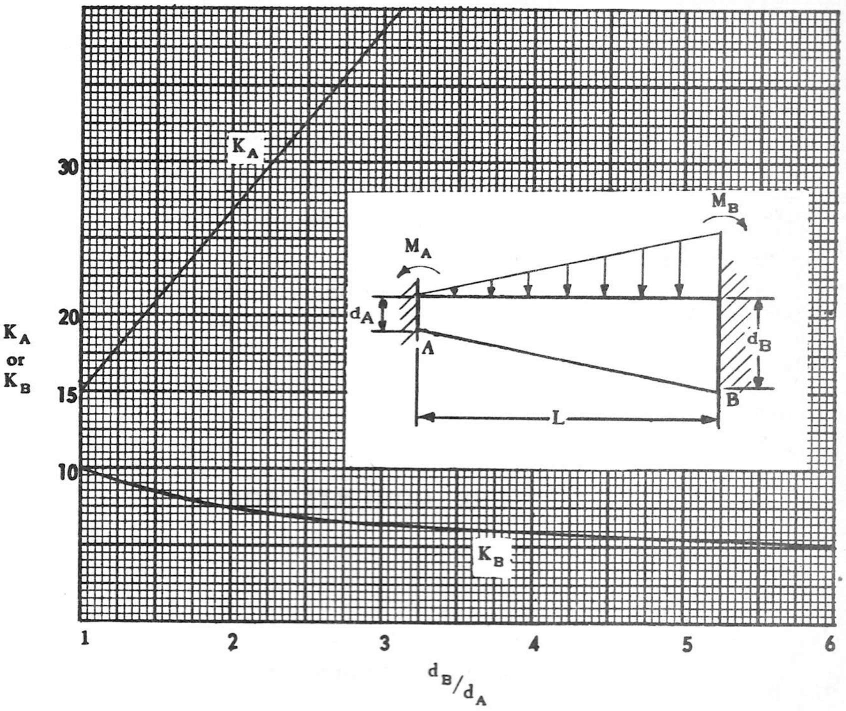


Figure 3.3.5–2 Triangular Load

W = total load on beam

$M_A = \frac{WL}{K_A}$

$M_B = \frac{WL}{K_B}$

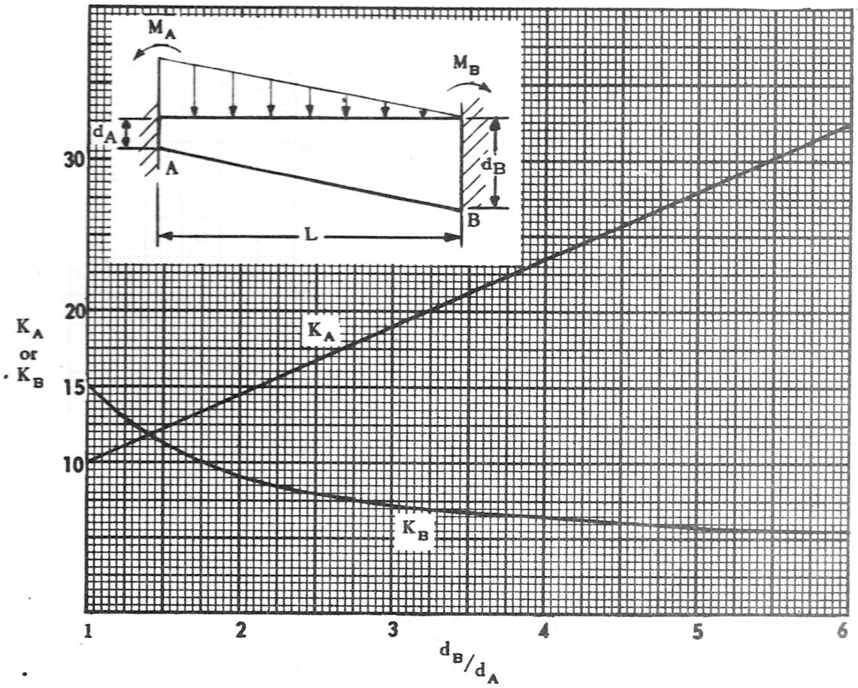


Figure 3.3.5–3 Triangular Load

End Moments for Linear Tapered Beams (Cont'd)

When $d^1 = 0$, beam is prismatic

For central load, $K = .5$

and $M_A = .125\ WL = \frac{WL}{8}$

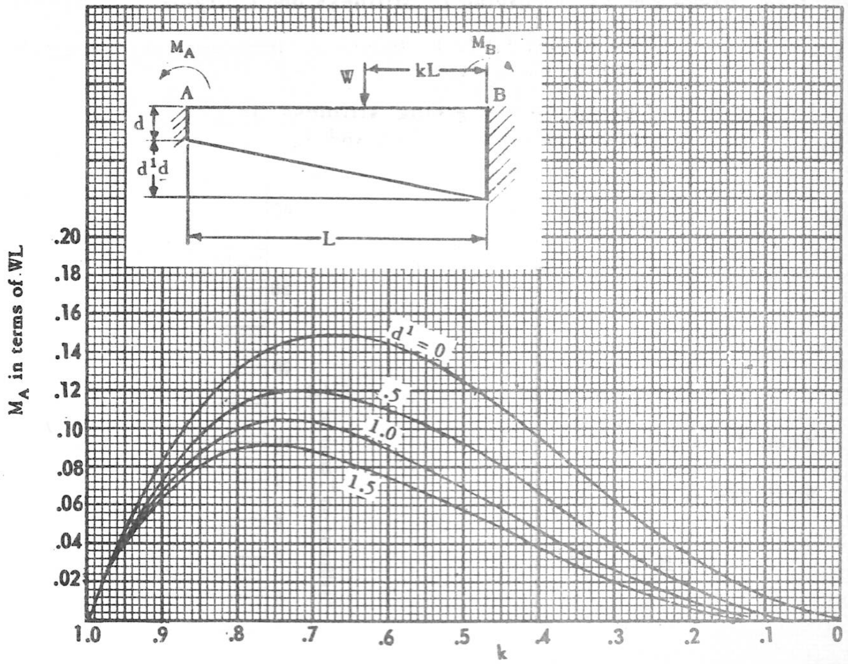


Figure 3.3.5–4

When $d^1 = 0$

and $K = .5$

$M_B = .125\ WL = \frac{WL}{8}$

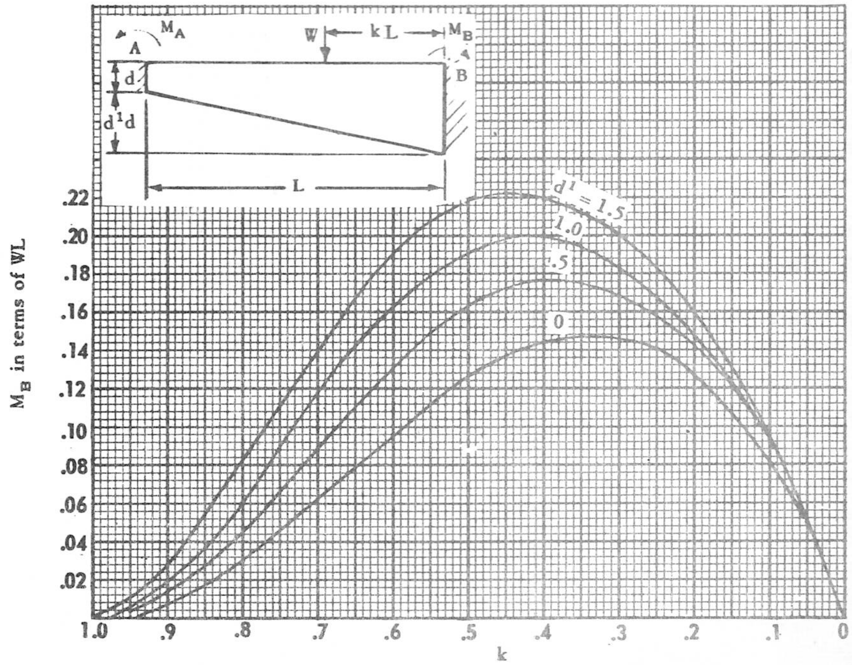


Figure 3.3.5–5

3.3.6 Fixed Beams of Varying I: Stiffness and Carry-over

See Report GEN/1090/805 for the solution of a typical problem by Column Analogy methods.

Design charts giving stiffness and carry-over for straight-tapered beams appear in Figs. 3.3.6–1 and 2.

When $d^1 = 0$

Stiffness at A $= \frac{4EI}{L}$

B $= \frac{4EI}{L}$

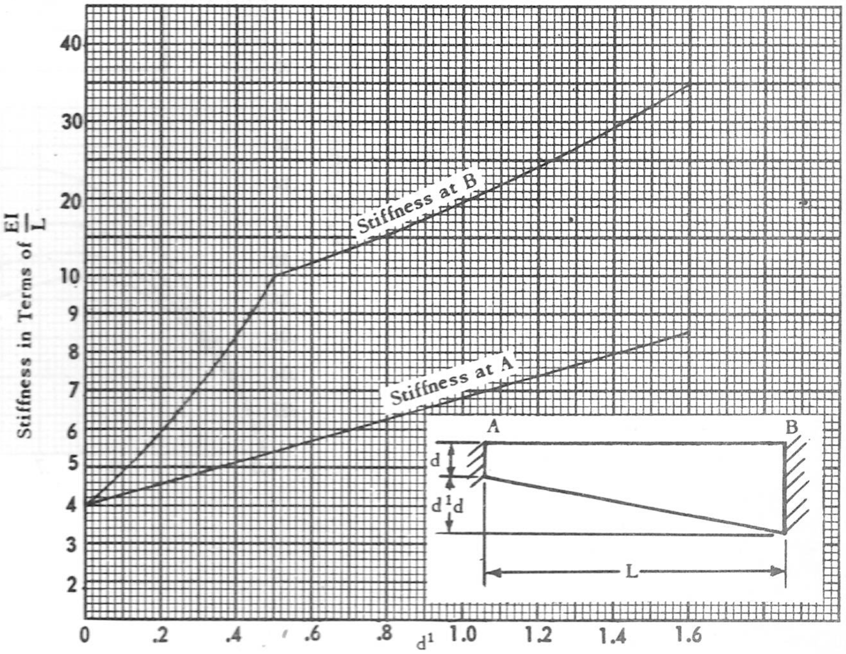


Figure 3.3.6–1

When $d_A = d_B$ (non-tapered beam),

carry-over factor. A to B = 0.5

B to A = 0.5

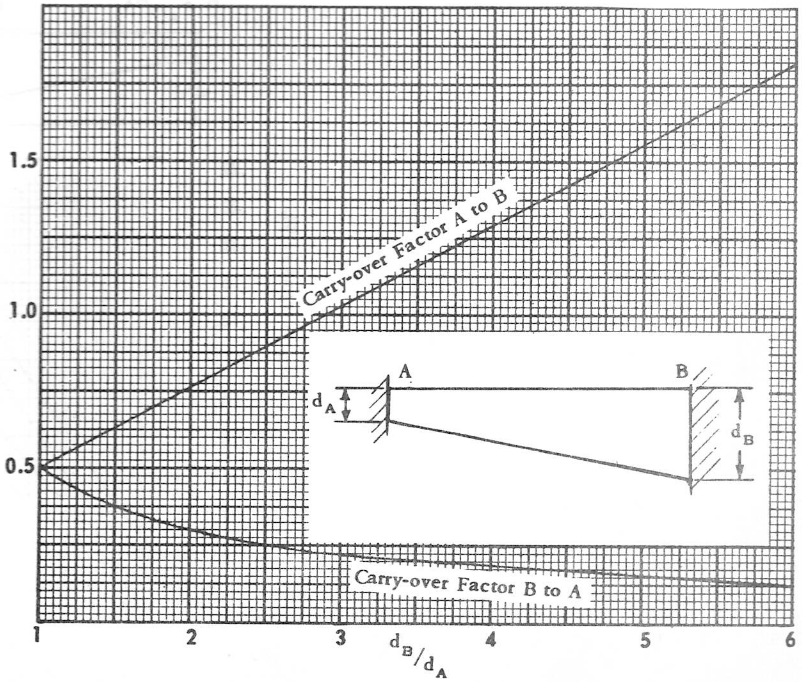


Figure 3.3.6–2

3.3.7 Fixed Beam of Varying I: Effect of Sinking Support.

The end moments resulting from a vertical deflection at one support equal to δ can be obtained by column analogy methods in the manner shown in the numerical example below.

NUMERICAL EXAMPLE:

To determine the end moments induced by a deflection $\delta = 1.0$ of support B relative to support A for the tapered beam shown in Fig. 3.3.7–1. The moment of inertia of the beam varies linearly from $I_A = 1.0$ at A to $I_B = 2.0$ at B.

$$A_{COL} = \frac{1.0 + 0.5}{2} \times 10 = 7.5$$

From Fig. 4.1.5–1, Section Properties of Trapezoid,

$$\text{for} \quad {}^{a}/_{b} = .50, \quad {}^{\rho_x}/_{d} = .2838$$

$$\rho_x = 2.838 \qquad \rho_x^{\,2} = 8.0$$

$$I_{na} = A\rho_x^{\,2} = 7.5 \times 8 = 60$$

$$I_{COL} = \frac{60}{E}$$

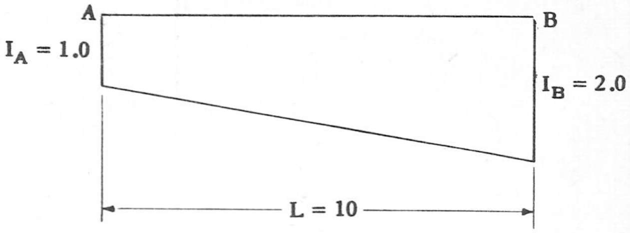
$$\bar{y}/_{d} = .445 \qquad \bar{y} = 4.45$$

For $\delta = 1.0$,

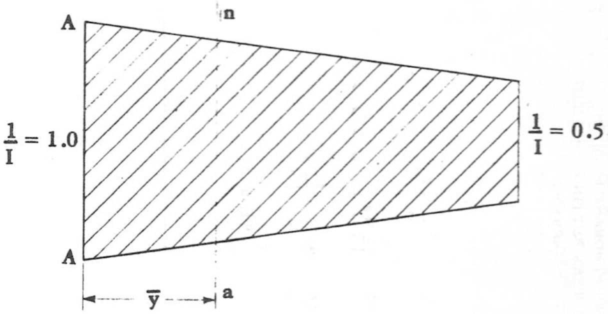
$$M_A = 1.0 \times 4.45 \frac{E}{60} = .0741\,E$$

$$M_B = 1.0 \times 5.55 \frac{E}{60} = .0924\,E$$

as explained in GEN/1090/805.5

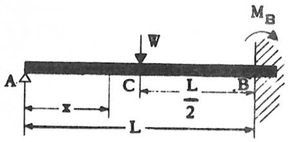
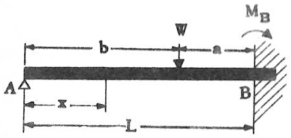
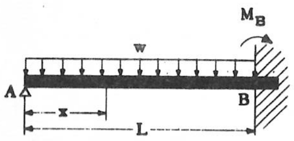


ORIGINAL BEAM

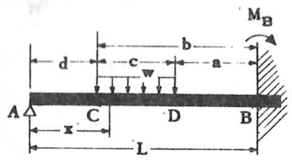
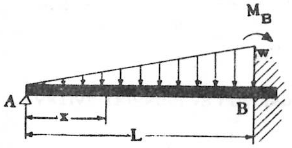
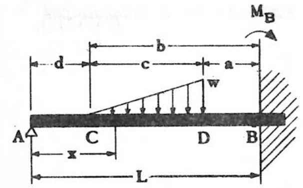


ANALOGOUS COLUMN

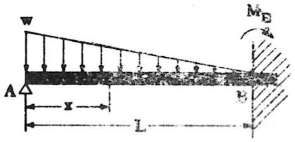
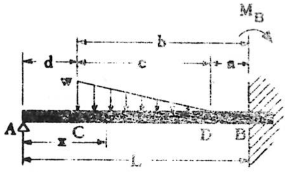
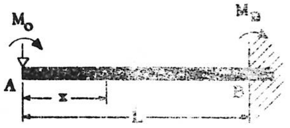
Figure 3.3.7–1

PROPPED CANTILEVER	REACTIONS R_A AND R_B (LB.) [POSITIVE UPWARD]	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD] AND SLOPE θ (RADIANS)
(1) CENTRE LOAD 	$R_A = \frac{5}{16}W$ $R_B = \frac{11}{16}W$	$M_A = 0$ $M_B = -\frac{3}{16}WL$ (A to C) $M = \frac{5}{16}Wx$ (C to B) $M = W(\frac{L}{2} - \frac{11}{16}x)$ Max. $+M = \frac{5}{32}WL$ at C	(A to C) $y = \frac{1}{96}\frac{W}{EI}(5x^3 - 3L^2x)$ (C to B) $y = \frac{1}{96}\frac{W}{EI}[5x^3 - 16(x - \frac{L}{2})^3 - 3L^2x]$ Max. $y = -0.00932\frac{WL^3}{EI}$ at $x = 0.4472L$ $\theta = -\frac{1}{32}\frac{WL^2}{EI}$ at A
(2) INTERMEDIATE LOAD 	$R_A = \frac{W}{2}(\frac{3a^2L - a^3}{L^3})$ $R_B = W - R_A$	$M_A = 0$ $M_B = -\frac{W}{2}(\frac{a^3 + 2aL^2 - 3a^2L}{L^2})$ (A to C) $M = R_Ax$ (C to B) $M = R_Ax - W(x - L + a)$ Max. $+M = R_A(L - a)$ at C Max. possible value is $M = 0.174WL$ when $a = 0.634L$	(A to C) $y = \frac{1}{6EI}[R_A(x^3 - 3L^2x) + 3Wa^2x]$ (C to B) $y = \frac{1}{6EI}\left[R_A(x^3 - 3L^2x) + W[3a^2x - (x - b)^3]\right]$ $\theta = \frac{1}{4}\frac{W}{EI}(\frac{a^3}{L} - a^2)$ at A
(3) UNIFORM LOADING 	$R_A = \frac{3}{8}W$ $R_B = \frac{5}{8}W$ $W = wL$	$M_A = 0$ $M_B = -\frac{WL}{8}$ $M = W(\frac{3}{8}x - \frac{1}{2}\frac{x^2}{L})$ Max. $+M = \frac{9}{128}WL$ at $x = \frac{3}{8}L$	$y = \frac{1}{48}\frac{W}{EIL}(3Lx^3 - 2x^4 - L^3x)$ Max. $y = -0.0054\frac{WL^3}{EI}$ at $x = 0.4215L$ $\theta = -\frac{1}{48}\frac{WL^2}{EI}$ at A

3.4.1 Propped Cantilevers of Constant I

PROPPED CANTILEVER	REACTIONS R_A AND R_B (LB) [POSITIVE UPWARD]	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD] AND SLOPE θ (RADIANS)
(4) PARTIAL UNIFORM LOADING  	$R_A =$ $\frac{W}{8L^3}[(a^2+ab+b^2)(4L-a)-b^3]$ $R_B = W - R_A$ $W = wc$	$M_A = 0$ $M_B = R_AL - \frac{1}{2}W(a+b)$ (A to C) $M = R_Ax$ (C to D) $M = R_Ax - \frac{W(x-d)^2}{2c}$ (D to B) $M = R_Ax - W(x-d-\frac{c}{2})$ Max. + $M = R_A(d + \frac{1}{2}\frac{R_A}{W}c)$ at $x = (d + \frac{R_A}{W}c)$	For y and θ, see Roark, p. 104, Case 24.
(5) TRIANGULAR LOADING  	$R_A = \frac{1}{5}W$ $R_B = \frac{4}{5}W$ $W = \frac{1}{2}wL$	$M_A = 0$ $M_B = -\frac{2}{15}WL$ $M = W(\frac{1}{5}x - \frac{1}{3}\frac{x^3}{L^2})$ Max. + $M = 0.06\ WL$ at $x = 0.4474\ L$	$y = -\frac{Wx}{60EIL^2}(L^2-x^2)^2$ Max. $y = -0.00477\frac{WL^3}{EI}$ at $x = L\sqrt{\frac{1}{5}}$ $\theta = -\frac{1}{60}\frac{WL^2}{EI}$ at A
(6) PARTIAL TRIANGULAR LOADING  	$R_A =$ $\frac{W}{20L^3}[(3a^2+2ab+b^2)(5L-b)-4a^3]$ $R_B = W - R_A$ $W = \frac{1}{2}wc$	$M_A = 0$ $M_B = R_AL - \frac{W}{3}(2a+b)$ (A to C) $M = R_Ax$ (C to D) $M = R_Ax - \frac{W}{3}\frac{(x-d)^3}{c^2}$ (D to B) $M = R_Ax - W(x-d-\frac{2}{3}c)$ Max. + $M = R_A(d + \frac{2}{3}c\sqrt{\frac{R_A}{W}}$ at $x = d + c\sqrt{\frac{R_A}{W}}$	For y and θ, see Roark p. 104, Case 26.

3.4.1 Propped Cantilevers of Constant I (Cont'd)

PROPPED CANTILEVER	REACTIONS R_A AND R_B (LB) [POSITIVE UPWARD]	BENDING MOMENT M (LB. IN.) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION y (IN.) [POSITIVE UPWARD] AND SLOPE θ (RADIANS)
(7) TRIANGULAR LOADING 	$R_A = \frac{11}{20} W$ $R_B = \frac{9}{20} W$ $W = \frac{1}{2} wL$	$M_A = 0$ $M_B = -\frac{7}{60} WL$ $M = W(\frac{11}{20} x - \frac{x^2}{L} + \frac{1}{3}\frac{x^3}{L^2})$ Max. + M = 0.0846 WL at x = 0.329 L	$y = \frac{1}{120}\frac{W}{EIL}(11Lx^3 - 3L^3x - 10x^4 + \frac{2x^5}{L})$ Max. $y = -0.00609 \frac{WL^3}{EI}$ at $x = 0.402 L$ $\theta = -\frac{1}{40}\frac{WL^2}{EI}$ at A
(8) PARTIAL TRIANGULAR LOADING 	$R_A =$ $\frac{W}{20L^3}[(a^2 + 2ab + 3b^2)(5L - a) - 4b^3]$ $R_B = W - R_A$ $W = \frac{1}{2} wc$	$M_A = 0$ $M_B = R_A L - \frac{W}{3}(a + 2b)$ (A to C) $M = R_A x$ (C to D) $M = R_A x - \frac{(x-d)^2}{c} W + \frac{(x-d)^3}{3c^2} W$ (D to B) $M = R_A x - W(x - d - \frac{c}{3})$	For y and θ, see Roark p. 105, Case 28.
(9) END COUPLE 	$R_A = -\frac{3}{2}\frac{M_o}{L}$ $R_B = \frac{3}{2}\frac{M_o}{L}$	$M_A = M_o$ $M_B = -\frac{M_o}{2}$ $M = \frac{1}{2} M_o (2 - 3\frac{x}{L})$ Max. + M = M_o at A	$y = -\frac{1}{4}\frac{M_o}{EI}\frac{x}{L}(L - x)^2$ Max. $y = -\frac{1}{27}\frac{M_o L^2}{EI}$ at $x = \frac{L}{3}$ $\theta = -\frac{1}{4}\frac{M_o L}{EI}$ at A

3.4.1 Propped Cantilevers of Constant I (Cont'd)

3.4.2 Propped Cantilever of Constant I: Irregular loading and prop at intermediate position.

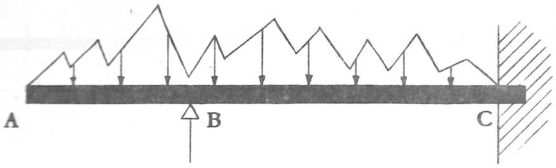


Figure 3.4.2–1

The general case of a propped cantilever with the prop at some intermediate position, such as B in Fig. 3.4.2–1, carrying an irregular load can be solved as follows:–

(a) Find the bending moment at the support B due to the overhung portion AB.

(b) Find the fixed end moments for span BC, by the method of 3.3.2 for irregular load.

(c) Balance Joint B by moment distribution methods.

NUMERICAL EXAMPLE:

By way of illustration, take the fairly straightforward case of Fig. 3.4.2–2 wherein the load varies linearly from zero at the tip A to 30 lb. per in. at the root C.

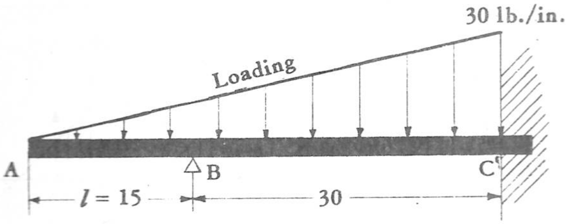


Figure 3.4.2–2

(a) B.M. at B due to overhang $= \frac{Wl}{3} = \frac{1}{2}\, 15.10 \times 5 =$ + 375 lb. in. (Section 3.2, Case 5).

(b) Fixed end moments on span BC assuming that it is an isolated fixed beam are, from Section 3.3.1, Case 11,

$$M_B = \frac{L^2}{60}\left[5w_A + 2(w_B - w_A)\right]$$

$$= -15\,(50 + 40) = -1350 \text{ lb. in.}$$

$$M_C = \frac{L^2}{60}\left[5w_A + 3(w_B - w_A)\right]$$

$$= 15\,(50 + 60) = +1650 \text{ lb. in.}$$

(c) Moment Distribution	B		C
Initial moments	+ 375	− 1350	+ 1650
Balance Joint B	+	+ 975	+ 488
Final moments	+ 375	− 375	+ 2138

The B.M. diagram is shown in Fig. 3.4.2–3.

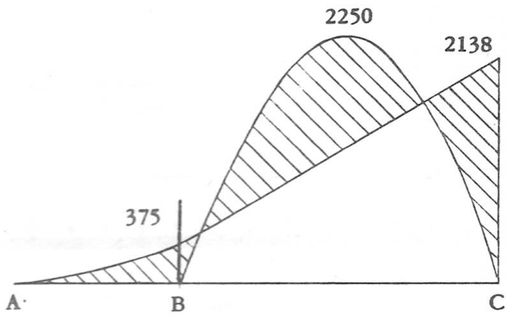


Figure 3.4.2–3

3.4.3 Propped Cantilever of Varying I: Irregular loading and prop at intermediate position.

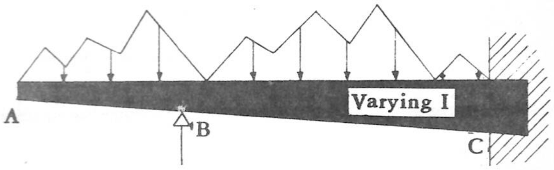


Figure 3.4.3–1

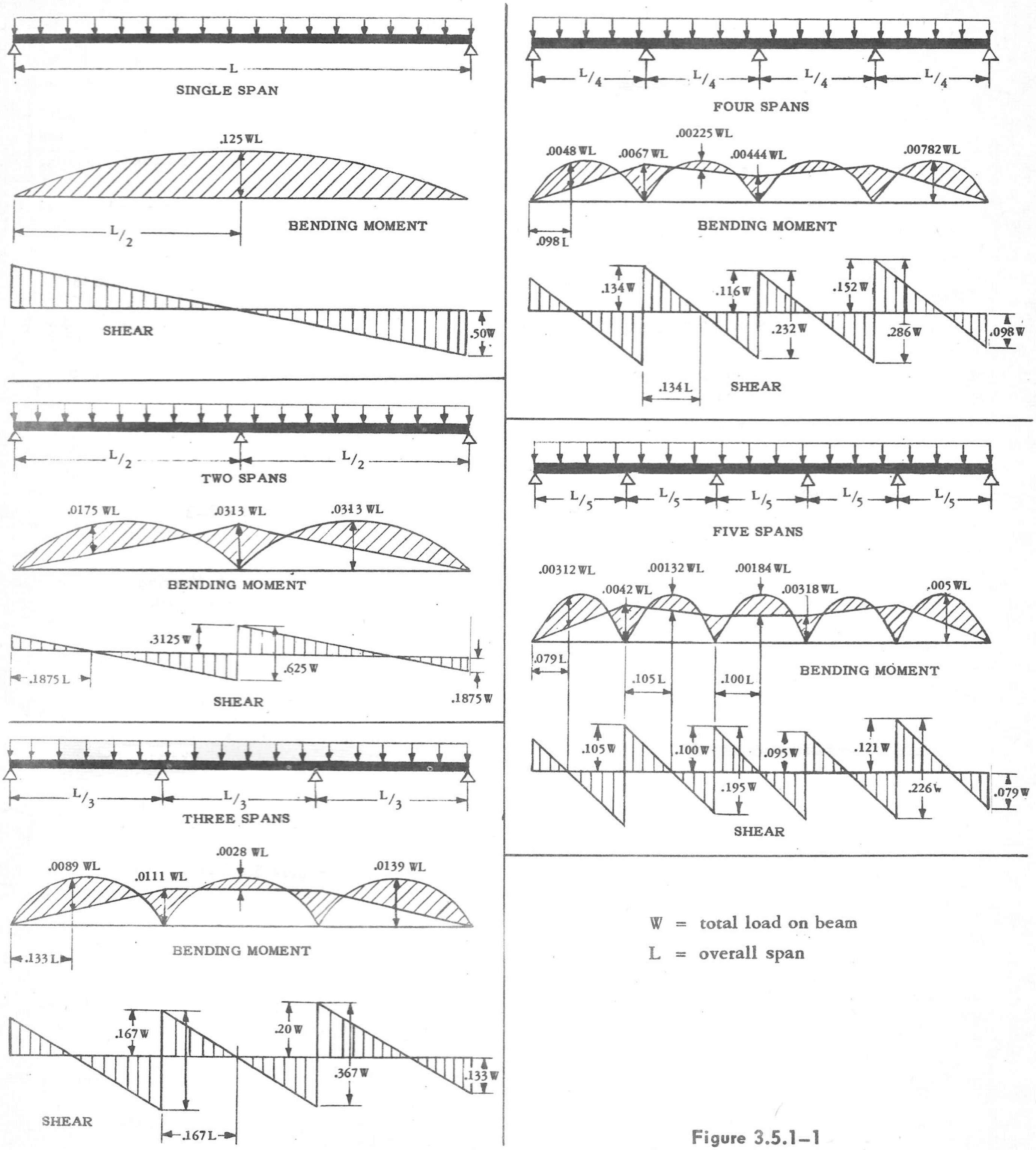
The method is basically the same as that of 3.4.2 and is as follows:–

(a) Find the bending moment at support B due to the overhung portion AB.

(b) Find the fixed end moments, stiffness and carry-over for span BC as an isolated fixed beam by the methods of 3.3.5, 3.3.6 and 3.3.7.

(c) Balance Joint B by moment distribution methods.

3.5.1 Continuous Beam: I Constant throughout. Tabular Summary for Uniform Loading on Beams up to 5 Spans.



W = total load on beam
L = overall span

Figure 3.5.1–1

3.5.2 Continuous Beam: I Constant Throughout. Design chart for uniform load on 2 spans. (Simply supported at ends)

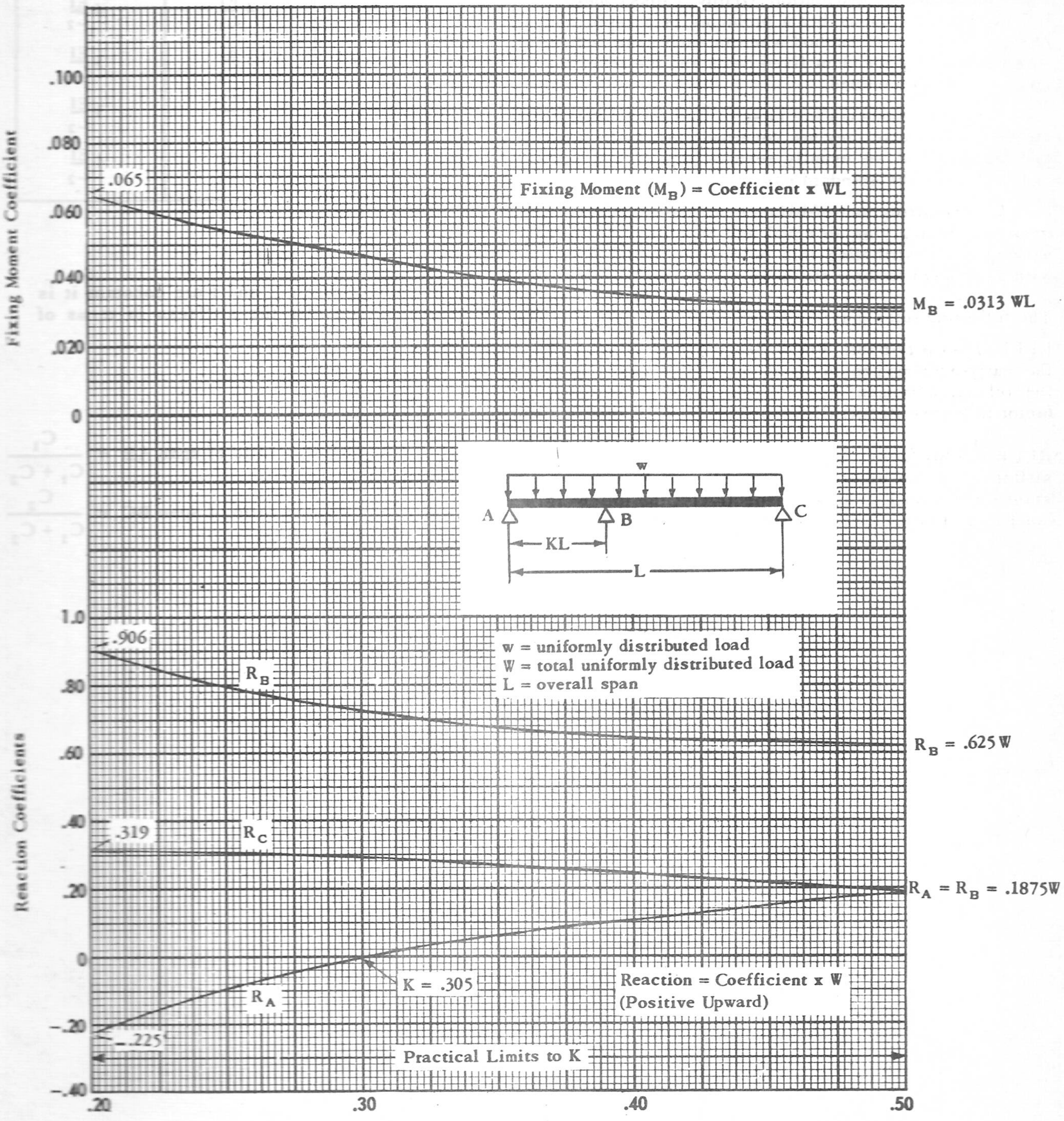


Figure 3.5.2–1

3.5.3 Continuous Beam: I Constant throughout. Tabular Summary for Various Loadings by Moment Distribution.

The moment distribution (Hardy Cross) method for solving continuous beam problems is a rapid and essentially practical method wherein each span is first treated as an independent fixed beam and the resulting end moments determined.

These end moments are easy to find for a beam of constant I, because they are governed entirely by span and loading and so usually resolve themselves into one or more of the standard cases given in 3.3.1.

Any resulting out-of-balance at the supports is overcome by applying an equal and opposite releasing moment and distributing this between adjacent spans with due regard to their stiffness.

The following rules must be followed:–

RULE 1 – Carry-over: For a beam AB of constant I the carry-over moment to B when A is released is ½ the releasing moment at A. That is, the carry-over factor is ½ (See Report GEN/1090/804).

RULE 2 – Stiffness: When considering the relative stiffness of two adjacent spans AB and BC of constant I and span L_1 and L_2 respectively, the following conditions apply. (See Report GEN/1090/804).

CONDITION	STIFFNESS	
	of Span AB = C_1	of Span BC = C_2
A and C both fixed	$4\frac{EI}{L_1}$	$4\frac{EI}{L_2}$
A and C both pinned	$3\frac{EI}{L_1}$	$3\frac{EI}{L_2}$
A fixed, C pinned	$4\frac{EI}{L_1}$	$3\frac{EI}{L_2}$
A pinned, C fixed	$3\frac{EI}{L_1}$	$4\frac{EI}{L_2}$

NOTE

For convenience when E and I are constant it is customary to consider the stiffness in terms of $\frac{I}{L}$ or $\frac{I}{L}$

Stiffness Ratio:

The stiffness ratio of span AB $= \frac{C_1}{C_1 + C_2}$

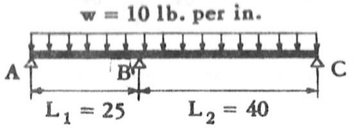
The stiffness ratio of span BC $= \frac{C_2}{C_1 + C_2}$

E and I Assumed Constant Throughout – Clockwise Moments are Positive

CONTINUOUS BEAM | **COMMENTS**

(1)

2 SPANS, FREELY SUPPORTED AT A, B AND C, DISTRIBUTED LOAD ON AB AND BC.



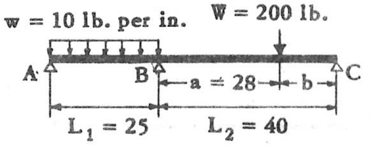
	A	B		C
Stiffness ratio		.615	.385	
Fixing moments	− 521	+ 521	− 1333	+ 1333
Release A and C	+ 521 →	+ 260	− 666	← − 1333
Nett moments		+ 781	− 1999	
Distribution		+ 749	+ 469	
Final moments	0	+ 1530	− 1530	0

End moments on AB as a fixed beam = $\pm\frac{wL_1^2}{12} = \pm\frac{10\times625}{12} = \pm 521$.

Corresponding end moments on BC are $\pm$ 1330. Stiffness of AB = I/25 = .04I and of BC = .025I. Stiffness ratio of AB = .04/.065 = .615 and of BC = .385. Since beam is freely supported at A and C, final moments there must be zero. Hence release A and C by applying equal and opposite cancelling moments. This gives carry-over moments – as arrowed – to each side of B. Nett out of balance at B, viz., −1999 +781 = −1218, is divided between AB and BC in proportion to their stiffness ratios. If $L_1 = L_2$, the stiffness ratio of each span =.50

(2)

2 SPANS, FREELY SUPPORTED AT A, B AND C. DISTRIBUTED LOAD ON AB, CONCENTRATED LOAD ON BC.

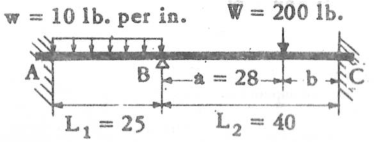


	A	B		C
Stiffness ratio		.615	.385	
Fixing moments	− 521	+ 521	− 504	+ 1176
Release A and C	+ 521 →	+ 260	− 588	← − 1176
Nett moments		+ 781	− 1092	
Distribution		+ 191	+ 120	
Final moments	0	+ 972	− 972	0

The procedure is exactly the same as that in Case (1). End moments on AB are ±521 and on BC are $M_{BC} = \frac{W}{L^2}ab^2 = \frac{200}{1600}\times 28\times 144 = -504$, $M_{CB} = \frac{W}{L^2}a^2b = \frac{1}{8}784\times 12 = +1176$. Stiffness values are as before.

(3)

2 SPANS. FIXED AT A AND C. DISTRIBUTED LOAD ON AB, CONCENTRATED LOAD ON BC.



	A	B		C
Stiffness ratio		.615	.385	
Fixing moments	− 521	+ 521	− 504	+ 1176
Release B	− 5.2 ←	− 10.4	− 6.6 →	− 3.3
Final moments	− 526.2	+ 510.6	− 510.6	+ 1172.7

Initial end moments and stiffness ratios will be as for Case (2). Since beam is freely supported at B the final moments on each side of it must be equal but opposite. Therefore apply there a releasing moment equal and opposite to the out-of-balance moment, viz., +521 − 504 = +17 and divide this between AB and BC in proportion to their stiffness ratios. There are carry-over values to A and C, as arrowed. If $L_1 = L_2$, the stiffness ratio of each span = .50

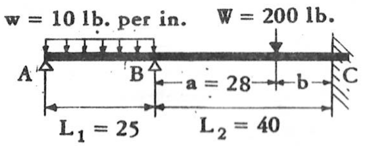
3.5.3 Continuous Beam: Tabular Summary

E and I Assumed Constant Throughout – Clockwise Moments are Positive

CONTINUOUS BEAM | COMMENTS

(4)

2 SPANS, FREELY SUPPORTED AT A AND B, FIXED AT C. DISTRIBUTED LOAD ON AB, CONCENTRATED LOAD ON BC.



	A	B		C
Stiffness ratio		.546	.454	
Fixing moments	− 521	+ 521	− 504	+ 1176
Release A	+ 521 →	+ 260		
		+ 781	− 504	
Release B		− 151	− 126 →	− 63
Final moments	0	+ 630	− 630	+ 1113

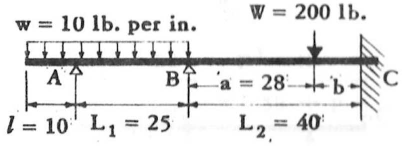
Since AB is freely supported but BC is fixed at C, stiffness of AB = ¾ × .041 = .031 (in accordance with Rule 2) and of BC = .025I, as before Stiffness ratio of AB = .03/.055 = .546 and of BC = .454.

Initial fixing moments are as before. First release A, since the final moment there must be zero, giving carry-over moment to B. Then release B, giving carry-over moment to C.

If $L_1 = L_2$, stiffness ratio of AB = .429 and of BC = .571 (still in accordance with Rule 2).

(5)

2 SPANS, WITH OVERHANG AT A. FREELY SUPPORTED AT A AND B, FIXED AT C. DISTRIBUTED LOAD ON AB, CONCENTRATED LOAD ON BC.



		A	B		C
Stiffness ratio			.546	.454	
Fixing moments	+ 500	− 521	+ 521	− 504	+ 1176
Release A		+ 21 →	+ 10.5		
			+ 531.5	− 504	
Release B			− 15	− 12.5 →	− 6.3
Final moments	+ 500	− 500	+ 516.5	− 516.5	+ 1170

Stiffness ratios and initial fixing moments on AB and BC are as for Case (4). Fixing moment to left of A due to overhang = $\frac{1}{2}wl^2 = \frac{1}{2} \times 10 = 100 = +500$ lb. in.

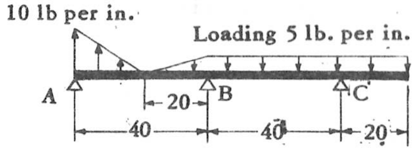
Release A by adding to M_{AB} a moment of +21 : this gives a carry-over of +10.5 to B.

Release B and add carry-over to C.

If $L_1 = L_2$ stiffness ratio of AB = .429 and of BC = .571.

(6)

2 SPANS, WITH OVERHANG AT C. FREELY SUPPORTED AT A, B AND C. IRREGULAR LOAD ON AB. DISTRIBUTED LOAD ON BC.



	A	B		C	
Stiffness ratio		.50	.50		
Down load moments	− 58	+ 192	− 667	+ 667	− 1000
Up load moments	+ 384	− 116			
Nett moments	+ 326	+ 76	− 667	+ 667	− 1000
Release A	− 326 →	− 163			
Release C			+ 167	← + 333	
Release B		+ 294	+ 294		
Final moments	0	+ 207	− 206	+ 1000	− 1000

Initial fixing moments on AB due to up load are $M_{AB} = 384$ and $M_{BA} = -116$: due to down load are $M_{AB} = -58$ and $M_{BA} = +192$. Initial end moments on BC are ±667; fixing moment to right of C due to overhang = −1000. Stiffness ratio = ½ since A, B and C are all simply supported.

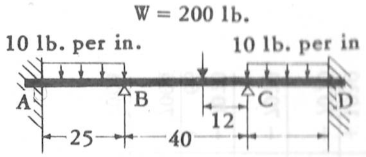
After finding the nett initial fixing moments, the procedure is similar to that in earlier examples.

3.5.3 Continuous Beam: Tabular Summary (Cont'd)

E and I Assumed Constant Throughout – Clockwise Moments are Positive

CONTINUOUS BEAM	COMMENTS
(7)	

3 SPANS, FIXED AT
ENDS A AND D.
LOADED AS SHOWN.



	A		B		C	D
Stiffness ratio		.615	.385	.385	.615	
Fixing moments	− 521	+ 521	− 504	+ 1176	− 521	+ 521
Release B	− 5.2 .←	− 10.4	− 6.6→	− 3.3		
	− 526.2	+ 510.6	− 510.6	+ 1172.7	− 521	+ 521
Release C			− 125	← − 249.7	− 402→	− 201
	− 526.2	+ 510.6	− 635.6	+ 923	− 923	+ 320
Release B	+ 38.4	← + 76.8	+ 48.2→	+ 24.1		
	− 487.8	+ 587.4	− 587.4	+ 947.1	− 923	+ 320
Release C			− 4.7	← − 9.3	− 14.8→	− 7.4
	− 487.8	587.4	− 592.1	+ 937.8	− 937.8	+ 312.6
Release B	+ 1.4	← + 2.9	+ 1.8→	+ .9		
Final moments	− 486.4	590.3	− 590.3	+ 938.7	− 937.8	+ 312.6

Stiffness ratios AB : BC and CD : BC will be .615 : .385 as earlier. First release B, giving carry-over to A and C : then release C, giving carry-overs to D and B, which will now be unbalanced again. Continue to release B and C until the resultant out of balance moment is so small as to be negligible.

If the spans are all equal, the stiffness ratios = .50.

3.5.3 Continuous Beam: Tabular Summary (Cont'd)

3.5.4 Continuous Beams: I Constant Per Span.

Numerical example of 4 span beam.

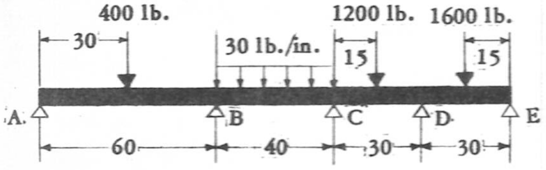


Figure 3.5.4–1

MEMBER	STIFFNESS	STIFFNESS RATIO
AB	$\frac{3}{4}\frac{I}{L} = .75 \times \frac{60}{60} = .75$	$\frac{.75}{2.25} = .33$
BC	$\frac{I}{L} = \frac{60}{40} = 1.5$	$\frac{1.5}{3.0} = .50$
CD	$\frac{I}{L} = \frac{45}{30} = 1.5$	$\frac{1.5}{2.625} = .571$
DE	$\frac{3}{4}\frac{I}{L} = .75 \times \frac{45}{30} = 1.125$	

$I_{AB} = I_{BC} = 60$ in.^{4}

$I_{CD} = I_{DE} = 45$ in.^{4}

E constant

	A	B		C		D		E
Stiffness ratio		.33	.67	.50	.50	.571	.429	
Initial moments	− 3000	+ 3000	− 4000	+ 4000	− 4500	+ 4500	− 6000	+ 6000
Release A & E	+ 3000 →	+ 1500					− 3000	←− 6000
		+ 4500	− 4000	+ 4000	− 4500	+ 4500	− 9000	
Release B		− 167	− 334 →	− 167				
		+ 4333	− 4334	+ 3833	− 4500	+ 4500	− 9000	
Release D					+ 1285	← + 2570	+ 1930	
				+ 3833	− 3215	+ 7070	− 7070	
Release C			− 154	←− 309	− 309 →	− 154		
		+ 4333	− 4488	+ 3524	− 3524	+ 6916	− 7070	
Release B		+ 51	+ 102 →	+ 51				
		+ 4384	− 4384	+ 3575	− 3524	+ 6916	− 7070	
Release D					+ 44	←+ 88	+ 66	
				+ 3575	− 3480	+ 7004	− 7004	
Release C			− 24	←− 48	− 48 →	− 24		
		+ 4384	− 4408	+ 3527	− 3528	+ 6980	− 7004	
Release B		+ 8	+ 16 →	+ 8				
		+ 4392	− 4392	+ 3535	− 3528			
Release D					+ 7	←+ 14	+ 10	
				+ 3535	− 3521	+ 6994	− 6994	
Release C			− 2	←− 3.5	− 3.5 →	− 2		
Final moments	0	+ 4392	− 4394	+ 3531	− 3535	+ 6992	− 6994	0

Note that A and E released first and kept released throughout.

Further accuracy in the final moments will be achieved by continuing with the distribution calculations.

3.5.5 Continuous Beams: I Constant Per Span. Numerical Example on Effect of Axial Load.

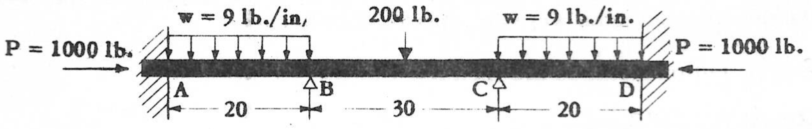


Figure 3.5.5–1

Given that $EI = 10^6$

	$j^2 = \frac{EI}{P}$	j	L	$\frac{L}{j}$	FIXED END MOMENT COEFFICIENT C	WL	FIXED END MOMENT $\frac{WL}{C}$	CARRY-OVER FACTOR	STIFFNESS COEFFICIENT	ABSOLUTE STIFFNESS	STIFFNESS RATIO
AB	100	10	20	2.0	11.176	3600	322	.6263	.859	$\frac{.859}{20} \times 4EI = .171\,EI$	$\frac{.171}{.2585} = .66$
BC	100	10	30	3.0	6.44	6000	931	.9189	.656	$\frac{.656}{30} \times 4EI = .0875\,EI$	
CD	100	10	20	2.0	11.176	3600	322	.6263	.859	$= .171\,EI$	$\frac{.0875}{.2585} = .34$

The fixed-end moments, stiffness and carry-over are obtained by the method of 3.3.3. Proceed now as for a normal moment distribution calculation. If the end load P in each bay is different, the procedure is basically the same, because the effect of P is automatically taken care of in finding $j^2 = \frac{EI}{P}$.

The above example is taken from "Aircraft Structures", Peery, p.552. There is an error in the value of his fixed-end moment coefficient for span BC: 6.9 should be 6.44.

This affects the moment distribution in Peery, Fig. 18.39, p. 553.

3.5.6 Continuous Beams: I Constant Per Span. Effect of Sinking Supports

To determine the moments induced when the supports sink, each joint of the *unloaded* beam is allowed to sink without angular rotation and the end moments found as though each span were an isolated fixed beam.

Moment distribution methods are then used for balancing.

NUMERICAL EXAMPLE:

The same beam as in 3.5.4. Take E = 10^7

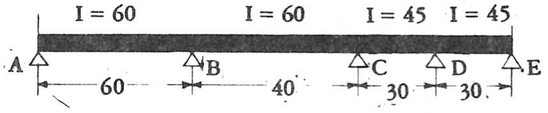
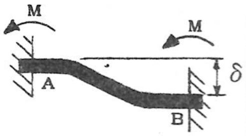
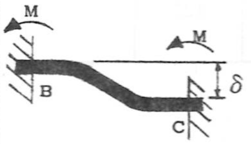
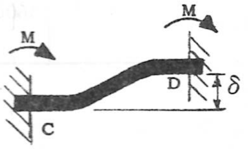
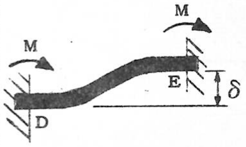


Figure 3.5.6–1

Let the supports sink as follows:–

A	B	C	D	E
.01"	.02"	.025"	.02"	.015"

MEMBER	AB	BC	CD	DE
				
NETT DEFLECTION δ	.01	.005	.005	.005
L	60	40	30	30
L^2	3600	1600	900	900
I	60	60	45	45
$\frac{6EI}{L^2}$	1.0×10^6	2.25×10^6	3.0×10^6	3.0×10^6
$M = \frac{6EI}{L^2}\,\delta$	−10,000	−11,250	+15,000	+15,000

Proceed now by ordinary moment distribution methods.

3.5.7 Continuous Beams: I Varying Per Span. General Notes

Treat each span as an isolated fixed beam and determine the end moments, stiffness and carry-over by the methods of 3.3.5, 3.3.6 and 3.3.7.

Proceed then by the ordinary technique of moment distribution.

3.6 Curved Beams: Circumferential Stresses

Ordinary bending theory cannot be used for a beam which is initially curved to such an extent that the radius of curvation is small compared with the dimensions of the beam.

In such a case the circumferential bending stress (for sections other than thin-walled) can be obtained by the method outlined below.

Consider the beam shown in Fig. 3.6–1 which has an initial curvative R measured to the centroidal axis.

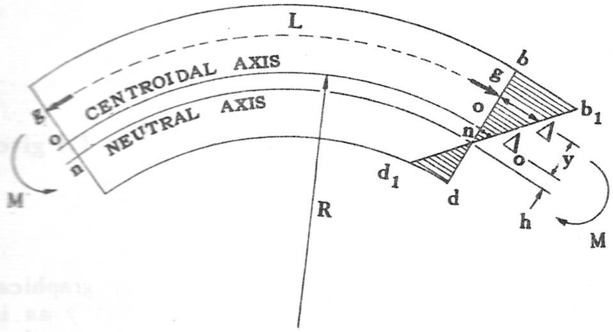


Fig. 3.6–1

Assume that a plane cross-section such as bnd remains plane after bending and, under the action of an applied bending moment M, tending to increase the curvature of the beam, moves to the position b_1nd_1.

Note that in Fig. 3.6–1 the position of the neutral axis nn, that is, the position of zero bending stress, is not coincident with the centroidal axis oo. Later it will be shown that this is because the stress distribution across the section is hyperbolic and not linear, as in ordinary beam theory.

For the time being, however, it will be necessary to accept the fact that n is below o in this particular case.

3.6.1 Development of Formulas

Let a longitudinal fibre gg of the beam of original length L situated at a distance y above the centroidal axis undergo an extension Δ. Then, if L_o is the original length of a fibre on the centroidal axis,

$$\frac{L}{R+y} = \frac{L_o}{R}$$

$$\text{or } L = \frac{L_o}{R}(R+y) = K_1(R+y) \text{ say.}$$

Also, if Δ_o is the extension of a longitudinal fibre on the centroidal axis and h is the distance of the neutral axis below the centroidal axis,

$$\frac{\Delta}{h+y} = \frac{\Delta_o}{h}$$

$$\Delta = \Delta_o + \frac{\Delta_o}{h}y$$

$$= K_2 + K_3y, \text{ say.}$$

The normal or circumferential bending stress f developed in fibre gg is clearly E × strain or

$$f = E\frac{\Delta}{L} = E\frac{(K_2 + K_3y)}{K_1(R+y)}$$

By regrouping, $f = a + \dfrac{K}{R+y}$ Eqn. (1)

If dA (Fig. 3.6–2) is the area of an element of cross-section, $A = \Sigma dA$ is the total area of the section and it is clear that ΣfdA is the total axial load P applied. That is, $P = \Sigma fdA$

$$= aA + K\Sigma\frac{dA}{R+y} \qquad \text{Eqn. (2)}$$

In addition,

$$M = \Sigma fydA$$

$$= a\Sigma yda + K\Sigma\frac{ydA}{R+y}$$

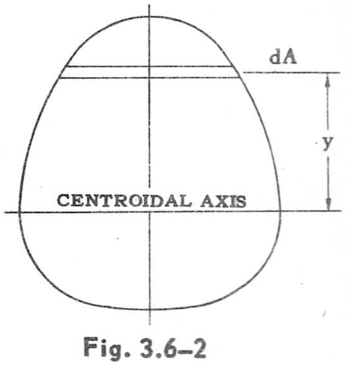


Fig. 3.6–2

But $\Sigma ydA = 0$ because it is the sum of the moments of all elements above the centroidal axis.

$$M = K\Sigma\frac{ydA}{R+y}$$

$$= K\Sigma(1 - \frac{R}{R+y})\ dA$$

$$= K[A - R\Sigma\frac{dA}{R+y}] \qquad \text{Eqn. (3)}$$

3.6.2 Procedure in Practice – When the section is such that the curves of 3.6.3 cannot be used, proceed as follows:–

(1) For a given value of M calculate A and $\Sigma \frac{dA}{R + y}$ and hence K from Equation (3).

(2) Find "a" from Equation (2), assuming P = zero, as it is for pure bending.

(3) Find the circumferential bending stress f from Equation (1) and hence

(4) The position of the neutral axis given by $y = h$ when $f = 0$.

3.6.3 Numerical Example – The method will be demonstrated by reference to a solid circular section of diameter d = 2.0 in. having an initial radius of curvature R = 3.0 in. measured to the centroidal axis, under the action of a bending moment M.

(1) The integral $\Sigma \frac{dA}{R + y}$ can be re-written as:–

$$\sum_{-\frac{d}{2}}^{+\frac{d}{2}} \frac{2w\,dy}{R + y}$$

where w is the half width of an elemental strip of thickness dy (Fig. 3.6–3). That is $dA = 2w dy$.

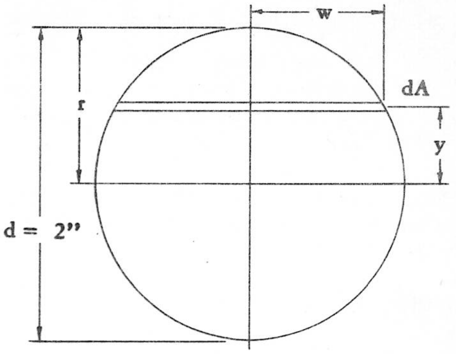


Fig. 3.6–3

The mathematical solution of this is:–

$$2\pi[R - \sqrt{R^2 - (\frac{d}{2})^2}]$$

which for R = 3.0, d = 2.0 and R/d = 1.5 gives

$$\Sigma \frac{dA}{R + y} = 1.079.$$

This result could have been obtained by graphical integration by tabulating values of 2w and y as in Tables 3.6–1 and 3.6–2. In fact, this procedure would have to be adopted in the general case of a section which did not lend itself to a mathematical approach.

TABLE 3.6–1

Clearly $w = \sqrt{r^2 - y^2}$

y	0	0.2	0.4	0.6	0.8	0.9	1.0
w	1.0	.98	.916	.80	.60	.435	0
2w	2.0	1.96	1.832	1.60	1.20	.87	0
R + y	3.0	3.2	3.4	3.6	3.8	3.9	4.0
$\frac{2w}{R + y}$	.66	.612	.538	.444	.316	.223	0

TABLE 3.6–2

y	0	–0.2	–0.4	–0.6	–0.8	–0.9	–1.0
R + y	3.0	2.8	2.6	2.4	2.2	2.1	2.0
$\frac{2w}{R + y}$	.66	.70	.704	.66	.545	.413	0

$$R\Sigma\frac{dA}{R+y} = 3.0 \times 1.079 = 3.237$$

$$[A - R\Sigma\frac{dA}{R+y}] = \pi - 3.237 = -0.0954$$

$$K = \frac{M}{-0.0954} = -10.48\ M$$

(2) For P = 0,

$$aA = -K\Sigma\frac{dA}{R+y} = 10.48\ M \times 1.079 = 11.31\ M$$

$$a = \frac{11.31\ M}{\pi} = 3.606\ M$$

(3) $f = a + \frac{K}{R+y} = M[\,3.606 - \frac{10.48}{3+y}\,]$

When y = +1.0,
stress at extreme outer fibre s_o = .985 M

When y = −1.0,
stress at extreme inner fibre s_i = −1.632 M

$$s = \frac{M}{Z} = \frac{32M}{\pi d^3} = 1.272\ M$$

$$\frac{s_o}{s} = \frac{.985}{1.272} = .773$$

$$\frac{s_i}{s} = \frac{1.632}{1.272} = 1.28$$

Check from Roark, p.144, Case 2.

$\frac{R}{c} = \frac{3.0}{1.0} = 3.0$ (In Roark's notation $c = \frac{d}{2}$)

$$K_o = \frac{s_o}{s} = 0.79$$

$$K_i = \frac{s_i}{s} = 1.33$$

(4) $\frac{f}{M} = 3.606 - \frac{10.48}{(3+y)}$

When f = 0, y = h = − 0.094

CHECK. From Roark, p.144, Case 2, for $\frac{R}{c}$ = 3.0,

$\frac{h}{R}$ = .03, h = .09

DISTRIBUTION OF BENDING STRESS

y	+1.0	+0.5	0	− .09	−0.5	−1.0
3 + y	4.0	3.5	3.0	2.91	2.5	2.0
10.48	2.62	2.99	3.49	3.6	4.17	5.24
3 + y						
f/M	.985	.616	.12	0	− .56	−1.632

The distribution of bending stress is hyperbolic, as in Fig. 3.6–4.

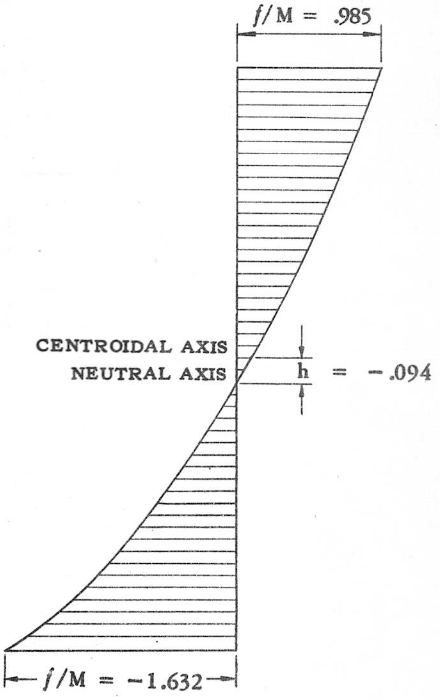


Fig. 3.6–4

3.6.4 Data Sheets – Curves for rectangular and solid circular sections appear in Fig. 3.6–5 and I, T and trapezoidal sections in Fig. 3.6–6.

When there is an applied axial load, in addition to the bending moment, the effect of the resulting direct stress and the bending moment due to the offset of this axial load from the neutral axis – because of its non-coincidence with the centroidal axis – must be included.

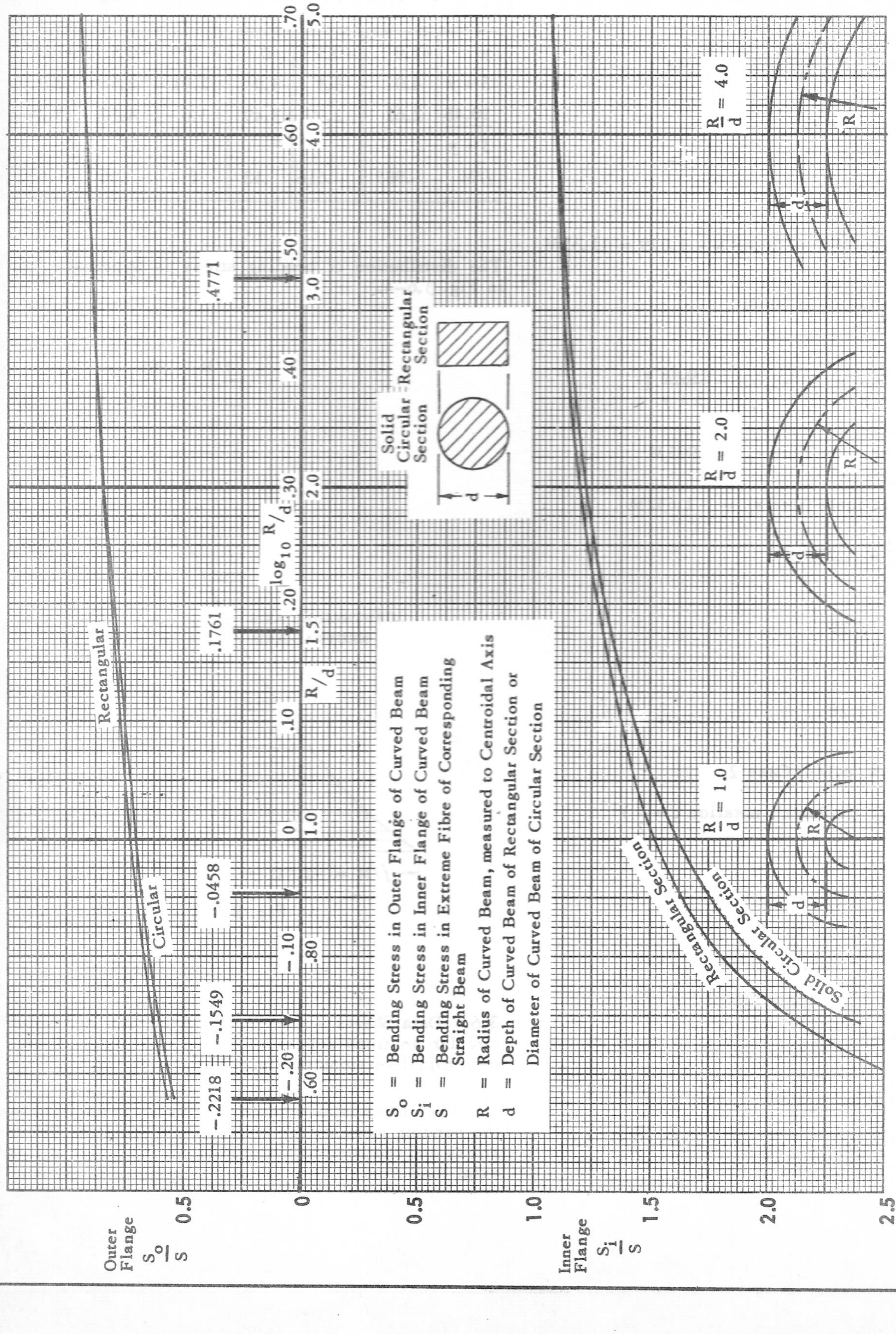


Fig. 3.6–5 Circumferential Bending Stresses in Curved Beams

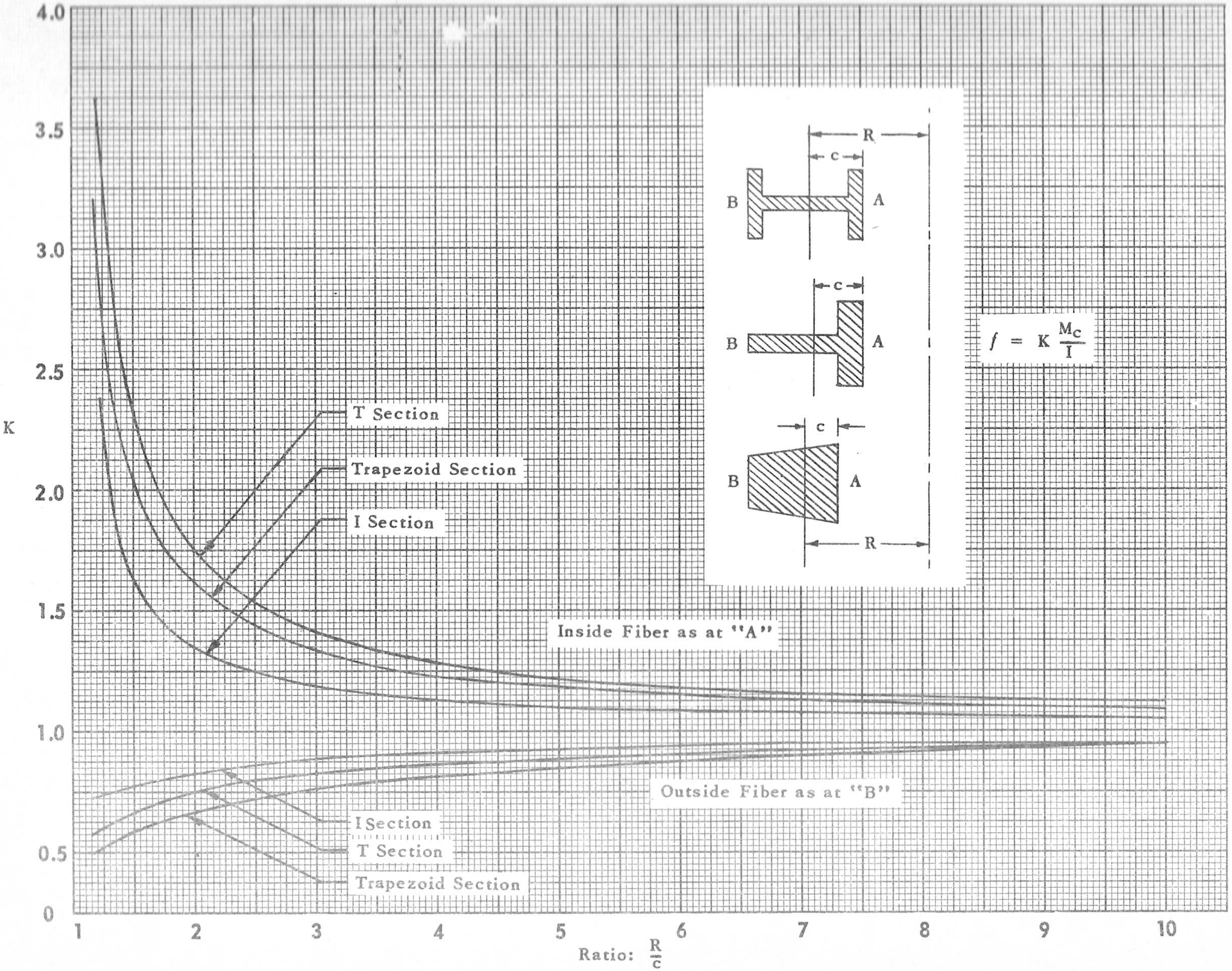


Fig. 3.6–6 I, T and Trapezoidal Sections

3.7

UNSYMMETRICAL BENDING

TABLE OF CONTENTS

3.7.1 Introduction

The bending stresses on a section subjected simultaneously to bending moments in two planes can be found by the formulas given in 3.7.3, without having to determine the principal axes, even when the section is completely unsymmetrical.

The theory is developed in Report GEN/1090/807.

3.7.2 Notation

XX YY		Any two convenient rectangular axes through the centroid O of the section. When the section has an axis of symmetry, XX or YY would naturally be chosen to coincide with this axis.
NA		Neutral axis of the section.
x	IN.	Co-ordinate measured along the XX axis.
y	IN.	Co-ordinate measured along the YY axis. (x and y are positive in the first quadrant.)
M_x	LB.IN.	Component of bending moment about XX.
M_y	LB.IN.	Component of bending moment about YY. (M_x and M_y are positive when acting as shown in 3.7.3.)
I_x	$IN.^4$	Moment of Inertia of effective section about XX.
I_y	$IN.^4$	Moment of Inertia of effective section about YY.
I_{xy}	$IN.^4$	Product of Inertia of effective section about axes XX and YY. (The "effective section" is the area effective in bending)
A_1 A_2 B m n		See 3.7.3 for these coefficients.
β	DEG.	angle between the neutral axis of the section and axis XX such that $\tan \beta = \frac{m}{n}$
σ	P.S.I.	bending stress at station whose co-ordinates are x and y.

3.7.3 Formulas

Bending stress at station having co-ordinates x, y is: $\sigma = mx + ny$ where m and n are as below.

Positive σ is a compressive stress.

M_x and M_y are positive when acting as shown.

When $\tan \beta = \frac{m}{n}$ is positive, β is as shown.

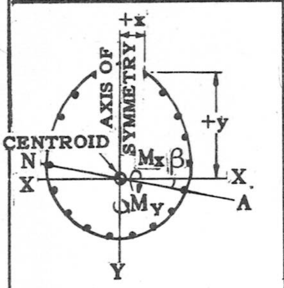
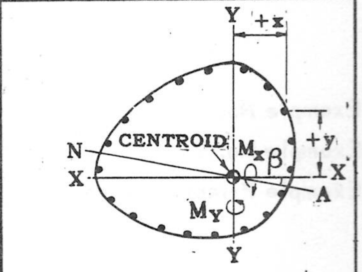
SECTION HAVING ONE AXIS OF SYMMETRY	COMPLETELY UNSYMMETRICAL SECTION
	
$m = \frac{M_y}{I_y}$ $n = \frac{M_x}{I_x}$ $\text{Tan}\,\beta = \frac{m}{n}$	$m = \frac{A_2}{B}$ $n = \frac{A_1}{B}$ $\text{Tan}\,\beta = \frac{m}{n}$ Where $A_1 = M_x I_y - M_y I_{xy}$ $A_2 = M_y I_x - M_x I_{xy}$ $B = I_x I_y - I^2_{xy}$ Note that I_{xy} must be found with due regard to sign.

Fig. 3.7–1

3.7.4 Procedure

(a) Find the position of the centroid of the effective section, that is, the section effective in bending, which in many cases will consist of stringers plus effskin.

(b) Decide on any convenient rectangular axes XX and YY through the centroid.

(c) Find I_x, I_y and I_{xy}, the last-mentioned with due regard to sign.

(d) Find the coefficients A_1, A_2 and B and calculate m, n and the ratio $\frac{m}{n}$.

(e) Calculate the bending stress $\sigma = mx + ny$.

A numerical example appears below.

3.7.5 Numerical Example

Fig. 3.7–2 shows a completely unsymmetrical wing section for which the stringer positions relative to axes XX and YY through the centroid are as in the Table below. (The example is taken from BRUHN, Fig. A 13–4). It is required to find the distribution of bending stress with the following given data:

$I_x = 186.5\ \text{IN}^4.$

$I_y = 431.7\ \text{IN}^4.$

$I_{xy} = -36.41\ \text{IN}^4.$

$M_x = 71.3 \times 10^4\ \text{LB. IN.}$

$M_y = -3.8 \times 10^4\ \text{LB. IN.}$

Steps (a), (b), and (c) of Procedure have been covered.

(d) $A_1 = M_x I_y - M_y I_{xy}$

$= 71.3 \times 10^4 \times 431.7 - 3.8 \times 10^4 \times 36.41$

$= 308 \times 10^6 - 1.383 \times 10^6 = +\underline{306.617 \times 10^6}$

$A_2 = M_y I_x - M_x I_{xy}$

$= -\ 3.8 \times 10^4 \times 186.5 + 71.3 \times 10^4 \times 36.41$

$= -\ 7.1 \times 10^6 + 25.95 \times 10^6 = +\underline{18.85 \times 10^6}$

$B = I_x I_y - I^2_{xy}$

$= 186.5 \times 431.7 - (36.41)^2$

$= 8.06 \times 10^4 - 1320$

$= \underline{7.928 \times 10^4}$

$$m = \frac{A_2}{B} = \frac{18.85 \times 10^6}{7.928 \times 10^4} = +\underline{237.5}$$

$$n = \frac{A_1}{B} = \frac{306.617 \times 10^6}{7.928 \times 10^4} = +\underline{3870}$$

$$\text{Tan}\,\beta = \frac{m}{n} = \frac{237.5}{3870} = \underline{0.0613}$$

$$\beta = \underline{3^\circ\ 30'}$$

3.7.5 (cont'd)

(e) $\sigma = mx + ny$

$= 237.5x + 3870y$ (+ Compression)

Values of σ for various values of x and y are given.

The distribution of σ is shown in Fig. 3.7–2.

+ COMPRESSION

ITEM	x (IN.)	y (IN.)	237.5x	3870y	σ (P.S.I.)
1	−17.41	4.40	−4120	17,000	12,880
2	−13.54	6.45	−3210	25,000	21,790
3	− 9.11	7.40	−2165	28,600	26,430
4	− 5.44	7.77	−1290	30,000	28,710
5	− 0.86	7.95	− 205	30,800	28,750
6	3.14	7.90	745	30,600	31,350
7	7.14	7.70	1700	29,800	31,500
8	11.74	7.30	2790	28,300	31,090
9	15.39	6.90	3650	26,700	30,350
10	−17.51	−2.90	−4160	−11,220	−15,380
11	−13.54	−4.50	−3210	−17,420	−20,630
12	− 9.11	−5.55	−2165	−21,500	−23,670
13	− 2.96	−7.00	− 700	−27,100	−27,800
14	3.32	−7.73	790	−29,900	−29,110
15	9.64	−8.22	2290	−31,800	−29,510
16	15.39	−8.47	3650	−32,800	−29,150

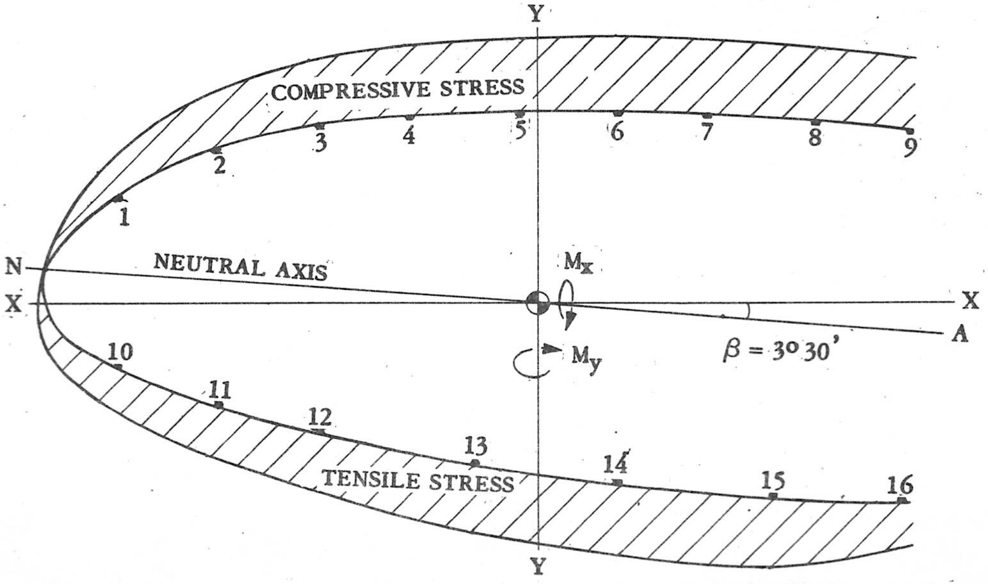


Fig. 3.7–2

3.7.6 Numerical Example No. 2 (Fig. 3.7–3)

Same Example but $M_y = 0$

$$A_1 = M_x I_y \qquad = 308 \times 10^6$$

$$A_2 = - M_x I_{xy} \qquad = 25.95 \times 10^6$$

$$B = I_x I_y - I^2_{xy} \qquad = 7.928 \times 10^4 \text{ as before}$$

$$m_1 = \frac{A_2}{B} = \frac{25.95 \times 10^6}{7.928 \times 10^4} = + 327$$

$$n_1 = \frac{A_1}{B} = \frac{308 \times 10^6}{7.928 \times 10^4} = + 3885$$

$$\text{Tan } \beta_1 = \frac{m_1}{n_1} = \frac{327}{3885} = 0.084$$

$$\beta_1 = 4^\circ \; 48'$$

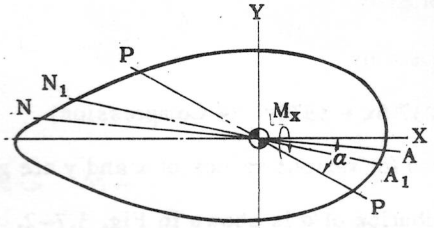


Fig. 3.7–3

Hence neutral axis moves clockwise from NOA to N_1OA_1. This is as would be expected since removal of M_y will reduce the compressive stress, and increase the tensile stress, at N and hence zero value could only be obtained by having N_1 clockwise relative to N.

3.7.7 Principal Axes

As a matter of interest the Principal axis PP for this section is at an angle $a = - 8^\circ \; 16.25'$ to XX (See Bruhn, A13.6).

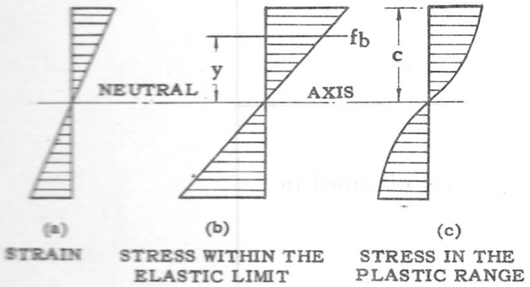
PP is thus located as in Fig. 3.7–3 (not to scale).

3.8 PLASTIC BENDING

TABLE OF CONTENTS

3.8.1 Introduction

Classic bending theory is based on the assumption that plane sections before bending remain plane after bending. Since, within the elastic limit, stress is proportional to strain, it follows that the stress distribution over the cross-section will be linear, varying from zero at the neutral axis to a maximum at the extreme fibre, as in Fig. 3.8–1 (b). This distribution is expressed mathematically for a symmetrical cross-section subjected to a bending moment m by: $\frac{my}{I}$



(a) STRAIN (b) STRESS WITHIN THE ELASTIC LIMIT (c) STRESS IN THE PLASTIC RANGE

Fig. 3.8–1

When the extreme fibre stress exceeds the proportional limit stress of the material, the stress distribution on the cross-section is no longer linear but tends to follow the stress-strain characteristics of the material (Fig. 3.8–1 (c)). Consequently, the convential formula for linear bending stress distribution no longer applies.

It has been standard aircraft practice, in such cases, to use a FORM FACTOR to indicate the amount by which the allowable ultimate stress of the material (or the allowable bending moment) could be considered to be increased in order to apply the convential formula given above. To use this method it is necessary to have form factors for all the cross-sectional shapes commonly used, information which is not readily available.

A procedure has therefore been developed for symmetrical sections, (see "Bending Strength in the Plastic Range" Cozzone, J. Ae. Sc., May 1943, Page 137) whereby the engineers' bending stress is reduced by using a SECTION FACTOR in conjunction with a series of Design Charts for various materials; this reduced value is then compared with the allowable bending stress of the material.

3.8.2 Section Factor

The section factor K is given by the formula:

$$K = \frac{2Q_m}{I/_c}$$

where Q_m = the static (first) moment about the neutral axis of the area between the neutral axis and the extreme fibre.

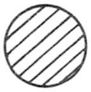
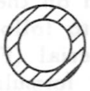
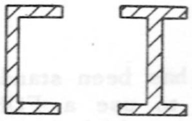
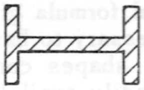
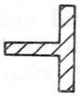
I = moment of inertia about the neutral axis.

c = distance from the neutral axis to the extreme fibre of the section.

Values of K for various common symmetrical sections appear in Table 3.8–1 and in Figs. 3.8–2, 3 and 4.

If K by calculation exceeds 2.0, use 2.0.

TABLE 3.8–1

(1) 	K = 1.5
(2) 	K = 1.7
(3) 	See Fig. 3.8–2
(4) 	See Fig. 3.8–3
(5) 	See Fig. 3.8–4
(6) 	Half the value obtained in Case 5.

Do not use a value of K greater than 2.0.

3.8.2 Section Factor (cont'd)

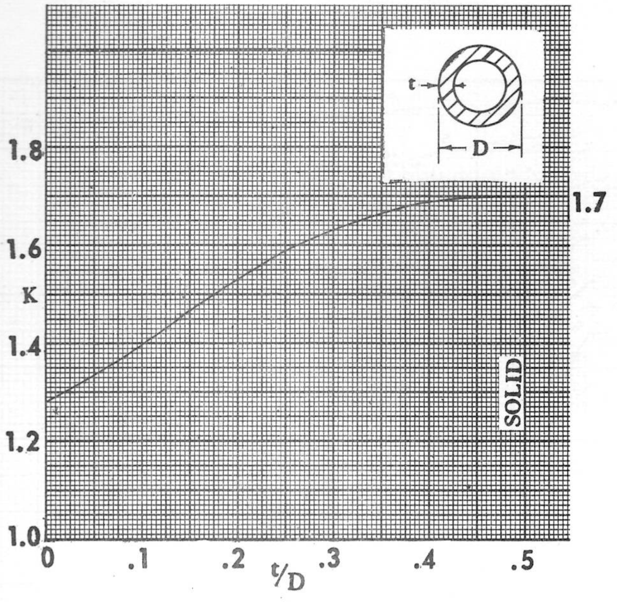


Fig. 3.8–2 K value for Round Tube

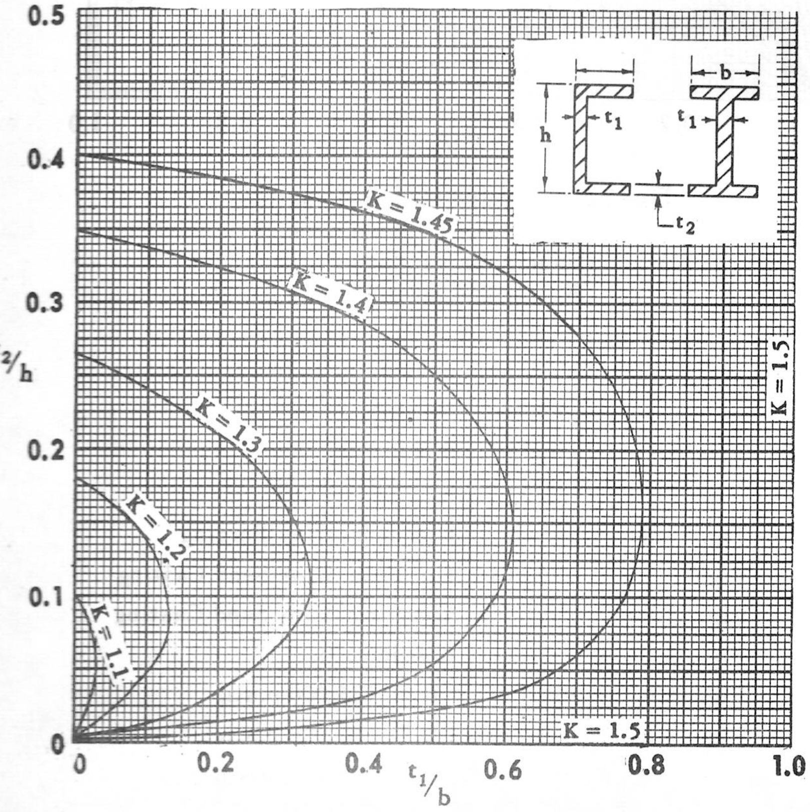


Fig. 3.8–3 K value for I or Channel Section

3.8.2 Section Factor (cont'd)

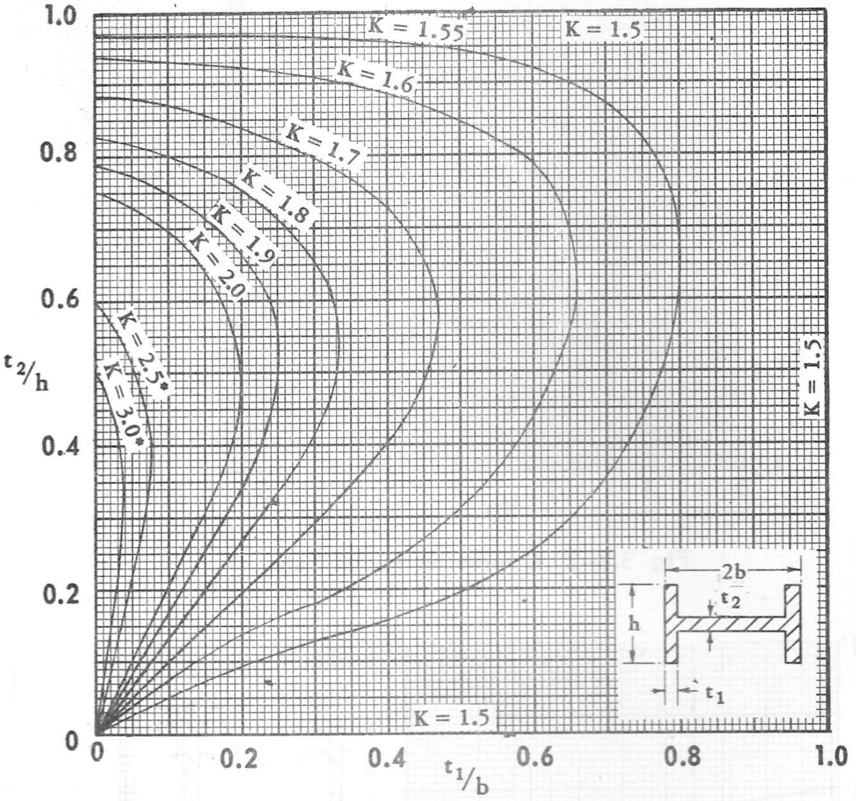


Fig. 3.8–4 Kvalue for I or T section

*DO NOT USE VALUES OF
K GREATER THAN 2.0.

3.8.3 Design Charts

The following Design Charts are given:

Fig. 3.8– 5	14S–T	Aluminum Alloy Forging
Fig. 3.8– 6	220–T4	Aluminum Alloy Sand Casting
Fig. 3.8– 7	195–T6	Aluminum Alloy Sand Casting
Fig. 3.8– 8	75S–T	Aluminum Bar
Fig. 3.8– 9	Alloy Steel H.T. 125,000 p.s.i.	
Fig. 3.8–10	Alloy Steel H.T. 150,000 p.s.i.	
Fig. 3.8–11	Alloy Steel H.T. 180,000 p.s.i.	
Fig. 3.8–12	HTA Magnesium Sand Castings	

3.8.3 Design Charts (cont'd)

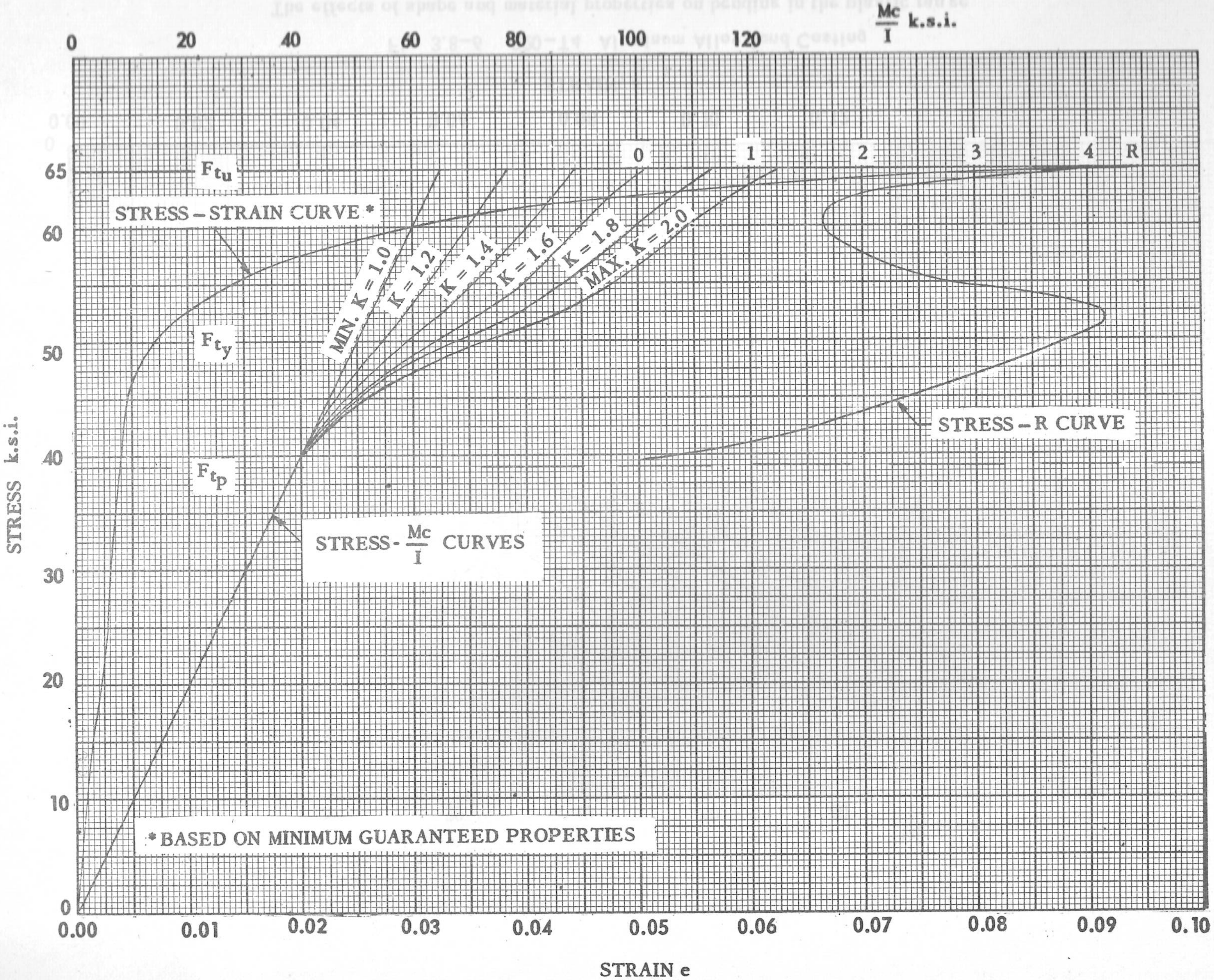


Fig. 3.8–5 14S–T Aluminum Alloy Forging The effects of shape and material properties on bending in the plastic range

3.8.3 Design Charts (cont'd)

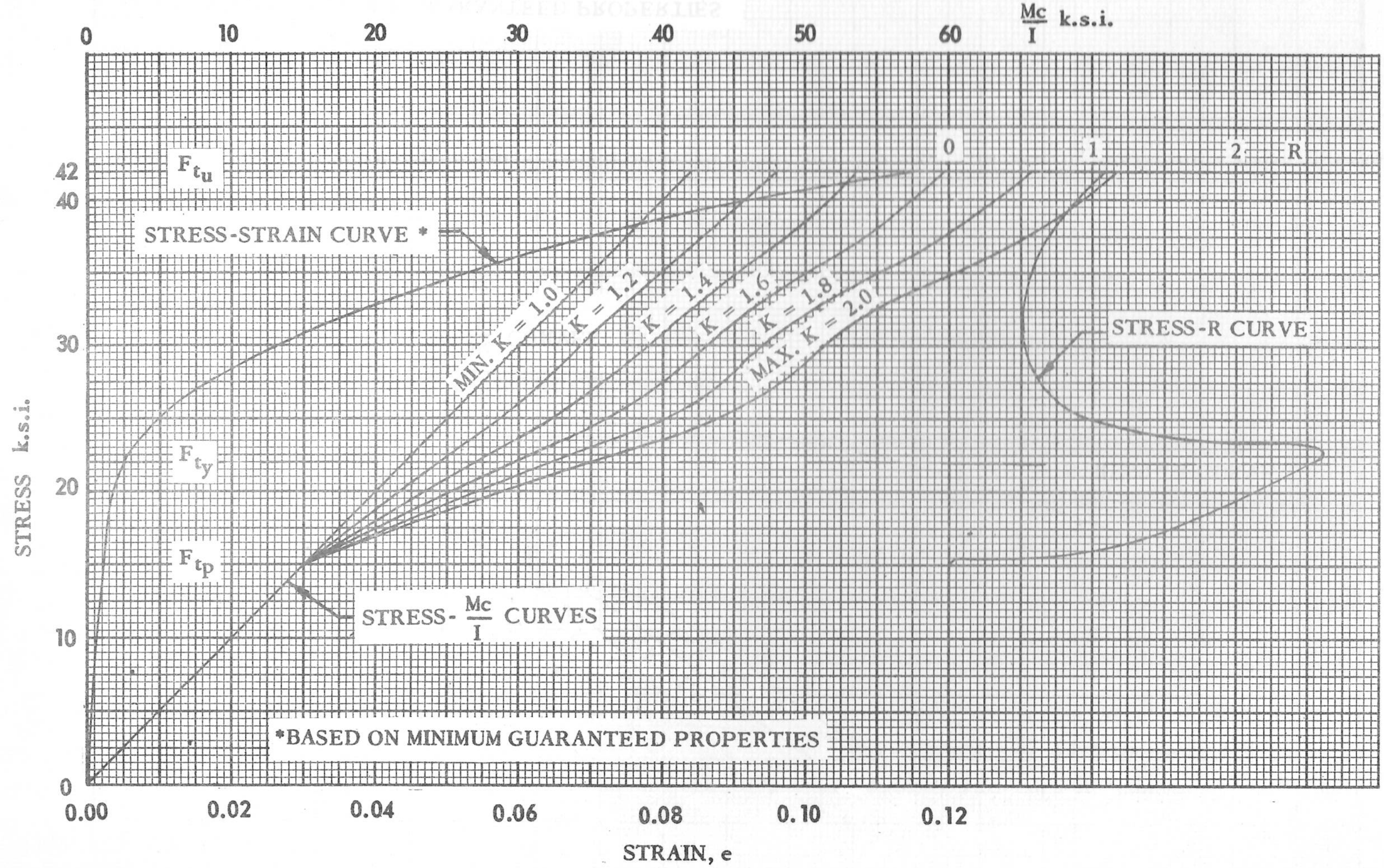


Fig. 3.8–6 220–T.4 Aluminum Alloy Sand Casting

The effects of shape and material properties on bending in the plastic range

3.8.3 Design Charts (cont'd)

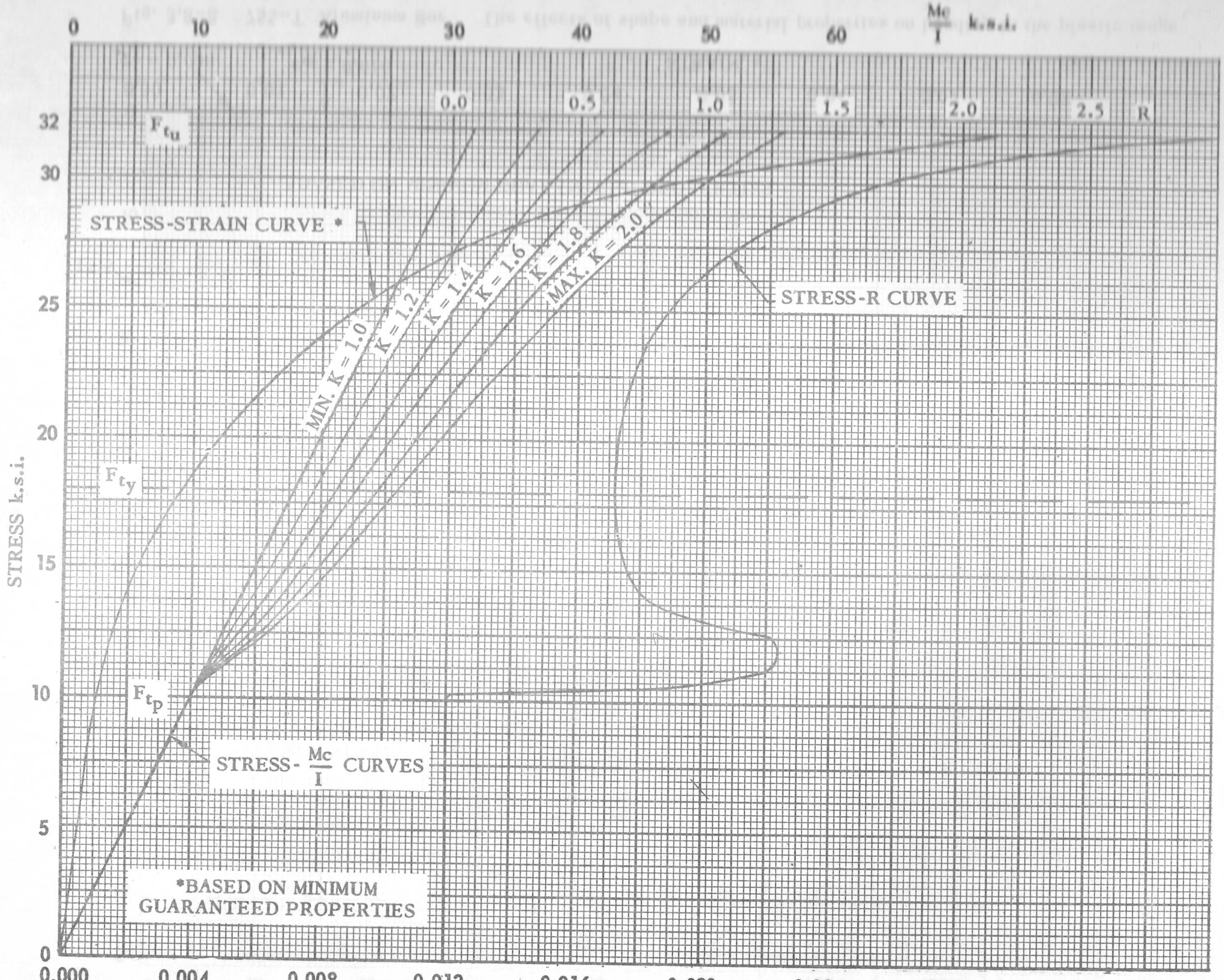


Fig. 3.8–7 **195–T6 Aluminum Alloy Sand Casting** The effects of shape and material properties on bending in the plastic range

3.8.3 Design Charts (cont'd)

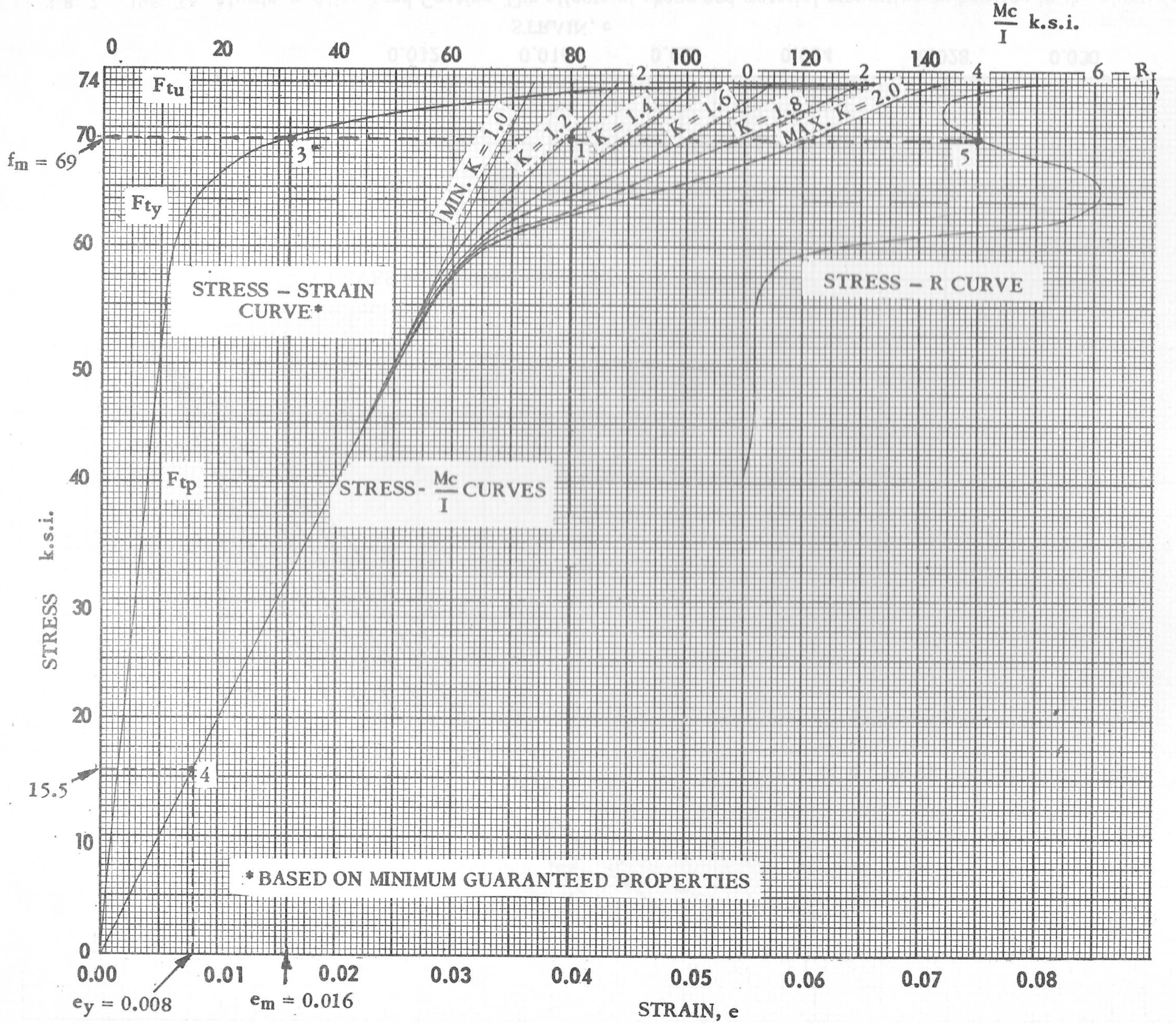


Fig. 3.8–8 75S–T Aluminum Bar The effects of shape and material properties on bending in the plastic range

3.8.3 Design Charts (cont'd)

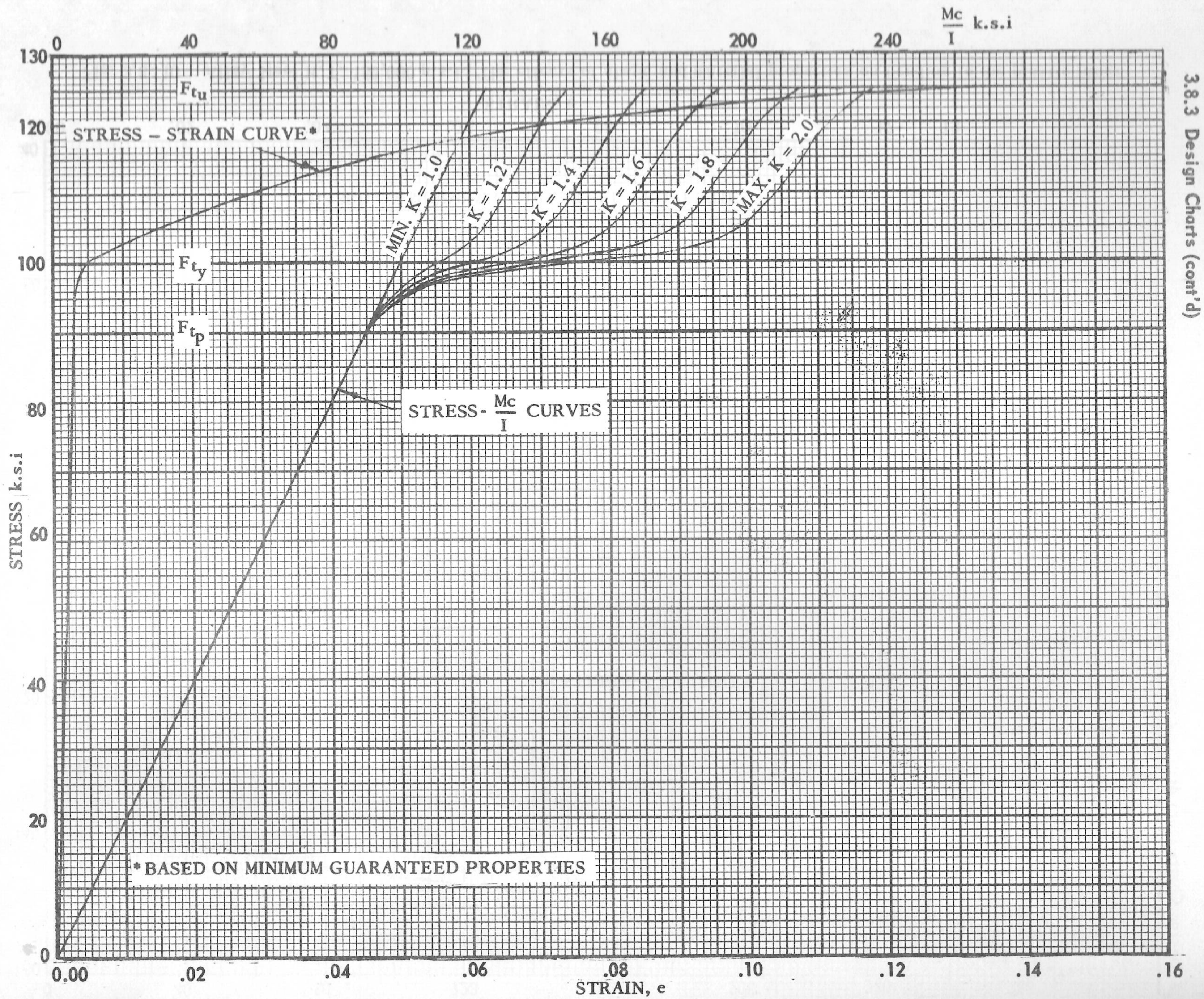


Fig. 3.8–9 Alloy Steel H. T. 125,000 p.s.i. The effects of shape and material properties on bending in the plastic range

3.8.3 Design Charts (cont'd)

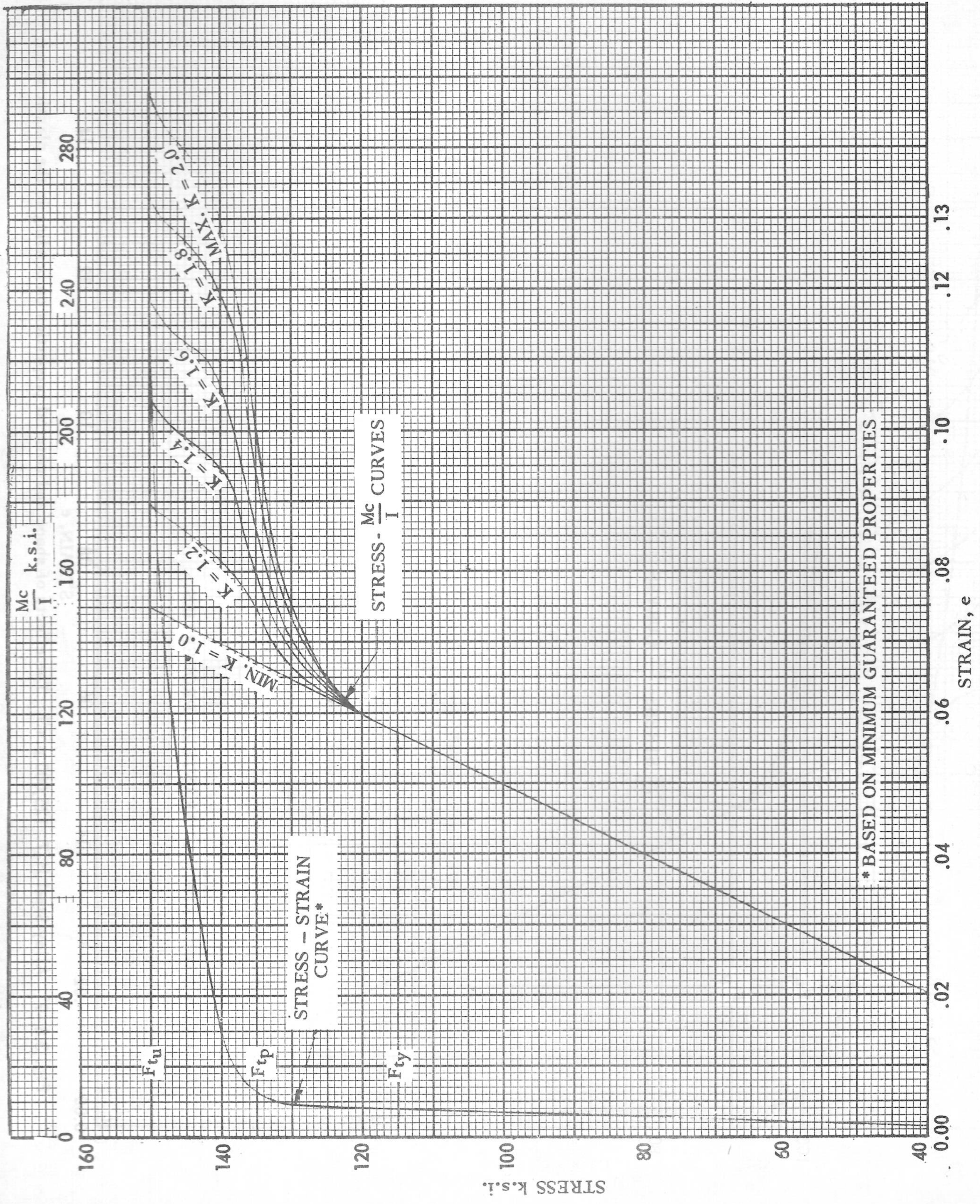


Fig. 3.8–10 Alloy Steel H.T. 150,000 p.s.i. The effects of shape and material properties on bending in the plastic range

3.8.3 Design Charts (cont'd)

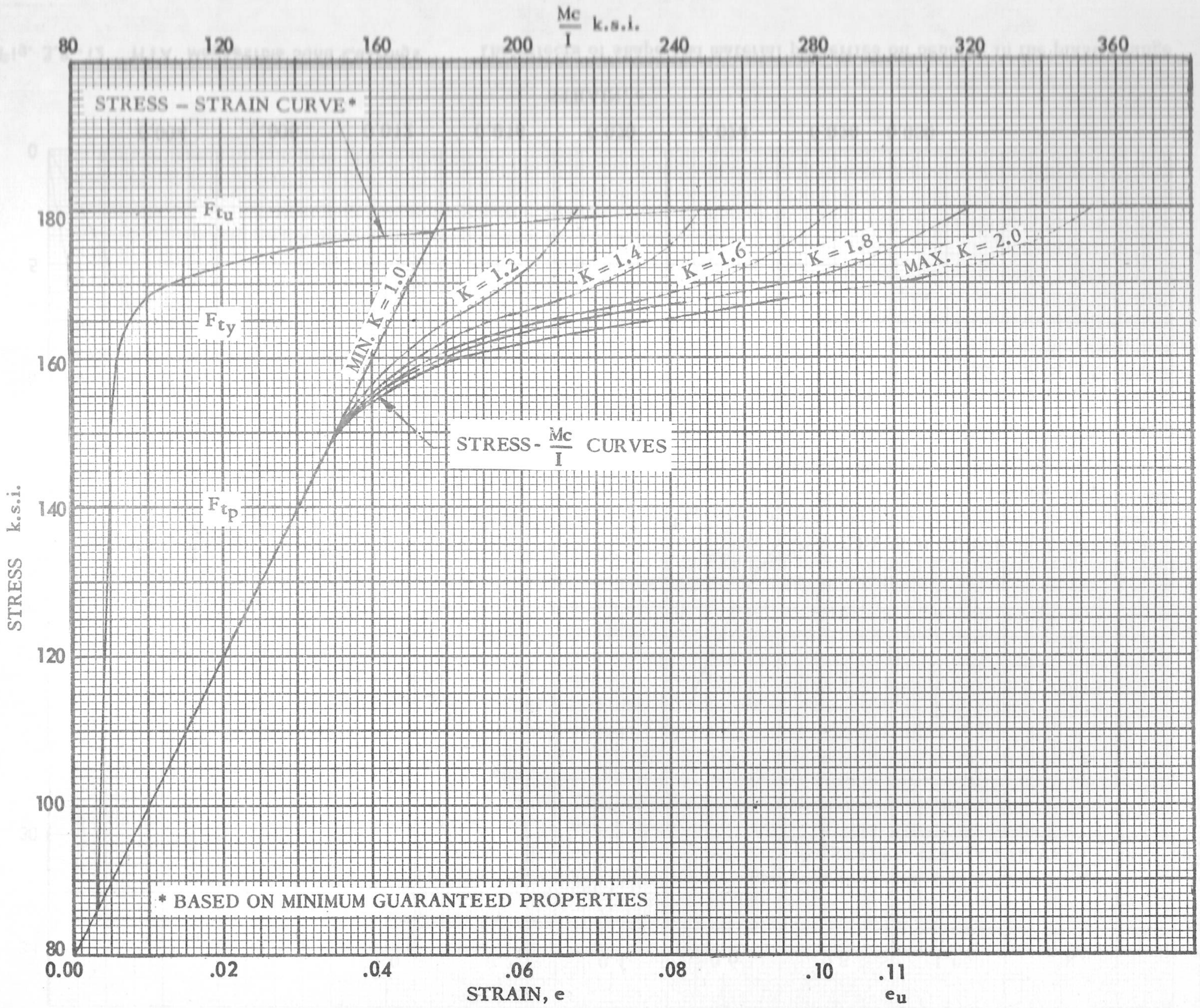


Fig. 3.8–11 **Alloy Steel H.T. 180,000 p.s.i.** The effects of shape and material properties on bending in the plastic range

3.8.3 Design Charts (cont'd)

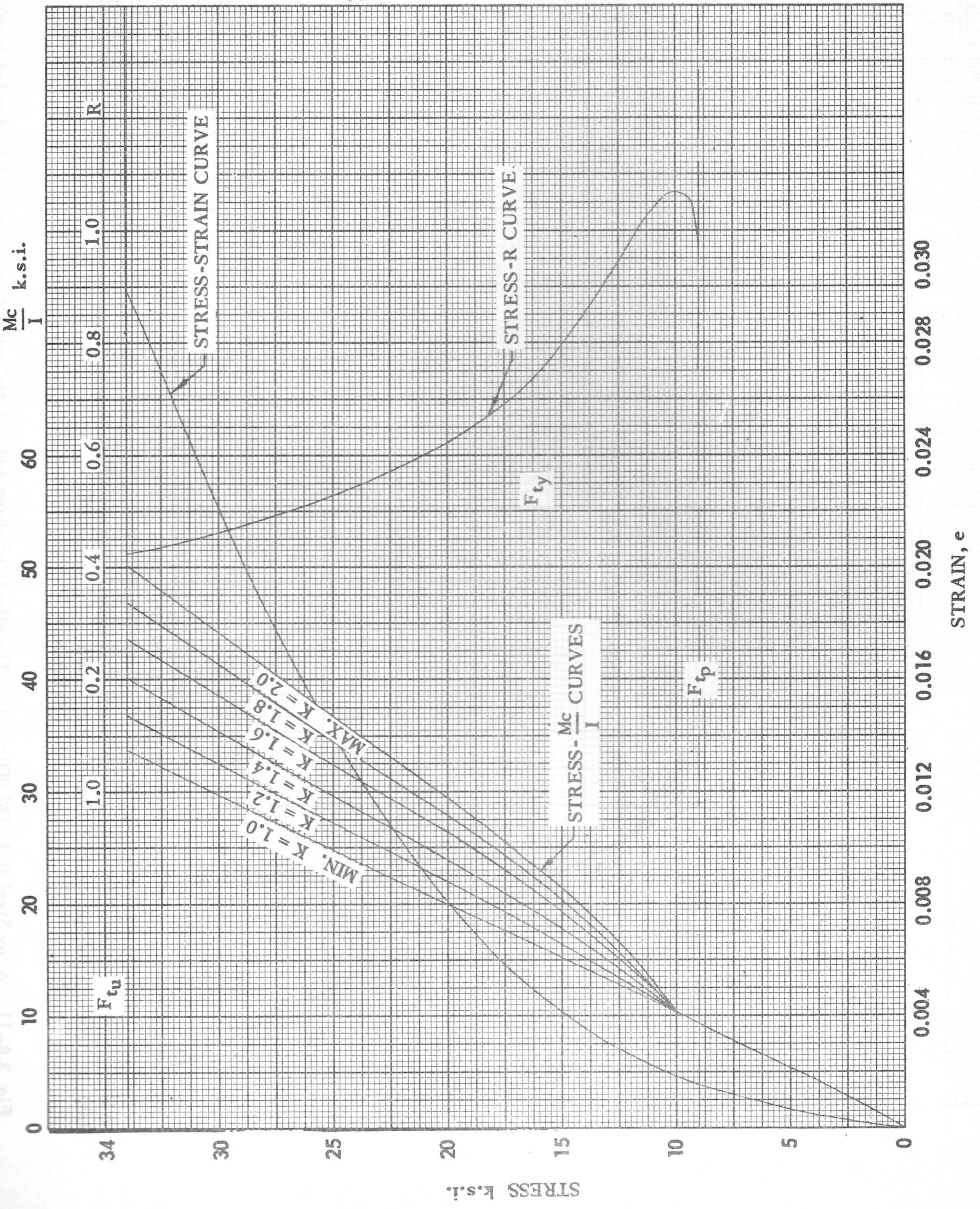


Fig. 3.8–12 HTA Magnesium Sand Castings The effects of shape and material properties on bending in the plastic range

3.8.4 Bending Stress at Extreme Fibre

(a) Find K from Table 3.8–1 or when this is not possible, use the formula $K = \dfrac{Q_m}{I/c}$

(b) Find $\dfrac{mc}{I}$, the engineers' bending stress at the extreme fibre.

(c) Enter the appropriate Design Chart on the upper horizontal $\dfrac{mc}{I}$ scale and obtain the maximum applied stress f_m at the extreme fibre for the particular value of K found in (a).

NUMERICAL EXAMPLE

I section beam shown in Fig. 3.8–13 subjected to a bending moment of 31600 lb. in. about the neutral axis. Specification 75S–T Aluminum Alloy Bar.

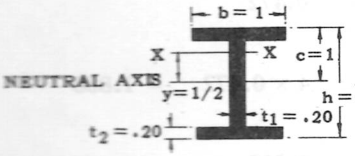


Fig. 3.8–13

(a) Enter Fig. 3.8–3 on the base line at $t_{1/b} = 0.20$. Read the value of $t_{2/h} = 0.10$ on the vertical scale. The intersection gives K = 1.234 by interpolation between the curves of K = 1.2 and K = 1.3.

As a check on this value, the general method, using the formula $K = \dfrac{2Q_m}{I/c}$, will be used.

$$Q_m = 1.0 \times .20 \times .90 \quad + .80 \times .20 \times .40$$
$$= .18 + .064 \quad = \underline{0.244 \text{ in}^3.}$$

$$I = \frac{1}{12}\left[1 \times 2^3 - .80 \times 1.6^3\right]$$
$$= \frac{1}{12}\left[8.0 - 3.26\right] = \underline{0.395 \text{ in}^4.}$$

$$I/_c = .395 \text{ in}^3$$

$$K = \frac{2 \times .244}{.395} \quad = \underline{1.236}$$

(b) Stress developed by engineers' formula

$$= \frac{mc}{I} = \frac{31600}{.395} = 80{,}000 \text{ p.s.i.}$$

(c) Enter Fig. 3.8–8 on the upper $\dfrac{Mc}{I}$ scale at 80,000 p.s.i. and proceed vertically downwards (as indicated by the dotted line) until point (1) on the $\dfrac{Mc}{I}$ curve representative of K = 1.236 (by interpolation between K = 1.2 and K = 1.4) is reached.

Corresponding stress value on the vertical scale is 69,000 p.s.i.

RESERVE FACTOR

Since stress is no longer proportional to bending moment, the reserve factor must be obtained on a bending moment basis.

Enter Fig. 3.8–8 with the allowable ultimate tensile stress F_{tu} to obtain an $\dfrac{Mc}{I}$ value for the proper section factor (K = 1.236 in this case) as indicated by point (2). $\dfrac{Mc}{I} = 91{,}000$ p.s.i. by interpolation.

Multiply this value by $I/_c$ to find the allowable ultimate bending moment viz M = 91000 × .395 = 36000 lb. in.

Reserve Factor on an ultimate basis

$$= \frac{\text{allowable ultimate B.M.}}{\text{applied B.M.}}$$

$$= \frac{36000}{31600} = 1.13$$

3.8.5 Bending Stress at Intermediate Fibre

(a) On the appropriate Design Chart find the strain e_m (on the lower horizontal scale) corresponding to the maximum applied stress f_m *on the stress-strain curve.*

(b) The strain at any point distant y from the neutral axis is:

$$e_y = e_m \times y/c$$

(c) Read off the vertical scale the stress corresponding to this strain on the stress-strain curve.

NUMERICAL EXAMPLE

To find the bending stress of the I section beam shown in Fig. 3.8–13 at Section XX, which is at a distance $y = 1/2$ in. from the neutral axis.

Applied bending moment = 31600 lb. in.

(a) $e_m = 0.016$ given by Point (3) on the stress-strain curve. $f_m = 69{,}000$ p.s.i.

(b) $e_y = 0.016 \times \dfrac{0.5}{1.0} = 0.008.$ This gives Point (4) on the stress-strain curve.

(c) Corresponding stress = 15500 p.s.i.

3.8.6 Shear Flow

The shear stress distribution shows a marked change from that obtained by the conventional formula for shear flow viz, $q = \dfrac{VQ}{I}$ lb. per in.

It is shown by Cozzone that the maximum shear flow in the plastic range for a doubly-symmetrical section is given by:

$$q = \frac{VQ}{I}\left[\frac{1 + R\left(\frac{A_c}{Q_m} - 1\right)}{1 + R(K-1)}\right]$$

Values of the coefficient R appear in Charts 3.8–5 to 12 in the form of a "Stress – R" curve.

Procedure

(a) Knowing f_m (from above) find the value of R on the stress – R curve.

(b) Substitute this value in the above expression for q.

NUMERICAL EXAMPLE

Consider the same I section as above.

(a) $f_m = 69{,}000$ p.s.i.

$R = 4$ for Point (5) on the Stress – R curve of Fig 3.8–8.

(b) $A = 1 \times 0.2 + 0.8 \times 0.2 = 0.36 \text{ in}^2.$

$$\frac{A_c}{Q_m} = \frac{0.36 \times 1.0}{0.244} = 1.472$$

$$\frac{A_c}{Q_m} - 1 = 0.472$$

$$R\left(\frac{A_c}{Q_m} - 1\right) = 4 \times 0.472 = 1.888$$

$$\left[1 + R\left(\frac{A_c}{Q_m} - 1\right)\right] = 2.888$$

$$(K-1) = 1.236 - 1 = 0.236$$

$$R(K-1) = 0.944$$

$$\left[1 + R(K-1)\right] = 1.944$$

$$q = \frac{VQ_m}{I} \times \frac{2.888}{1.944}$$

$$= 1.485 \ \frac{VQ_m}{I}$$

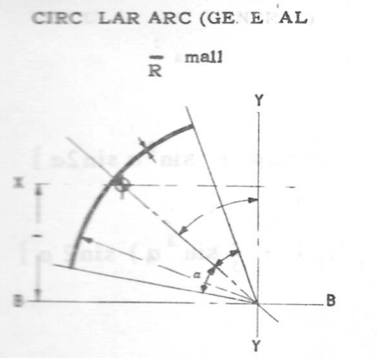
SECTION PROPERTIES

TABLE OF CONTENTS

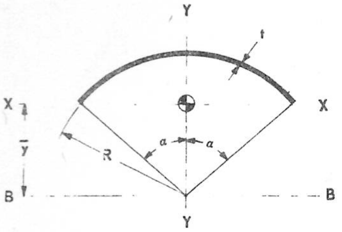
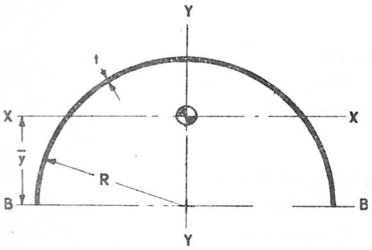
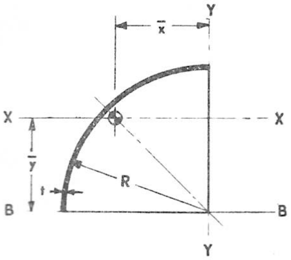
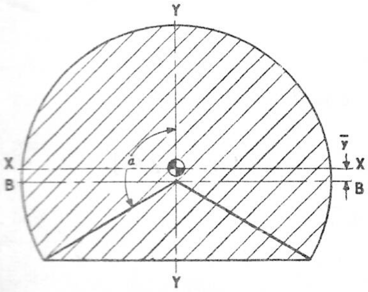
4.1 General Section Properties

4.1.1 Tabular Summary

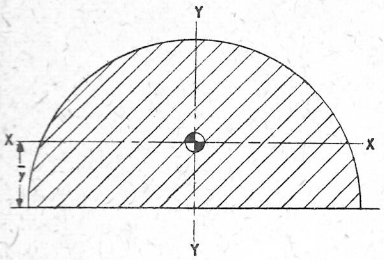
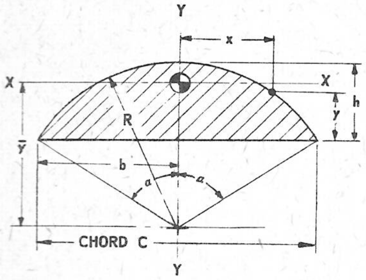
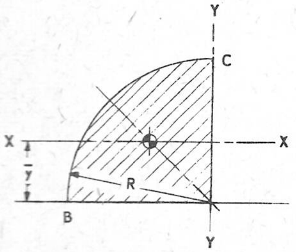
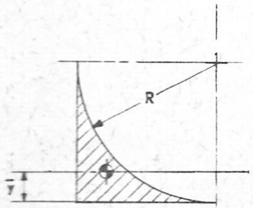
The following tabular summary is intended to supplement Roark, pp 68, 69 and 70, Table I.

	Area $A = 2\alpha Rt$ $y = \frac{R}{\alpha}\sin\alpha\cos\theta$ $I_B = R^3t\,[\,\alpha + \tfrac{1}{2}\sin 2\alpha\cos 2\theta\,]$ $I_X = I_B - A\bar{y}$

4.1.1 Tabular Summary (Cont'd)

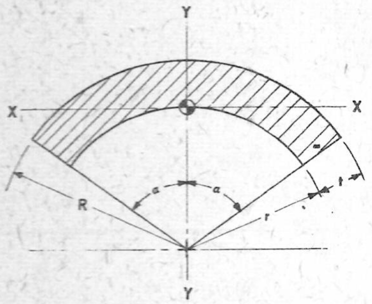
CIRCULAR ARC (SYMMETRICAL) $\frac{t}{R}$ small  	Area A $= 2\alpha R t$ $\bar{y} = \frac{R \sin\alpha}{\alpha}$ $I_B = R^3 t \, [\, \alpha + \tfrac{1}{2}\sin 2\alpha \,]$ $I_X = R^3 t \, [\, \alpha + \tfrac{1}{2}\sin 2\alpha - \frac{2\sin^2\alpha}{\alpha} \,]$ $I_Y = R^3 t \, [\, \alpha - \tfrac{1}{2}\sin 2\alpha \,]$
SEMI-CIRCULAR ARC $\frac{t}{R}$ small  	Area A $= \pi R t$ $\bar{y} = .637 R$ $I_B = 1.57 R^3 t$ $I_X = .298 R^3 t$ $I_Y = 1.57 R^3 t$
QUARTER-CIRCULAR ARC (90° Bend) $\frac{t}{R}$ small  	Area A $= \frac{\pi}{2} R t$ $\bar{x} = .637 R$ $\bar{y} = .637 R$ $I_B = .785 R^3 t$ $I_X = .149 R^3 t$ $I_Y = .785 R^3 t$
CIRCULAR SEGMENT $\alpha > 90^\circ$  	Area A $= \frac{R^2}{2} (2\alpha - \sin 2\alpha)$ $\bar{y} = \frac{2 R^3 \sin^3\alpha}{3A}$ $I_B = \frac{R^4}{4} \, [\, \alpha - \tfrac{1}{2}\sin 2\alpha + \sin^2\alpha \sin 2\alpha \,]$ $I_Y = \frac{R^4}{4} \, [\, \alpha - \tfrac{1}{2}(1 + \tfrac{2}{3}\sin^2\alpha) \sin 2\alpha \,]$ $I_X = I_B - A\bar{y}^2$

4.1.1 Tabular Summary (Cont'd)

SEMI-CIRCULAR SEGMENT  	Area $A = \frac{\pi R^2}{2}$ $\bar{y} = .4244 R$ $I_X = .1098 R^4$ $I_Y = \frac{\pi R^4}{8}$
CIRCULAR SEGMENT α less than 90°  	$R = \frac{b^2 + h^2}{2h}$ Ordinate $y = \sqrt{R^2 - x^2} - \sqrt{R^2 - b^2}$ Area $A = \frac{R^2}{2}(2\alpha - \sin 2\alpha)$ $\bar{y} = \frac{2}{3}\frac{b^3}{A}$ See also Fig. 4.1.2 $I_B = \frac{R^4}{4}[\alpha - \frac{1}{2}\sin 2\alpha + \sin^2\alpha \sin 2\alpha]$ $I_Y = \frac{R^4}{4}[\alpha - \frac{1}{2}(1 + \frac{2}{3}\sin^2\alpha)\sin 2\alpha]$ $I_X = I_B - A\bar{y}^2$
CIRCULAR QUADRANT  	Area $A = .785 R^2$ $\bar{y} = .4244 R$ $I_X = .0549 R^4$
CIRCULAR FILLET  	Area $A = .2146 R^2$ $\bar{y} = .2234 R$ $I_X = .00754 R^4$

4.1.1 Tabular Summary (Cont'd)

HOLLOW CIRCULAR SECTOR
(Not Thin-Walled)



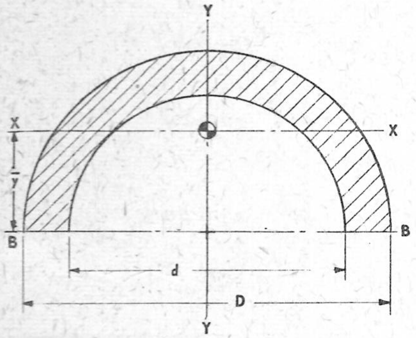
$$\text{Area } A = \alpha (R^2 - r^2)$$

$$\bar{y} = .667 \frac{\sin \alpha}{A} (R^3 - r^3)$$

$$I_X = .25A \; (R^2 + r^2) \left[1 + \frac{\sin 2\alpha}{2\alpha} \right] - A\bar{y}^2$$

$$I_Y = .25A \; (R^2 + r^2) \left[1 - \frac{\sin 2\alpha}{2\alpha} \right]$$

HOLLOW SEMI-CIRCLE
(Not Thin-Walled)



$$\text{Area } A = \frac{\pi}{2} (R^2 - r^2)$$

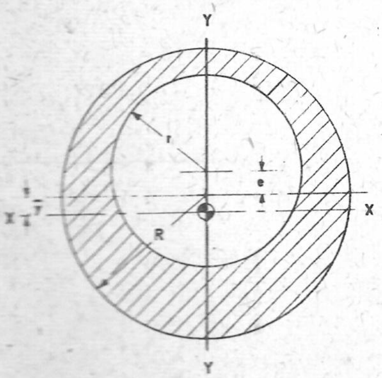
$$\bar{y} = \frac{4}{3\pi} \frac{R^3 - r^3}{(R^2 - r^2)}$$

$$I_Y = A/_4 \; (R^2 + r^2)$$

$$I_B = A/_4 \; (R^2 + r^2)$$

$$I_X = I_B - A\bar{y}^2$$

HOLLOW CIRCLE ECCENTRIC



$$\text{Eccentricity} = e$$

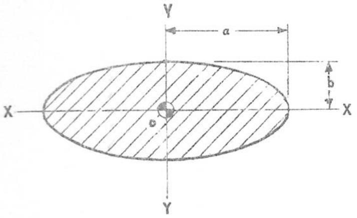
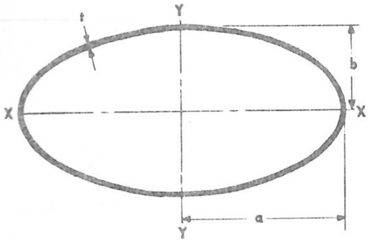
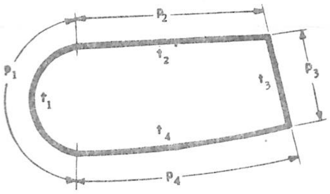
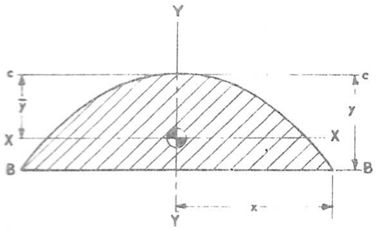
$$\text{Area } A = \pi (R^2 - r^2)$$

$$\bar{y} = \frac{r^2 e}{(R^2 - r^2)}$$

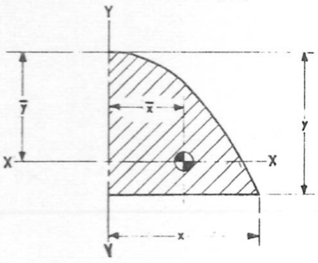
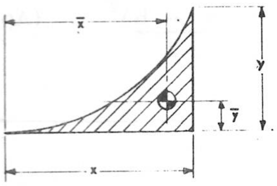
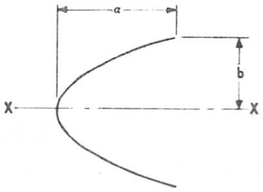
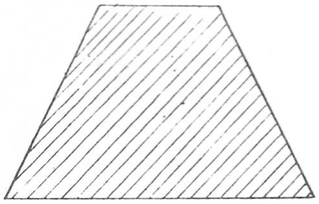
$$I_X = .25A \; (R^2 + r^2) - 9.87 \frac{R^2 r^2 e^2}{A}$$

$$I_Y = .7854 \; (R^4 - r^4)$$

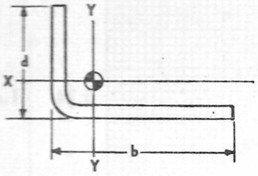
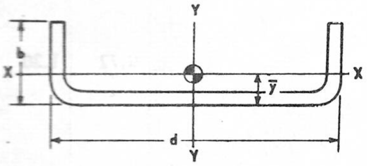
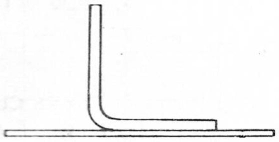
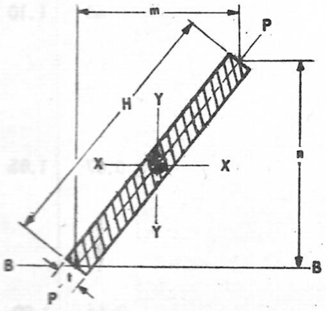
4.1.1 Tabular Summary (Cont'd)

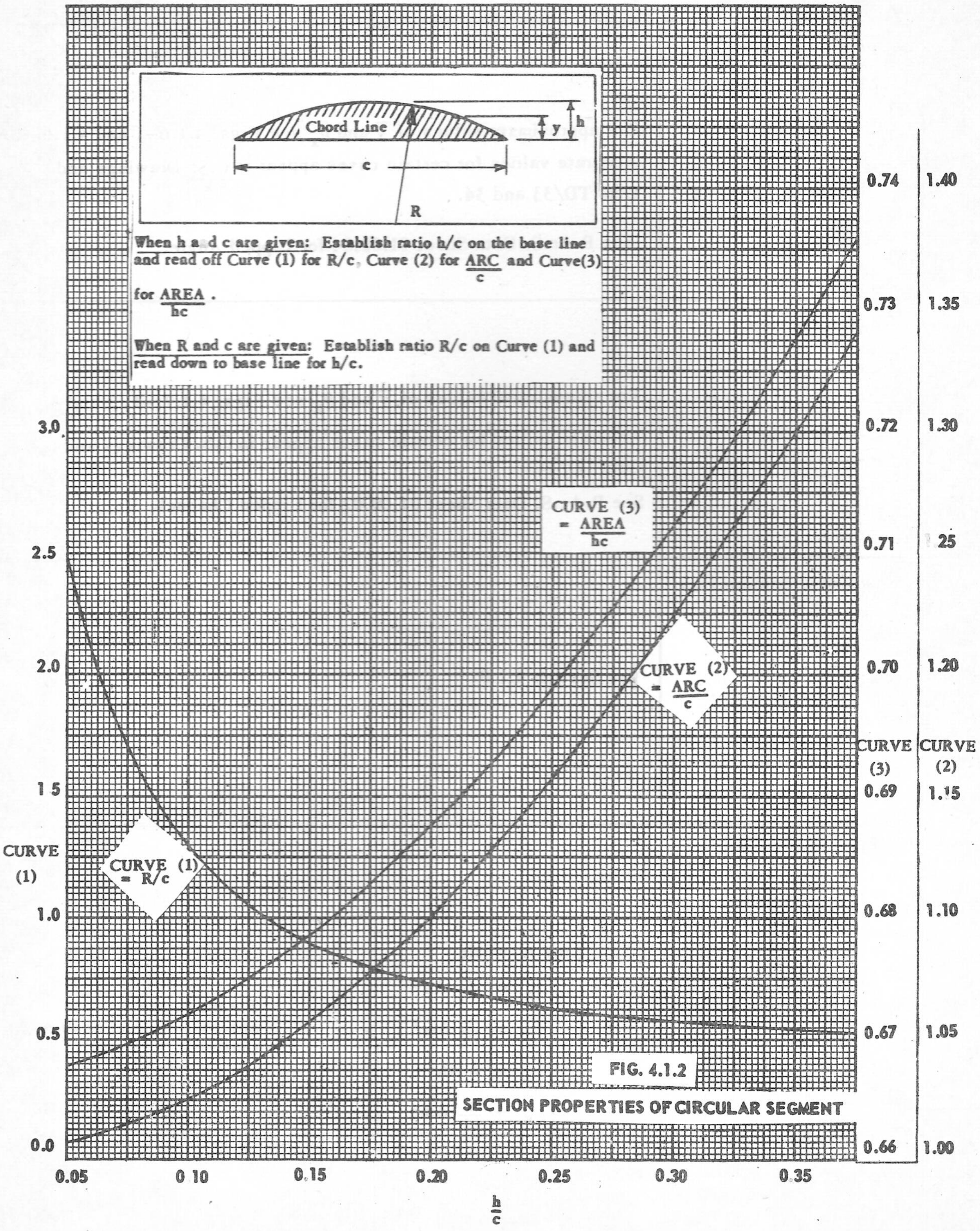
SOLID ELLIPSE  	Perimeter $= \pi\sqrt{2(a^2 + b^2) - \frac{(a-b)^2}{2.2}}$ (Approx.) Area $A = \pi a b$ $I_X = \pi/_4\, a b^3$ $I_Y = \pi/_4\, a^3 b$
THIN-WALLED ELLIPSE  	$I_X = \frac{\pi}{4} b^2 t\, [3a + b]$ (Approx.) $I_Y = \frac{\pi}{4} a^2 t\, [3b + a]$ (Approx.) Perimeter $= \pi\sqrt{2(a^2 + b^2) - \frac{(a-b)^2}{2.2}}$ (Approx.) See Fig. 4.1.3–1
ANY THIN-WALLED ENCLOSED SECTION  	Perimeter $p = p_1 + p_2 + p_3 + p_4$ Area $A = p_1 t_1 + p_2 t_2 + p_3 t_3 + p_4 t_4$ Torsional stiffness constant $J = \frac{4A_1^{\,2}}{\Sigma p/_t} = \frac{4A_1^{\,2} t}{p}$ (for constant t) where $\Sigma p/_t = \frac{p_1}{t_1} + \frac{p_2}{t_2} + \frac{p_3}{t_3} + \frac{p_4}{t_4}$ and A_1 = area enclosed by mean perimeter
PARABOLIC SEGMENT  	Area $= \frac{4}{3} xy$ $\bar{y} = .60\,y$ $I_C = .5714\, xy^3$ $I_X = .0914\, xy^3$ $I_B = .3048\, xy^3$ $I_Y = .2667\, x^3 y$

4.1.1 Tabular Summary (Cont'd)

PARABOLIC QUADRANT  	Area $= \frac{2}{3} xy$ $\bar{x} = .375 x$ $\bar{y} = .60 y$
PARABOLIC FILLET  	Area $= \frac{1}{3} xy$ $\bar{x} = .75 x$ $\bar{y} = .30 y$
PARABOLIC ARC  	See Fig. 4.1.4–1
TRAPEZOID 	See Fig. 4.1.5–1 and Fig. 4.1.5–2

4.1.1 Tabular Summary (Cont'd)

90° ANGLE SECTION  	For approximate values of $\bar{y}$ and ρ_x see Figs. 4.1.6–1 and 4.1.6.–2. Accurate values for certain cases appear in C.S. Drawings and Gen/TD/33 and 34. See R.Ae.S. Data Sheet 01.00.04 for Lipped Angle.
CHANNEL SECTION  	For approximate values of $\bar{y}$ and ρ_x see Figs. 4.1.7.–1, 2 and 3. Accurate values for certain cases appear in C.S. Drawings and Gen/TD/34. See R.Ae.S. Data Sheet 01.00.02 for Lipped Channel.
ANY STRINGER – SKIN COMBINATION SUCH AS: 	See Section 4.1.8.
INCLINED RECTANGLE  	$A = Ht$ (Area A) $I_X = \frac{1}{12} An^2$ $I_Y = \frac{1}{12} Am^2$ $I_B = \frac{1}{3} An^2$ $I_{XY} = \frac{1}{12} Amn$ Formulas are approximate and assume that ratio $\frac{t}{H}$ is small.



When h and c are given: Establish ratio h/c on the base line and read off Curve (1) for R/c, Curve (2) for $\frac{ARC}{c}$ and Curve(3) for $\frac{AREA}{hc}$.

When R and c are given: Establish ratio R/c on Curve (1) and read down to base line for h/c.

FIG. 4.1.2

SECTION PROPERTIES OF CIRCULAR SEGMENT

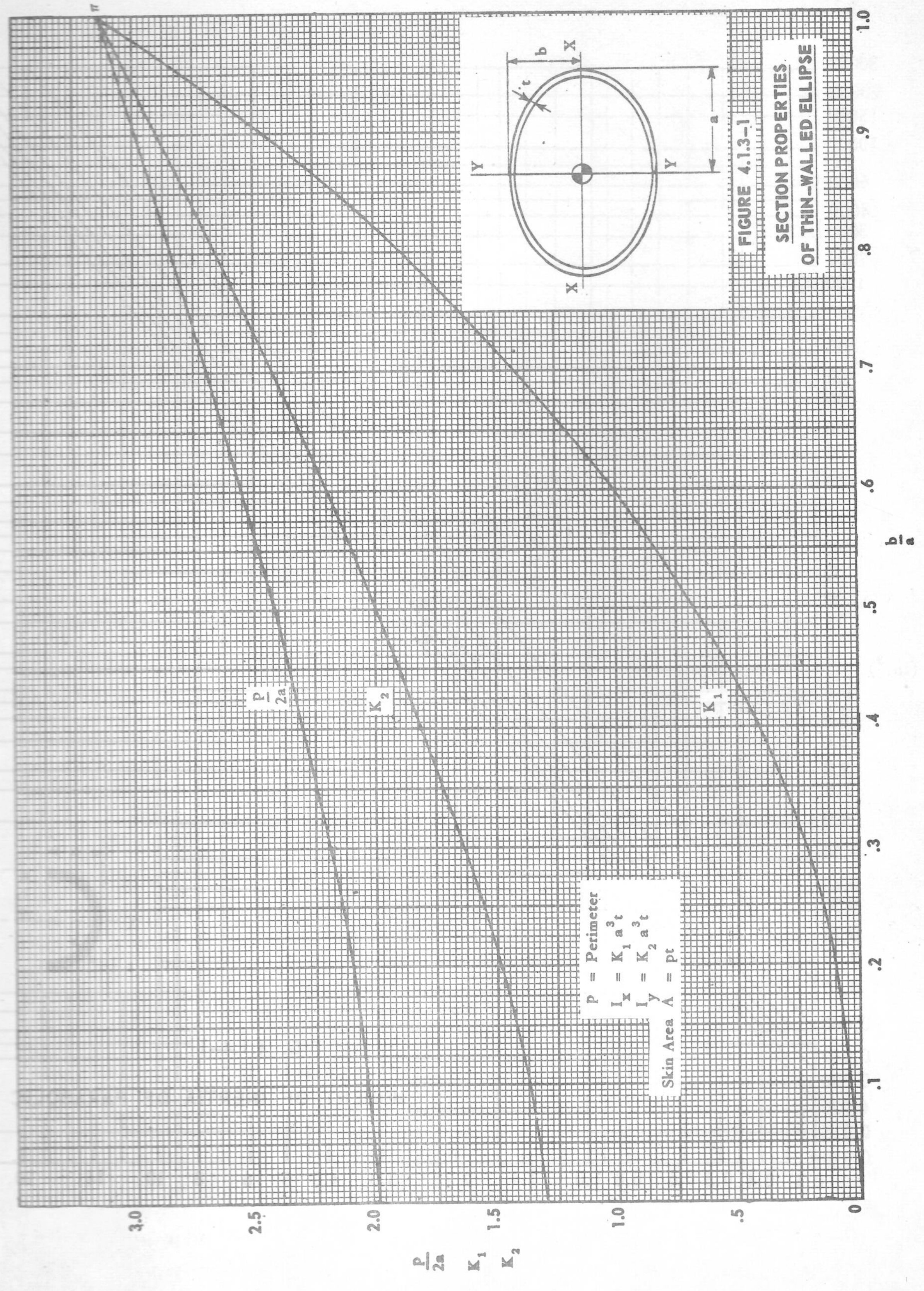


FIGURE 4.1.3-1

SECTION PROPERTIES OF THIN-WALLED ELLIPSE

P = Perimeter

$I_x = K_1 a^3 t$

$I_y = K_2 a^3 t$

Skin Area $A = pt$

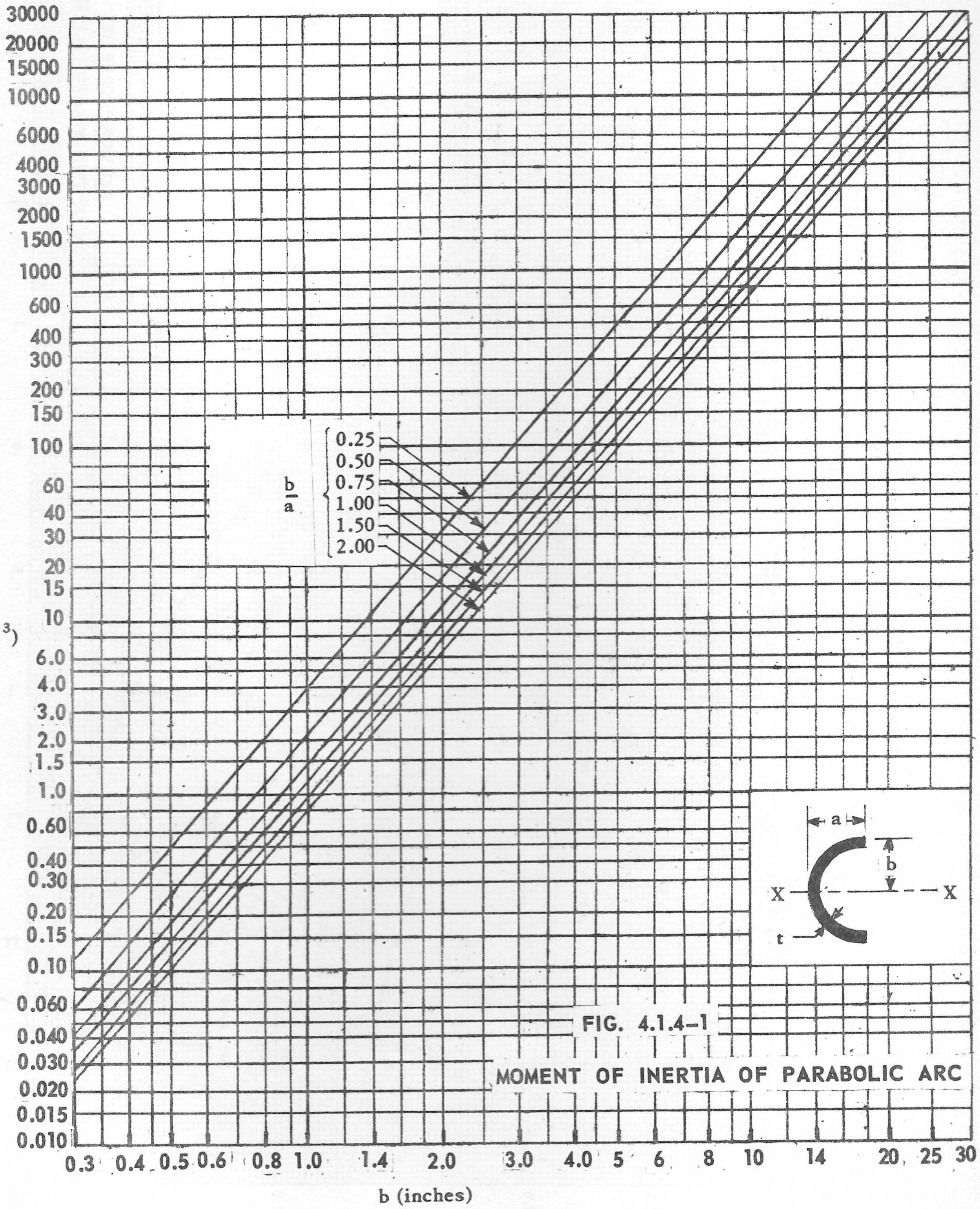


FIG. 4.1.4-1

MOMENT OF INERTIA OF PARABOLIC ARC

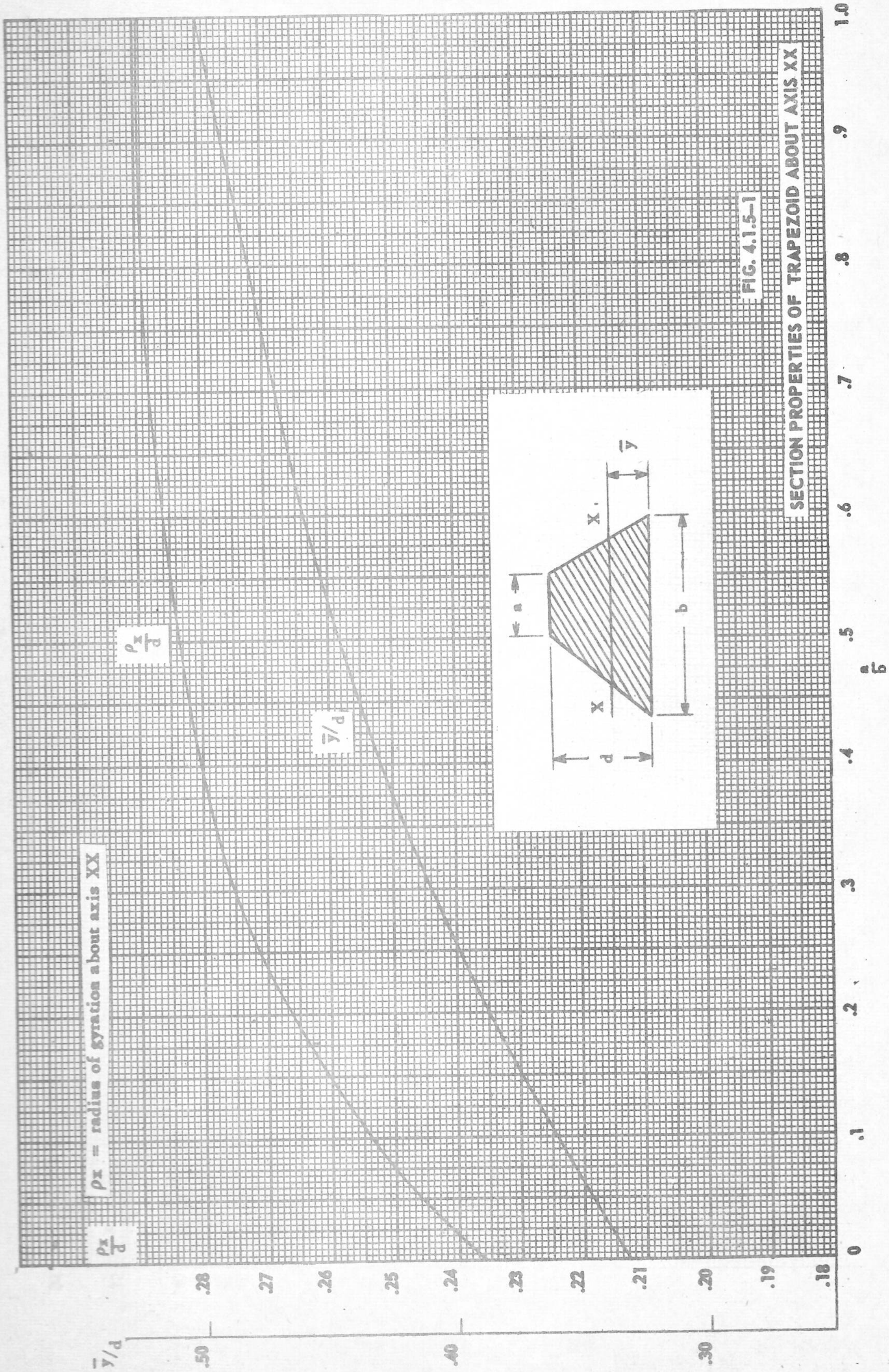


FIG. 4.1.5–1

SECTION PROPERTIES OF TRAPEZOID ABOUT AXIS XX

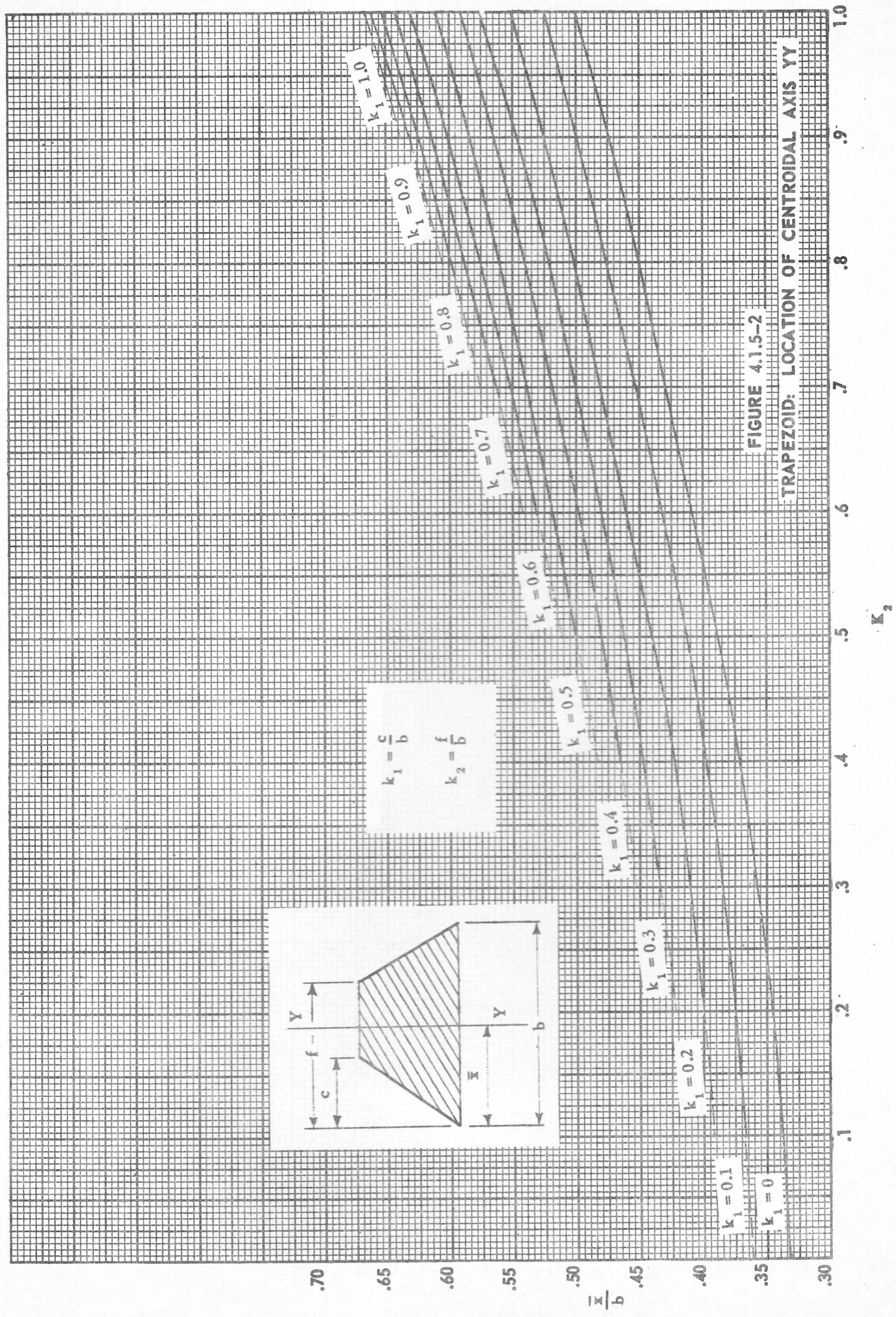


FIGURE 4.1.5–2

TRAPEZOID: LOCATION OF CENTROIDAL AXIS YY

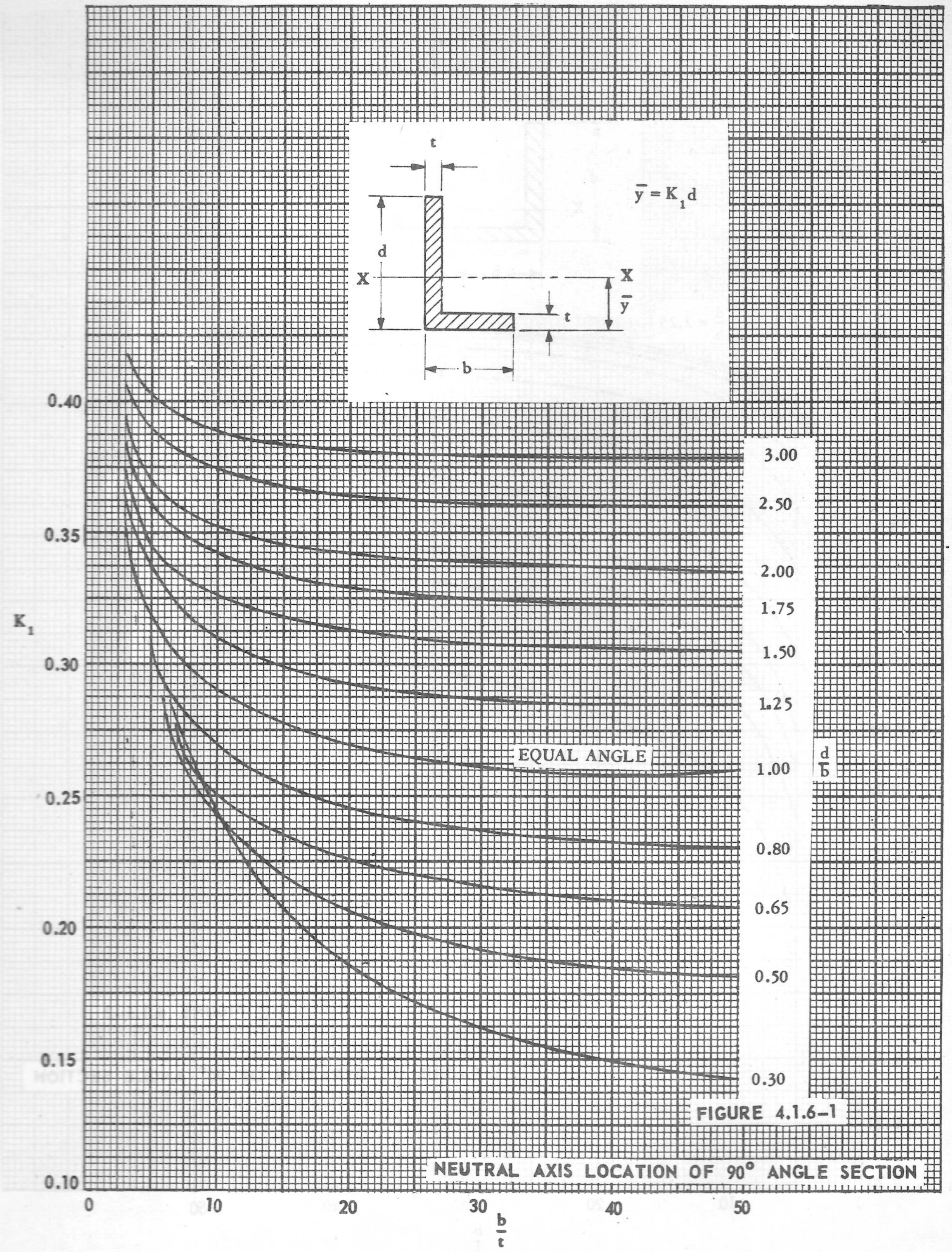


FIGURE 4.1.6–1

NEUTRAL AXIS LOCATION OF 90° ANGLE SECTION

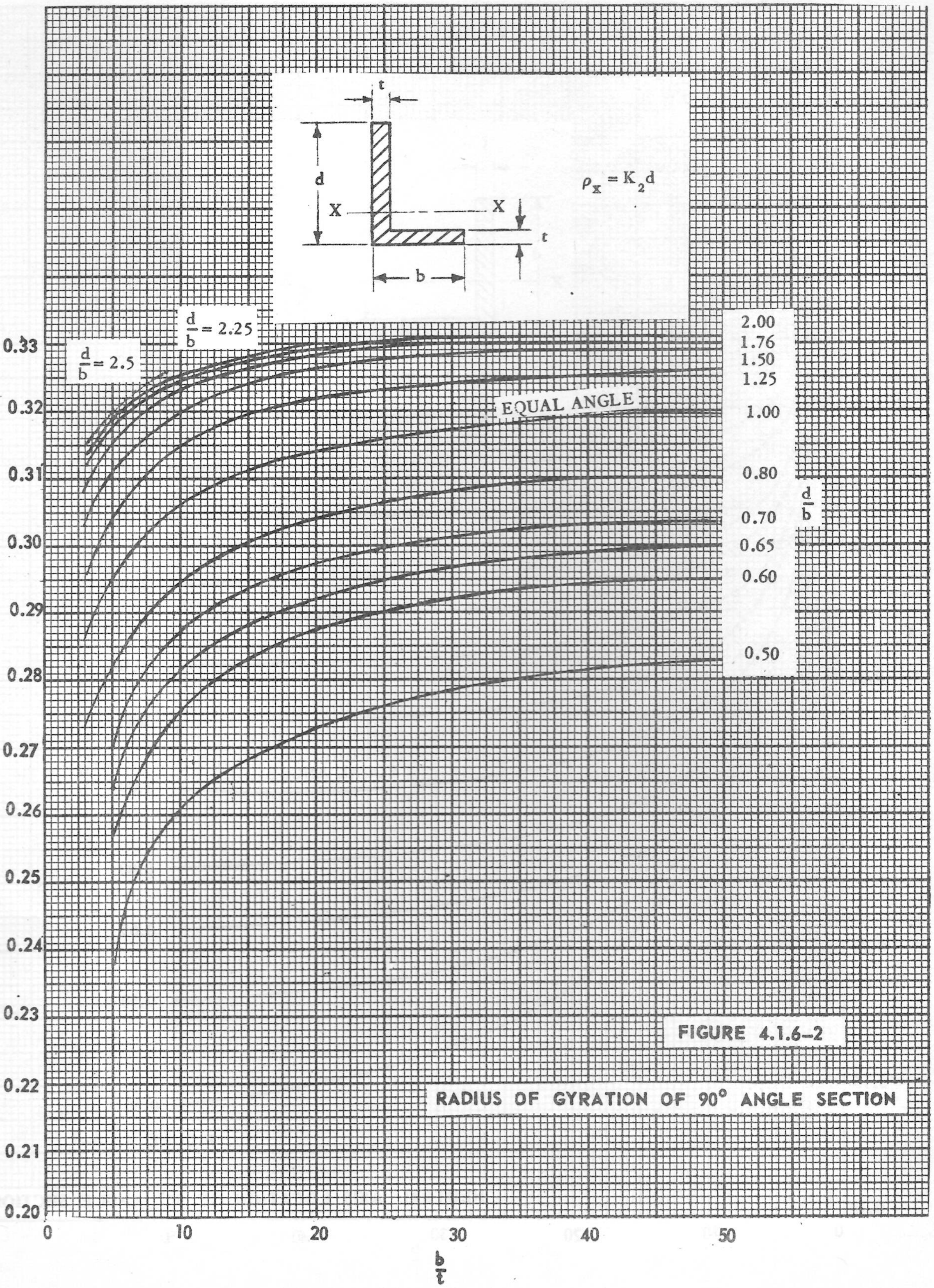


FIGURE 4.1.6–2

RADIUS OF GYRATION OF 90° ANGLE SECTION

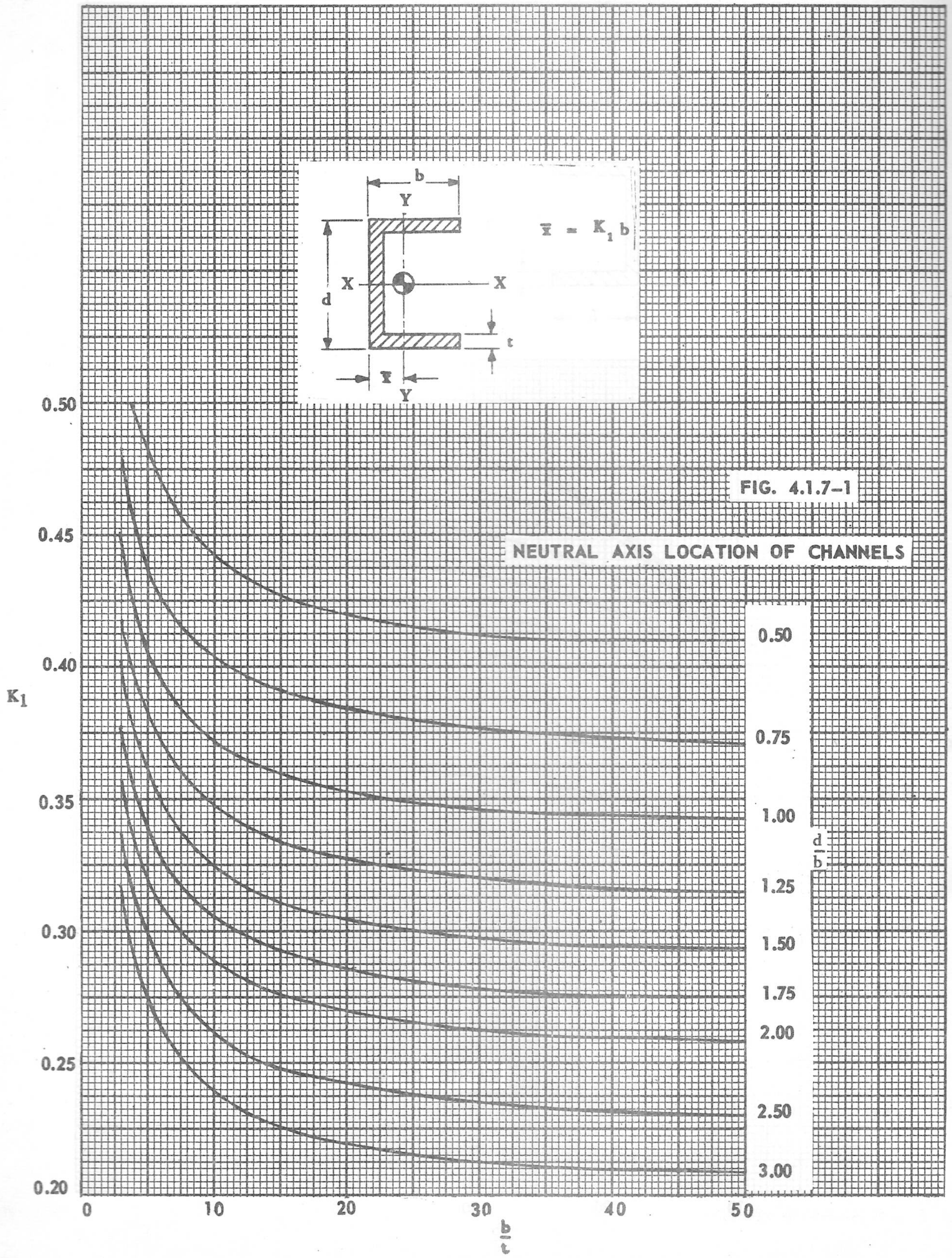


FIG. 4.1.7-1

NEUTRAL AXIS LOCATION OF CHANNELS

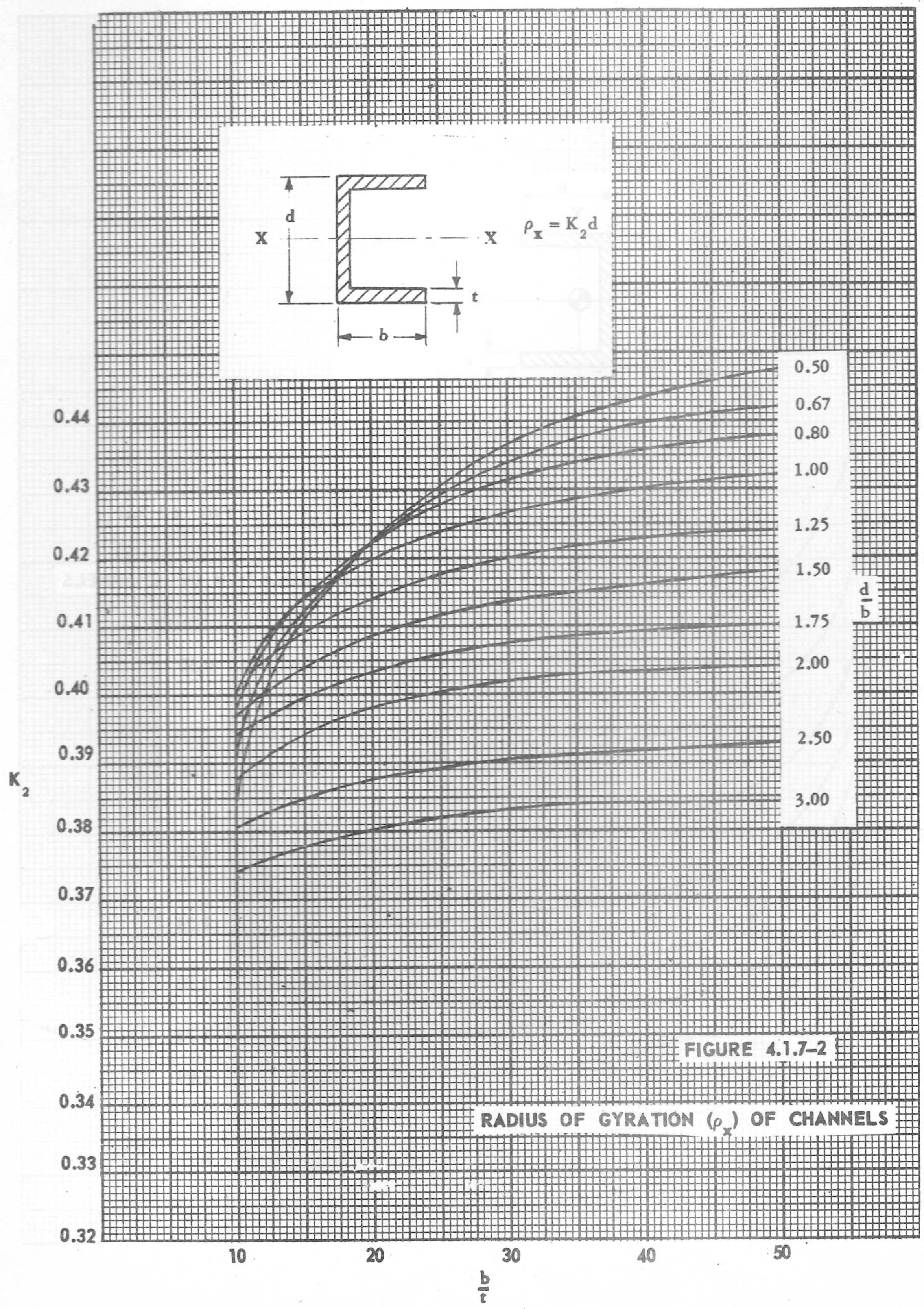


FIGURE 4.1.7–2

RADIUS OF GYRATION (ρ_x) OF CHANNELS

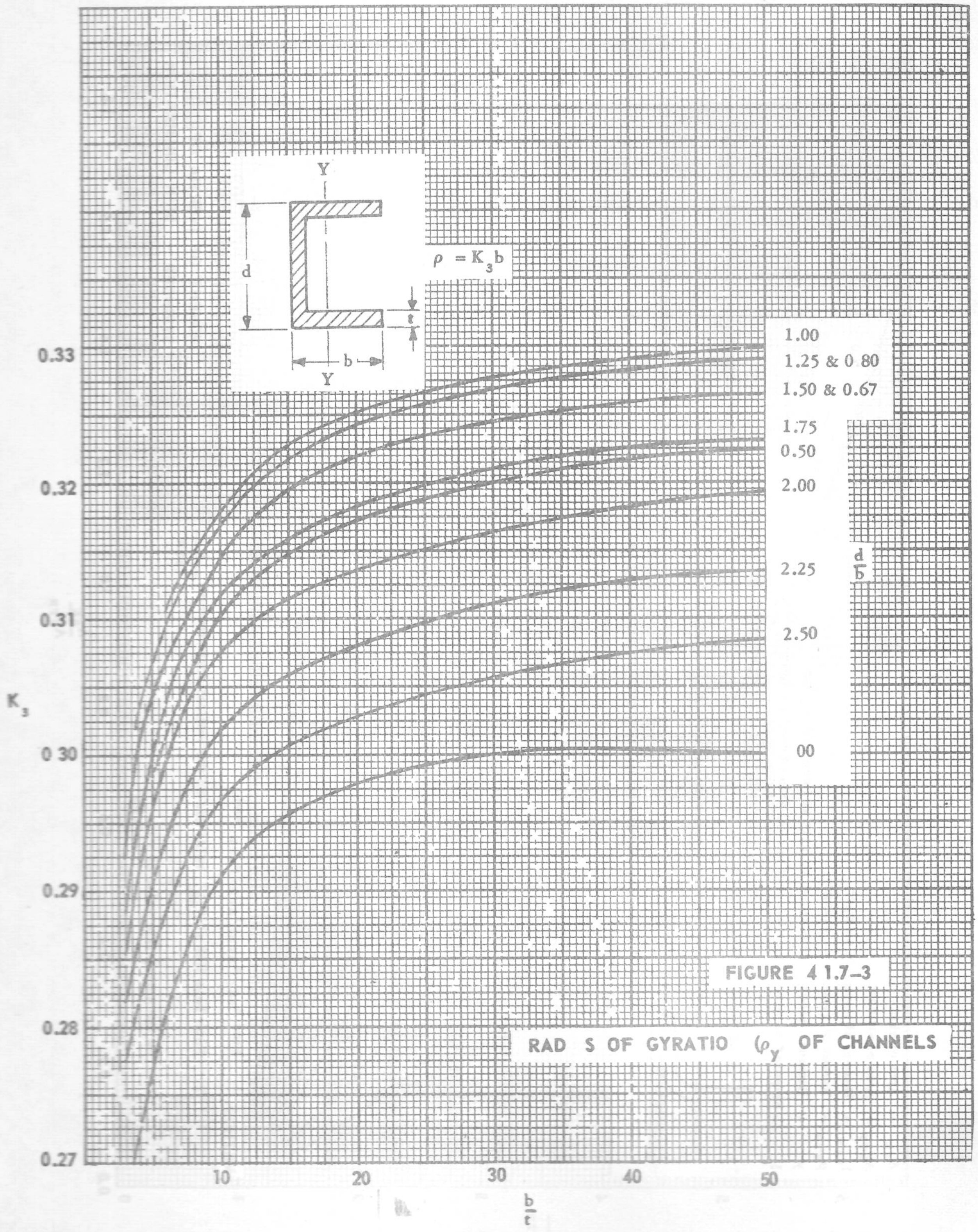
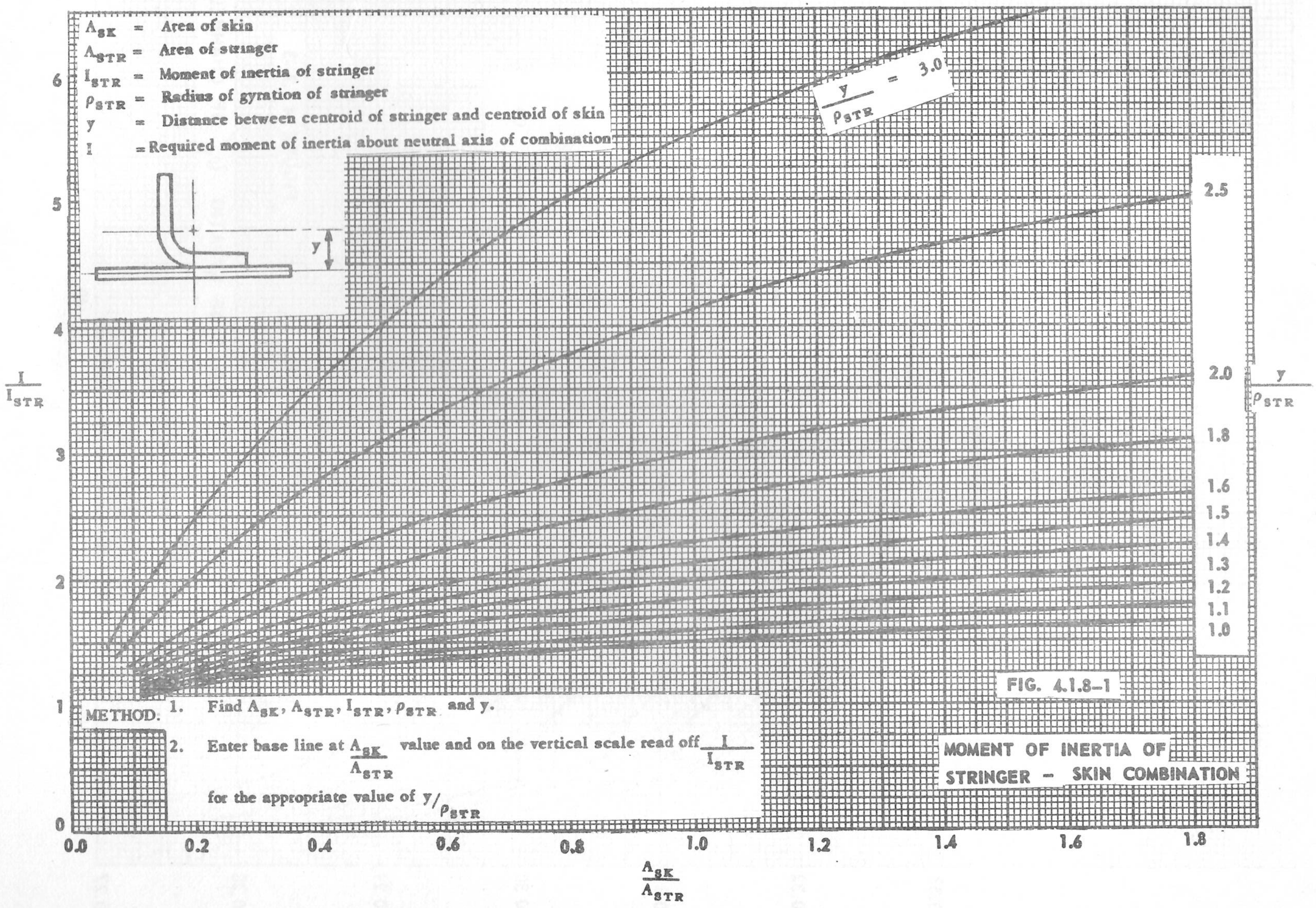


FIGURE 4 1.7–3

RAD S OF GYRATIO (ρ_y OF CHANNELS



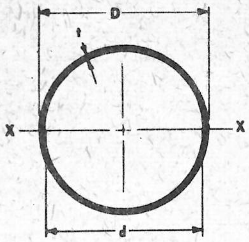
A_{SK} = Area of skin
A_{STR} = Area of stringer
I_{STR} = Moment of inertia of stringer
ρ_{STR} = Radius of gyration of stringer
y = Distance between centroid of stringer and centroid of skin
I = Required moment of inertia about neutral axis of combination

METHOD:
1. Find A_{SK}, A_{STR}, I_{STR}, ρ_{STR} and y.
2. Enter base line at $\frac{A_{SK}}{A_{STR}}$ value and on the vertical scale read off $\frac{I}{I_{STR}}$ for the appropriate value of y/ρ_{STR}

FIG. 4.1.8-1

MOMENT OF INERTIA OF STRINGER – SKIN COMBINATION

4.2.1 Round Tubes



$$A = \frac{\pi}{4}(D^2 - d^2)$$

$$I_x = \frac{\pi}{64}(D^4 - d^4)$$

$$\rho_x = \frac{1}{4}\sqrt{D^2 + d^2}$$

$$Z_x = \frac{\pi}{32}\frac{(D^4 - d^4)}{D}$$

D (in.)	t (in.)	A (in.^{2})	ρ_x (in.)	I_x (in.^{4})	Z_x (in.^{3})	D/t	WEIGHT lb/100 in.	
							Steel	Dural
3/16	.028	.01403	.05726	.000046	.000491	6.70	.40	.14
	.035	.01677	.05531	.000051	.000547	5.36	.48	.17
1/4	.022	.01576	.08100	.000103	.000825	11.38	.45	.16
	.028	.01953	.07911	.000122	.000978	8.93	.55	.20
	.035	.02364	.07701	.000140	.001122	7.14	.67	.24
	.049	.03094	.07314	.000166	.001324	5.10	.88	.31
	.058	.03498	.07091	.000176	.001407	4.31	.99	.35
	.065	.03778	.06933	.000182	.001453	3.85	1.07	.38
5/16	.022	.02008	.1030	.000213	.001363	14.20	.57	.20
	.028	.02503	.1011	.000256	.001636	11.16	.71	.25
	.035	.03051	.09889	.000298	.001910	8.93	.86	.31
	.049	.04056	.09476	.000364	.002331	6.38	1.15	.41
	.056	.04637	.09228	.000395	.002527	5.39	1.31	.47
	.065	.05054	.09048	.000414	.002648	4.81	1.43	.51
	.095	.06491	.08392	.000457	.002925	3.29	1.84	.66
3/8	.028	.03053	.1231	.000462	.002466	13.39	.86	.31
	.035	.03738	.1208	.000546	.002912	10.72	1.06	.38
	.049	.05018	.1166	.000682	.003636	7.65	1.43	.51
	.058	.05776	.1139	.000750	.003999	6.47	1.64	.58
	.065	.06330	.1120	.000794	.004234	5.77	1.79	.64
	.083	.07614	.1073	.000877	.004678	4.52	2.16	.77
	.095	.08357	.1043	.000913	.004870	3.95	2.37	.84
7/16	.028	.03602	.1451	.000759	.003468	15.63	1.02	.36
	.035	.04426	.1428	.000903	.004128	12.50	1.25	.45
	.049	.05981	.1384	.001146	.005240	8.93	1.69	.60
	.058	.06915	.1357	.001274	.005824	7.54	1.96	.70
	.065	.07607	.1337	.001360	.006215	6.73	2.16	.77
	.083	.09244	.1287	.001532	.007002	5.27	2.62	.93
	.095	.1022	.1257	.001614	.007379	4.61	2.90	1.03
1/2	.022	.03304	.1692	.000946	.003782	22.73	.94	.33
	.028	.04152	.1672	.001160	.004641	17.86	1.18	.42
	.035	.05113	.1649	.001390	.005559	14.28	1.45	.52
	.049	.06943	.1604	.001786	.007144	10.20	1.96	.70
	.058	.08054	.1576	.002001	.008002	8.62	2.28	.81
	.065	.08883	.1555	.002148	.008592	7.69	2.52	.90
	.083	.1087	.1503	.002457	.009828	6.02	3.08	1.10
	.095	.1209	.1471	.002615	.01046	5.26	3.42	1.22
	.120	.1433	.1409	.002844	.01137	4.17	4.06	1.45
9/16	.028	.04702	.1892	.001684	.005986	20.09	1.33	.48
	.035	.05800	.1869	.002026	.007205	16.07	1.64	.59
	.049	.07905	.1824	.002629	.009348	11.48	2.24	.80
	.058	.09193	.1795	.002963	.01054	9.70	2.60	.93
	.120	.1668	.1621	.004383	.01558	4.69	4.73	1.69
5/8	.028	.05252	.2113	.002345	.007503	22.32	1.49	.53
	.035	.06487	.2090	.002833	.009065	17.85	1.84	.66
	.049	.08867	.2044	.003704	.01185	12.77	2.51	.90
	.058	.1033	.2016	.004195	.01342	10.79	2.93	1.05
	.065	.1144	.1993	.004543	.01454	9.62	3.24	1.16
	.083	.1413	.1939	.005311	.01700	7.53	4.00	1.43
	.095	.1582	.1904	.005732	.01834	6.58	4.48	1.60
	.120	.1904	.1835	.006412	.02052	5.21	5.30	1.92
11/16	.028	.05801	.2334	.003160	.009192	24.55	1.64	.59
	.035	.07175	.2310	.003829	.01114	19.64	2.03	.73
	.049	.09829	.2264	.005038	.01466	14.03	2.78	.99
	.058	.1147	.2235	.005730	.01667	11.85	3.25	1.16
3/4	.028	.06351	.2555	.004145	.01105	26.80	1.80	.65
	.035	.07862	.2531	.005036	.01343	21.42	2.23	.80
	.049	.1079	.2485	.006661	.01776	15.30	3.06	1.09
	.058	.1261	.2455	.007601	.02027	12.94	3.57	1.28
	.065	.1399	.2433	.008278	.02208	11.53	3.96	1.42
	.083	.1739	.2376	.009820	.02619	9.04	4.93	1.76
	.095	.1955	.2340	.01070	.02854	7.89	5.54	1.98
	.120	.2375	.2267	.01221	.03256	6.25	6.73	2.40
	.156	.2911	.2171	.01372	.03660	4.81	8.25	2.94
7/8	.028	.07451	.2996	.006689	.01529	31.23	2.11	.76
	.035	.09236	.2973	.008161	.01865	25.00	2.62	.94
	.049	.1272	.2925	.01088	.02487	17.85	3.60	1.29
	.058	.1489	.2896	.01248	.02853	15.10	4.22	1.51
	.065	.1654	.2865	.01365	.03121	13.47	4.66	1.68
	.083	.2065	.2815	.01637	.03742	10.54	5.85	2.09
	.095	.2328	.2778	.01797	.04107	9.21	6.59	2.35
	.120	.2846	.2703	.02079	.04753	7.29	8.06	2.88
	.156	.3524	.2601	.02384	.05450	5.61	9.98	3.56
1	.028	.08550	.3438	.01011	.02021	35.71	2.42	.86
	.035	.1061	.3414	.01237	.02474	28.56	3.01	1.07
	.049	.1464	.3367	.01659	.03319	20.40	4.15	1.48
	.058	.1716	.3337	.01911	.03822	17.25	4.86	1.74
	.065	.1909	.3314	.02097	.04193	15.38	5.41	1.93
	.083	.2391	.3255	.02534	.05068	12.05	6.77	2.42
	.095	.2701	.3217	.02796	.05591	10.53	7.65	2.73
	.120	.3318	.3140	.03271	.06542	8.33	9.40	3.35
	.156	.4142	.3034	.03809	.07618	6.41	11.72	4.18
	.187	.4776	.2949	.04155	.08310	5.35	13.53	4.83
1-1/8	.028	.09640	.3880	.01453	.02582	40.18	2.73	.98
	.035	.1198	.3856	.01782	.03168	32.10	3.40	1.21
	.049	.1656	.3808	.02402	.04270	22.95	4.68	1.68
	.058	.1944	.3780	.02775	.04933	19.40	5.51	1.97
	.065	.2165	.3755	.03052	.05425	17.30	6.14	2.20
	.083	.2717	.3696	.03711	.06597	13.55	7.70	2.75
	.095	.3074	.3657	.04111	.07309	11.84	8.71	3.11
	.120	.3789	.3578	.04852	.08625	9.38	10.73	3.83

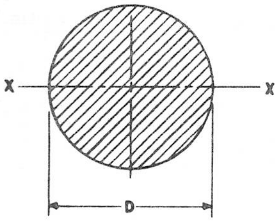
4.2.1 Round Tubes (Cont'd)

D (in.)	t (in.)	A (in.^{2})	ρ_X (in.)	I_X (in.^{4})	Z_X (in.^{3})	D/t	WEIGHT lb/100 in.	
							Steel	Dural
1-1/4	.028	.1075	.4321	.02007	.03212	44.64	3.05	1.09
	.035	.1336	.4297	.02467	.03948	35.70	3.78	1.35
	.049	.1849	.1250	.03339	.05342	25.50	5.23	1.87
	.058	.2172	.4219	.03867	.06187	21.55	6.15	2.20
	.065	.2420	.4196	.04260	.06816	19.22	6.86	2.45
	.083	.3043	.4136	.05206	.08330	15.06	8.62	3.08
	.095	.3447	.4097	.05787	.09259	13.16	9.77	3.49
	.120	.4260	.4018	.06876	.1100	10.42	12.07	4.31
	.156	.5362	.3907	.08184	.1310	8.01	15.19	5.42
1-3/8	.028	.1185	.4764	.02689	.03911	49.11	3.36	1.20
	.035	.1473	.4739	.03309	.04814	39.25	4.17	1.49
	.049	.2041	.4691	.04492	.06534	28.05	5.78	2.07
	.058	.2400	.4661	.05213	.07583	23.70	6.80	2.43
	.065	.2675	.4636	.05753	.08367	21.15	7.58	2.70
	.083	.3369	.4577	.07059	.1027	16.57	9.54	3.41
	.095	.3820	.4538	.07867	.1144	14.47	10.82	3.86
	.120	.4731	.4457	.09400	.1367	11.46	13.40	4.78
	.156	.5974	.4345	.1129	.1642	8.81	16.92	6.04
	.187	.6979	.4252	.1262	.1835	7.35	19.77	7.07
1-1/2	.028	.1295	.5205	.03508	.04678	53.57	3.67	1.31
	.035	.1611	.5181	.04324	.05765	42.80	4.56	1.63
	.049	.2234	.5132	.05885	.07847	30.60	6.32	2.26
	.058	.2628	.5102	.06841	.09121	25.85	7.45	2.66
	.065	.2930	.5079	.07558	.1008	23.05	8.30	2.97
	.083	.3695	.5018	.09305	.1241	18.08	10.47	3.74
	.095	.4193	.4979	.1039	.1386	15.79	11.98	4.24
	.120	.5202	.4897	.1248	.1664	12.50	14.74	5.26
	.156	.6587	.4784	.1507	.2010	9.62	18.66	6.66
	.187	.7714	.4689	.1696	.2261	8.02	21.85	7.80
1-5/8	.028	.1405	.5647	.04480	.05514	58.04	3.98	1.42
	.035	.1748	.5622	.05528	.06803	46.40	4.95	1.77
	.049	.2426	.5575	.07540	.09279	33.15	6.87	2.46
	.058	.2855	.5544	.08776	.1080	28.00	8.09	2.89
	.065	.3186	.5520	.09707	.1195	25.00	9.05	3.23
	.083	.4021	.5459	.1198	.1475	19.58	11.40	4.06
	.095	.4566	.5420	.1341	.1651	17.11	12.94	4.62
	.120	.5674	.5338	.1617	.1990	13.54	16.07	5.74
1-3/4	.028	.1515	.6089	.05616	.06418	62.50	4.29	1.53
	.035	.1885	.6065	.06936	.07927	50.00	5.32	1.91
	.049	.2618	.6017	.09478	.1083	35.70	7.42	2.65
	.058	.3083	.5986	.1105	.1262	30.20	8.73	3.12
	.065	.3441	.5962	.1223	.1398	26.90	9.75	3.48
	.083	.4347	.5901	.1514	.1730	21.10	12.32	4.40
	.095	.4939	.5861	.1697	.1939	18.42	13.99	4.99
	.120	.6145	.5778	.2052	.2345	14.58	17.41	6.21
1-7/8	.028	.1625	.6531	.06930	.07392	66.96	4.60	1.64
	.035	.2023	.6507	.08565	.09136	53.60	5.73	2.04
	.049	.2811	.6458	.1172	.1250	38.25	7.95	2.84
	.058	.3311	.6427	.1368	.1459	32.30	9.38	3.35
	.065	.3696	.6404	.1516	.1617	28.80	10.47	3.74
	.083	.4673	.6342	.1880	.2005	22.60	13.25	4.73
	.095	.5312	.6302	.2110	.2251	19.74	15.05	5.37
	.120	.6616	.6219	.2559	.2730	15.63	18.74	6.69
2	.028	.1735	.6973	.08434	.08434	71.43	4.91	1.75
	.035	.2161	.6948	.1043	.1043	57.14	6.12	2.18
	.049	.3003	.6900	.1430	.1430	40.80	8.50	3.04
	.058	.3539	.6869	.1670	.1670	34.45	10.03	3.58
	.065	.3951	.6845	.1851	.1851	30.75	11.19	4.00
	.083	.4999	.6783	.2300	.2301	24.10	14.16	5.06
	.095	.5685	.6744	.2586	.2586	21.05	16.11	5.76
	.109	.6476	.6697	.2904	.2904	18.35	18.35	6.55
2	.120	.7087	.6660	.3144	.3144	16.67	20.08	7.17
	.156	.9037	.6543	.3869	.3869	12.82	25.60	9.14
	.188	1.070	.6441	.4440	.4440	10.64	30.32	10.82
	.250	1.374	.6250	.5269	.5369	8.00	38.94	13.90
2-1/8	.035	.2298	.7390	.1255	.1181	60.71	6.51	2.32
	.049	.3196	.7342	.1723	.1621	43.37	9.05	3.23
	.058	.3766	.7311	.2013	.1895	36.64	10.67	3.81
	.065	.4207	.7287	.2234	.2102	32.69	11.92	4.25
	.083	.5325	.7226	.2780	.2616	25.60	15.08	5.38
	.095	.6059	.7185	.3128	.2944	22.37	17.16	6.13
	.120	.7559	.7101	.3812	.3588	17.71	21.41	7.64
	.156	.9650	.6983	.4706	.4429	13.62	27.34	9.76
2-1/4	.035	.2436	.7832	.1494	.1328	64.29	6.90	2.46
	.049	.3388	.7783	.2052	.1824	45.90	9.59	3.43
	.058	.3994	.7753	.2401	.2134	38.80	11.30	4.05
	.065	.4462	.7728	.2665	.2369	34.60	12.64	4.52
	.083	.5651	.7667	.3322	.2953	27.15	16.01	5.72
	.095	.6432	.7626	.3741	.3325	23.70	18.22	6.51
	.120	.8030	.7543	.4568	.4061	18.75	22.75	8.12
	.156	1.026	.7424	.5656	.5028	14.42	29.07	10.38
	.188	1.218	.7320	.6526	.5801	11.97	34.50	12.31
	.250	1.571	.7216	.7977	.7090	9.00	44.50	15.88
2-5/16	.250	1.620	.7345	.8740	.7559	9.25	45.89	16.38
2-3/8	.049	.3581	.8225	.2423	.2040	48.47	10.14	3.62
	.058	.4222	.8194	.2835	.2387	40.95	11.96	4.27
	.065	.4717	.8170	.3149	.2651	36.54	13.36	4.77
	.083	.5976	.8109	.3930	.3309	28.61	16.93	6.04
	.095	.6805	.8068	.4429	.3730	25.00	19.28	6.88
	.120	.8501	.7984	.5419	.4563	19.79	24.08	8.59
	.156	1.088	.7865	.6727	.5664	15.22	30.81	10.99
	.188	1.292	.7761	.7780	.6551	12.63	36.59	13.06
	.250	1.669	.7565	.9551	.8043	9.50	47.28	16.87
	.375	2.356	.7194	1.220	1.027	6.33	66.75	23.82
2-1/2	.049	.3773	.8667	.2834	.2267	51.00	10.68	3.82
	.058	.4450	.8635	.3318	.2655	43.10	12.60	4.50
	.065	.4972	.8613	.3688	.2950	38.45	14.09	5.03
	.083	.6302	.8550	.4607	.3686	30.10	17.85	6.38
	.095	.7178	.8509	.5197	.4158	26.30	20.34	7.27
	.120	.8972	.8425	.6369	.5095	20.83	25.42	9.07
	.156	1.149	.8306	.7925	.6340	16.03	32.54	11.61
	.188	1.366	.8201	.9184	.7347	13.30	38.68	13.81
	.250	1.767	.8004	1.132	.9057	10.00	50.06	17.87
2-3/4	.049	.4158	.9551	.3793	.2759	56.10	11.78	4.20
	.058	.4905	.9521	.4446	.3233	47.40	13.90	4.96
	.065	.5483	.9496	.4944	.3596	42.30	15.50	5.55
	.083	.6954	.9434	.6189	.4501	33.15	19.70	7.04
	.095	.7924	.9393	.6991	.5084	28.95	22.48	8.03
	.120	.9915	.9308	.8590	.6247	22.92	28.09	10.02
	.156	1.271	.9188	1.073	.7805	17.63	36.02	12.85
	.188	1.513	.9082	1.248	.9078	14.63	42.87	15.30
	.250	1.964	.8883	1.549	1.127	11.00	55.63	19.85
3	.049	.4543	1.044	.4946	.3298	61.22	12.87	4.59
	.058	.5361	1.040	.5802	.3868	51.70	15.18	5.42
	.065	.5993	1.038	.6457	.4305	46.20	16.95	6.06
	.083	.7606	1.032	.8097	.5398	36.15	21.55	7.70
	.095	.8670	1.028	.9156	.6104	31.58	24.56	8.78
	.120	1.086	1.019	1.128	.7518	25.00	30.76	11.00
	.156	1.394	1.007	1.413	.9423	19.23	39.49	14.09
	.188	1.661	.9964	1.649	1.099	15.96	47.05	16.79
	.250	2.160	.9763	2.059	1.372	12.00	61.19	21.84
	.375	3.093	.9375	2.718	1.812	8.00	87.61	31.27

4.2.1 Round Tubes (Cont'd)

D (in.)	t (in.)	A (in ^{2})	ρ_X (in.)	I_X (in.^{4})	Z_X (in.^{3})	$D/_t$	WEIGHT lb/100 in.	
							Steel	Dural
3-1/4	.049	.4928	1.132	.6313	.3885	66.33	13.96	4.98
	.058	.5816	1.129	.7410	.4560	56.10	16.47	5.89
	.065	.6504	1.126	.8251	.5077	50.00	18.40	6.58
	.083	.8258	1.120	1.036	.6376	39.15	23.38	8.35
	.095	.9416	1.116	1.173	.7217	34.20	26.66	9.52
	.120	1.180	1.107	1.447	.8906	27.10	33.43	11.95
	.156	1.516	1.095	1.819	1.119	20.83	42.96	15.33
	.188	1.808	1.085	2.127	1.309	17.29	51.23	18.28
	.250	2.356	1.064	2.669	1.643	13.00	66.75	23.82
3-1/2	.065	.7014	1.215	1.035	.5914	53.80	19.85	7.09
	.083	.8910	1.208	1.301	.7435	42.20	25.20	9.01
	.095	1.016	1.204	1.474	.8422	36.85	28.70	10.25
	.120	1.274	1.196	1.822	1.041	29.15	36.00	12.89
	.156	1.639	1.184	2.296	1.312	22.44	46.43	16.57
	.188	1.956	1.173	2.691	1.538	18.62	55.42	19.78
	.250	2.553	1.152	3.390	1.937	14.00	72.31	25.81
3-3/4	.065	.7525	1.303	1.278	.6814	57.60	21.30	7.60
	.083	.9562	1.297	1.608	.8576	45.20	27.06	9.67
	.095	1.091	1.293	1.823	.9722	39.50	30.84	11.04
	.120	1.368	1.284	2.256	1.204	31.25	38.70	13.82
	.156	1.761	1.272	2.849	1.520	24.04	49.90	17.81
	.188	2.104	1.261	3.346	1.784	19.95	59.60	21.27
	.219	2.429	1.251	3.801	2.027	17.12	68.82	24.56
	.250	2.749	1.241	4.231	2.256	15.00	77.88	27.79
4	.065	.8035	1.392	1.556	.7779	61.50	22.75	8.12
	.083	1.021	1.385	1.960	.9799	48.20	28.95	10.32
	.095	1.166	1.381	2.223	1.111	42.10	32.95	11.78
	.120	1.463	1.372	2.755	1.378	33.33	41.40	14.80
	.156	1.884	1.360	3.485	1.743	25.64	53.37	19.05
	.188	2.251	1.349	4.099	2.050	21.28	63.78	22.76
	.219	2.601	1.339	4.664	2.332	18.26	73.70	26.30
	.250	2.945	1.329	5.200	2.600	16.00	83.44	29.78
	.313	3.626	1.308	6.205	3.102	6.39	102.71	36.65
4-1/4	.065	.8546	1.480	1.871	.8806	65.38	24.21	8.64
	.083	-1.087	1.474	2.359	1.110	51.20	30.78	10.99
4-1/4	.095	1.240	1.469	2.677	1.260	44.74	35.13	12.54
	120	1.557	1.461	3.322	1.564	35.42	44.11	15.74
	134	1.733	1.456	3.673	1.741	31.75	49.10	17.55
	156	2.006	1.448	4 210	1.981	27.24	56.84	20.29
	.188	2.399	1.438	4.959	2.334	22.60	67.97	24.25
	.219	2.773	1.427	5.650	2.659	19.40	78.57	28.04
	.250	3.142	1.417	6.308	2.968	17.00	89.00	31.76
	.313	3.871	1.396	7.548	3.552	13.58	109.67	39.14
4-1/2	.065	.9056	1.568	2.227	.9898	69.23	25.66	9.16
	083	1.152	1.562	2.810	1.249	54.22	32.63	11.64
	.095	1.315	1.558	3.190	1.418	47.37	37.24	13.29
	.120	1.651	1.549	3.963	1.761	37.50	46.78	16.69
	.156	2.129	1.537	5.028	2.235	28.80	60.40	21.55
	.188	2.547	1.526	5.930	2.636	23.94	72.15	25.75
	.219	2.945	1.516	6.765	3.007	20.55	83.44	29.78
	.250	3.338	1.505	7.562	3.361	18.00	94.56	33.75
	.313	4.117	1.484	9.073	4.032	14.38	116.64	41.62
	.375	4.860	1.464	10.42	4.632	12.00	137.67	49.13
4-3/4	.065	.9567	1.657	2.025	1.105	73.08	27.10	9.67
	.083	1.217	1.650	3.314	1.396	57.23	34.48	12.30
	.095	1.389	1.646	3.765	1.585	50.00	39.36	14.05
	.120	1.745	1.638	4.680	1.971	39.58	49.45	17.65
	.156	2.251	1.625	5.946	2.504	30.45	63.78	22.76
	.188	2.694	1.614	7.021	2.956	25.25	76.25	27.20
	.219	3.117	1.604	8.019	3.376	21.69	88.32	31.52
	.250	3.534	1.593	8.974	3.778	19.00	100.13	35.73
	.313	4.363	1.573	10.79	4.543	15.18	123.60	44.11
	.375	5.154	1.552	12.42	5.230	12.6	146.02	52.11
5	.083	1.282	1.739	3.876	1.550	60.24	36.32	12.96
	.095	1.464	1.734	4.404	1.762	52.63	41.47	14.80
	.120	1.840	1.726	5.480	2.192	41.67	52.12	18.60
	.156	2.374	1.714	6.970	2.788	32.05	67.26	24.00
	.188	2.842	1.703	8.239	3.295	26.60	80.52	28.73
	.219	3.289	1.692	9.418	3.767	22.83	93.19	33.26
	.250	3.731	1.682	10.55	4.220	20.00	105.69	37.72
	.313	4.609	1.661	12.7	5.085	15.97	130.07	46.60
	.375	5.449	1.640	14.66	5.866	13.33	154.36	55.09

4.2.2 Round Bar



$$A = \frac{\pi}{4} D^2$$

$$\rho_x = D/4$$

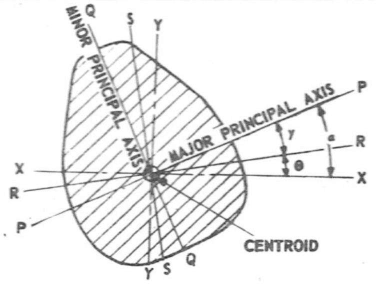
$$I_x = \frac{\pi}{64} D^4$$

$$Z_x = \frac{\pi}{32} D^3$$

D (in.)	A (in.^{2})	ρ_x (in.)	I_x (in.^{4})	Z_x (in.^{3})	WEIGHT lb/100 in.	
					Steel	Dural
1/8	.01227	.03125	.000012	.000192	.35	.12
1/4	.04909	.06250	.000192	.001534	1.39	47
3/8	.1104	.09375	.000971	.005177	3.13	1.07
1/2	.1963	.1250	.003068	.01227	5.56	1.90
5/8	.3068	.1562	.007490	.02397	8.69	2.96
3/4	.4418	.1875	.01553	.04142	12.52	4.27
7/8	.6013	.2188	.02877	.06577	17.04	5.81
1	.7854	.2500	.04909	.09817	22.25	7.59
1-1/4	1.227	.3125	.1198	.1917	34.77	11.86
1-1/2	1.767	.3750	.2485	.3313	50.06	17.07
1-3/4	2.405	.4375	.4604	.5262	68.14	23.24
2	3.142	.5000	.7854	.7854	89.00	30.35
2-1/4	3.976	.5625	1.258	1.118	112.64	38.41
2-1/2	4.909	.6250	1.917	1.534	139.06	47.42
2-3/4	5.940	.6875	2 807	2.042	168.27	57.38
3	7.069	.7500	3 976	2.651	200.25	68.29
3-1/2	9.621	.8750	7.366	4.209	272.57	92.95
4	12.57	1.000	12 57	6.283	356.00	121.40
4-1/2	15.90	1.125	20.13	8.946	450.57	153.64
5	19.63	1.250	30.68	12.27	556.26	189.68

4.3 Principal Axes

4.3.1 Tabular Summary of Formulas



Notation:

XX and YY = convenient rectangular axes through the centroid

PP = axis of greatest moment of inertia *(Major principal axis)*

QQ = axis of least moment of inertia *(Minor principal axis)*

RR and SS = any rectangular axes through the centroid

a = angle between major principal axis PP and axis XX

4.3.1 Tabular Summary of Formulas (Cont'd)

γ = angle between major principal axis PP and axis RR

θ = angle between axis RR and axis XX

I_X = moment of inertia about axis XX

I_Y = moment of inertia about axis YY

I_P = moment of inertia about axis PP

I_Q = moment of inertia about axis QQ

I_R = moment of inertia about axis RR

I_S = moment of inertia about axis SS

I_{RS} = product of inertia about axes RR and SS

I_{XY} = product of inertia about axes XX and YY

KNOWING	TO FIND	FORMULA
I_X and I_Y	I_P and I_Q	$I_P = \dfrac{I_X \cos^2 \alpha - I_Y \sin^2 \alpha}{\cos 2\alpha} = \dfrac{I_X + I_Y}{2} + \sqrt{\left(\dfrac{I_X - I_Y}{2}\right)^2 + I^2_{XY}}$ $I_Q = I_X + I_Y - I_P$ $\text{TAN } 2\alpha = \dfrac{-2 I_{XY}}{I_X - I_Y}$ *Note that I_{XY} must be determined with due regard to sign, as shown in the numerical example of 4.3.2.*
I_X and I_Y	I_R and I_S	$I_R = I_X \cos^2 \theta + I_Y \sin^2 \theta - I_{XY} \sin 2\theta$ $I_S = I_X + I_Y - I_R$ $I_{RS} = \frac{1}{2}(I_X - I_Y) \sin 2\theta + I_{XY} \sin 2\theta$
I_P and I_Q	I_R and I_S	$I_R = I_P \cos^2 \gamma + I_Q \sin^2 \gamma$ $I_S = I_P + I_Q - I_R$
I_P and I_Q	I_X and I_Y	$I_X = I_P \cos^2 \alpha + I_Q \sin^2 \alpha$ $I_Y = I_P + I_Q - I_X$

4.3.2 Numerical Example

To find I_P and I_Q for the unequal angle shown in Fig. 4.3.2–1 from the given data:–

$A = .2336$

$\bar{x} = .197$

$\bar{y} = .698$

$I_X = .0995$

$I_Y = .0178$

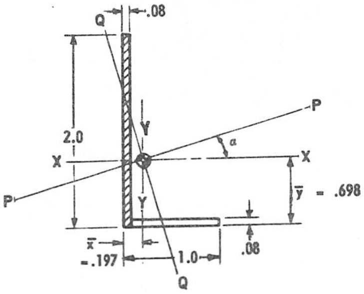


Fig. 4.3.2–1

Find I_{XY} as follows, with due regard to sign:–

ITEM	SHADED RECTANGLE	PLAIN RECTANGLE
Self-product of inertia	NIL	NIL
Area A	2 × .08 = .16	.92 × .08 = .0736
x	– .157	.343
y	.302	– .658
xy	– .0474	– .2255
Axy	– .0076	– .0166
	I_{XY} = – .0242	

$$\text{Tan } 2a = \frac{-2\,I_{XY}}{I_X - I_Y} = \frac{-(-.0484)}{.0817} = \underline{+0.592}$$

$2a = 30^\circ\ 38'$

$a = 15^\circ\ 19'$ $\qquad \sin a = .2642 : \quad \cos a = .9645$

$$I_P = \frac{I_X \cos^2 a - I_Y \sin^2 a}{\cos 2a} \qquad \sin^2 a = .0697 : \quad \cos^2 a = .931$$

$$= \frac{.0995 \times .931 - .0178 \times .0697}{.8604}$$

$$= \underline{0.1062}$$

$$I_Q = I_X + I_Y - I_P$$

$$= .0995 + .0178 - .1062$$

$$= \underline{0.0111}$$

4.3.2 Numerical Example (Cont'd)

Alternatively,

$$I_P = \frac{I_X + I_Y}{2} + \sqrt{\left(\frac{I_X - I_Y}{2}\right)^2 + I_{XY}^2}$$

$$= .0587 + \sqrt{.0411^2 + .0242^2} = \underline{.1063}$$

4.4 Moment of Inertia of a Series of Masses

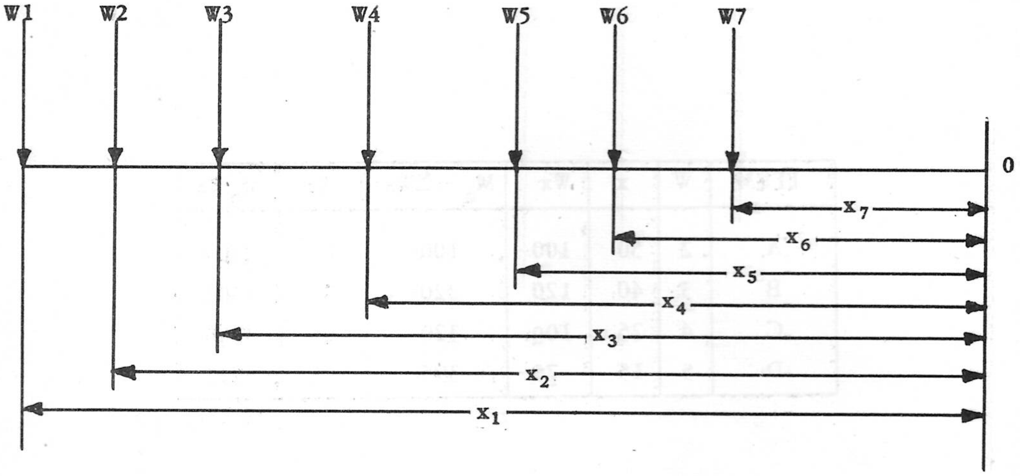


Fig. 4.4–1

Consider the system of masses shown in Fig. 4.4–1. The moment of inertia (second moment) about 0 of W_1 is $W_1x_1^2$ and of W_2 it is $W_2x_2^2$ and so on. The total moment of inertia of all the masses about 0 is:–

$$I_o = W_1x_1^2 + W_2x_2^2 + W_3x_3^2 + --- = \Sigma Wx^2$$

In practice, the amount of calculation is reduced by finding Wx for each item, then $M_o = \Sigma Wx$, that is, $W_1x_1 + W_2x_2 + W_3x_3 + ---$ and finally $M_o\Delta x$, where Δx is the increment of length between any two items, and $I_o = \Sigma M_o\Delta x$. This is because $M_o = \Sigma Wx$ is the first moment of the masses about 0. The area under the curve of M_o against $x = \Sigma M\Delta x$, the second moment.

EXAMPLE: Masses disposed as shown in Fig. 4.4–2.

4.4 Moment of Inertia of a Series of Masses (Cont'd)

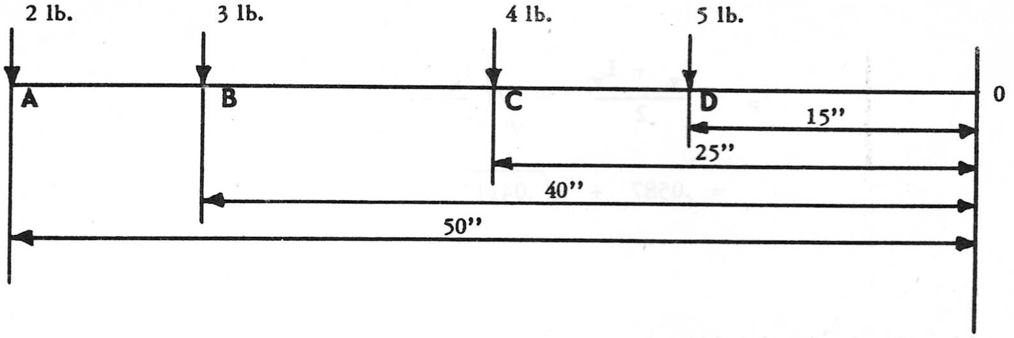


Fig. 4.4–2

ITEM	W	x	Wx	$M_o = \Sigma Wx$	Δx	$M_o \Delta x$
A	2	50	100	100	10	1000
B	3	40	120	220	15	3300
C	4	25	100	320	10	3200
D	5	15	75	395	15	5925

$I_o = 13425$

5 TORSION

TABLE OF CONTENTS

INTRODUCTION

Solid sections are dealt with in 5.1 and in 5.1.1 there is a summary of the results in relation to the few cross-sections for which a mathematical solution can be obtained.

A method based on the membrane analogy is given in 5.1.2 for obtaining the shear stresses in any solid section.

In 5.2 formulas for hollow sections, thin-walled and otherwise, are summarised, whilst 5.3 deals with thin-walled open sections. After giving formulas for the shear stress and torsion constant of many common sections, a method of calculating the warp of an open section is shown.

It is necessary to find the warp in order that the warping (or torsion bending) constant, and hence the torsional instability, can be calculated, as explained in 5.3.

5.4 deals with thin multi-walled sections having constant spanwise dimensions throughout and shows how to determine the shear stresses (or shear fluxes) due to a concentrated torque.

5.1 Solid Sections

Rigorous mathematical analysis can be employed for only a few solid sections such as those having circular, elliptical, rectangular, square or equilateral triangular form.

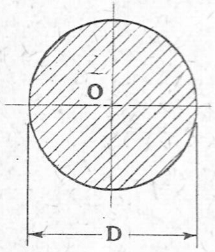
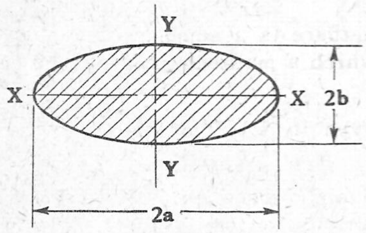
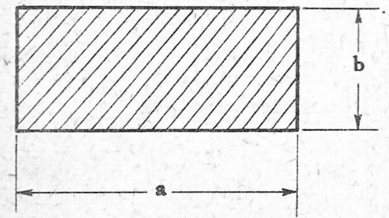
The resulting formulas (for small angles of twist) are given in the table of 5.1.1 (Cases 1 to 5 inclusive); Cases 3 and 4 are, however, simplified approximations of the exact results.

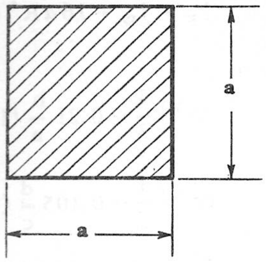
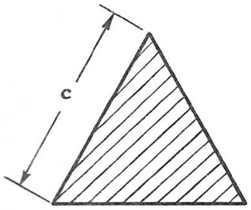
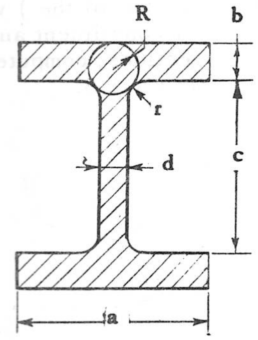
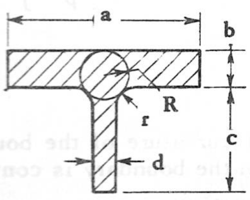
The formulas of Cases 6 to 10 inclusive are approximations based on the membrane analogy. For greater accuracy in these cases, and for the general case of tapering flanges, see 5.1.2, which employs Griffith's method based on soap-film technique and the membrane analogy.

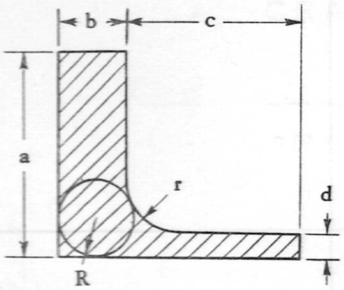
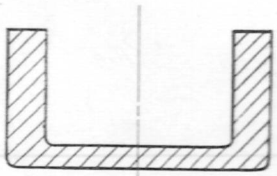
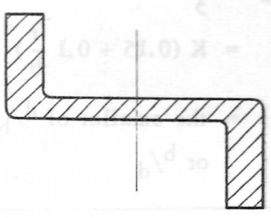
5.1.1 Summary of Common Cases

T = Torque (lb.in.)
J = Torsion Constant (in.^{4})
θ = Angle of Twist (Radians)
G = Shear Modulus (p.s.i.)
τ = Shear Stress (p.s.i.)
L = Length of Member (in.)

$$\frac{\theta G}{L} = \frac{T}{J}$$

	SECTION	SHEAR STRESS	TORSION CONSTANT J
1	SOLID CIRCULAR  	Max. $\tau = \frac{16T}{\pi D^3}$ at boundary	$J = \frac{\pi D^4}{32}$ In this particular case, J is equal to the polar moment of inertia of the section about the centre 0.
2	SOLID ELLIPTICAL  	Max. $\tau = \frac{2T}{\pi ab^2}$ at ends of minor axis YY $\tau = \frac{2T}{\pi a^2 b}$ at ends of major axis XX	$J = \frac{\pi a^3 b^3}{(a^2 + b^2)}$
3	SOLID RECTANGULAR  a > b	Max. $\tau = \frac{T}{ab^2}(3 + 1.8\frac{b}{a})$ This is the approximate value of τ at the mid. point of each longer side.	$J = ab^3 [\frac{1}{3} - 0.21\frac{b}{a}(1 - \frac{1}{12}\frac{b^4}{a^4})]$ (approx.)

SECTION	SHEAR STRESS	TORSION CONSTANT J
4 SOLID SQUARE 	Max. $\tau = 4.8 \frac{T}{a^3}$ (approx.) at mid. point of each side	$J = 0.1408\, a^4$ (approx.)
5 SOLID EQUILATERAL TRIANGLE 	Max. $\tau = \frac{20T}{c^3}$ at mid. point of each side	$J = \sqrt{3}\ \frac{c^4}{80}$
6 I SECTION Flange Thickness Constant 	R = radius of largest inscribed circle r = fillet radius See note below for maximum shear stress	$J = 2J_F + J_W + 32\alpha R^4$ (approx.) Where $J_F = ab^3[\frac{1}{3} - 0.21 \frac{b}{a} (1 - \frac{b^4}{12a^4})]$ $J_W = \frac{1}{3} cd^3$ $\alpha = K (0.15 + 0.1 \frac{r}{b})$ K = the smaller of $^d/_b$ or $^b/_d$
7 T SECTION Flange Thickness Constant 	R = radius of largest inscribed circle r = fillet radius See note below for maximum shear stress	$J = J_F + J_W + 16\, \alpha R^4$ (approx.) Where $J_F = ab^3[\frac{1}{3} - 0.21 \frac{b}{a} (1 - \frac{b^4}{12a^4})]$ $J_W = cd^3[\frac{1}{3} - 0.105 \frac{d}{c} (1 - \frac{d^4}{192c^4})]$ $\alpha = K (0.15 + 0.10 \frac{r}{b})$ K = the smaller of $^d/_b$ or $^b/_d$

SECTION	SHEAR STRESS	TORSION CONSTANT J
8 ANGLE SECTION b equal to, or greater than, d 	R = radius of largest inscribed circle r = fillet radius See note below for maximum shear stress.	$J = J_F + J_W + 16\alpha R^4$ (approx.) Where $J_F = ab^3 [\frac{1}{3} - 0.21 \frac{b}{a} (1 - \frac{b^4}{12a^4})]$ $J_W = cd^3 [\frac{1}{3} - 0.105 \frac{d}{c} (1 - \frac{d^4}{192c^4})]$ $\alpha = \frac{b}{d} (0.07 + 0.076 \frac{r}{b})$
9 CHANNEL SECTION 	See note below for maximum shear stress.	J = the sum of the J values for constituent angle sections, computed as for Case 8.
10 Z SECTION 		

NOTE REGARDING SHEAR STRESS (Cases 6 – 10 inclusive):–

The maximum shear stress occurs, in general, at or near one of the points of contact of the largest inscribed circle at the point where the radius of curvature is least.

Empirical formulae are:–

Equation (a): When the boundary is convex or straight,

$$\text{Max. } \tau = \frac{2R}{1 + m^2} [1 + 0.15 (m^2 - \frac{R}{\rho})] \frac{T}{J}$$

$$\text{Where } m = \frac{\pi R^2}{A}$$

ρ = radius of curvature of the boundary (positive when the boundary is convex)

A = area of section

R = radius of largest inscribed circle

Equation (b): When the boundary is concave,

$$\text{Max. } \tau = \frac{2R}{1+m^2}\left[1+\tanh\frac{2\beta}{\pi}\left\{0.118\log_e\left(1-\frac{R}{\rho}\right)-0.238\Big|\frac{R}{\rho}\right\}\right]\frac{T}{J}$$

Where β = angle turned through by the tangent to the boundary of the re-entrant portion.

The application of these formulas is given in Section 5.1.2.

5.1.2 Approximate Method for any Solid Section

It can be shown that the mean stress around the boundary of any solid section of area A and perimeter p is equal to the boundary stress in a circular bar of radius $\frac{2A}{p}$.

Since the torsion constant of a solid circular section of cross sectional-area A is $J = \frac{\pi D^4}{32} = \frac{\pi r^4}{2} = \frac{A}{2} r^2$ this value of $\frac{2A}{p}$ may be looked upon as an equivalent torsional radius. Denoting it by C gives

$$J = \frac{A}{2} C^2$$

A preliminary estimate of the torsion constant is consequently given by this expression.

In most practical cases it will be necessary to divide the section into suitable compartments and find the contribution to J of each part, the total J value then being the sum of the component J's.

The general rules for division into compartments are as follows:–

Imagine a circle of varying radius, always touching the profile at two points at least, to move inside the section. Then there will be some positions when the circle has three or more points of contact and between each such pair of positions there will be a position of the circle where its radius is a minimum.

The division lines are drawn through the points of contact (with the profile) of this minimum circle (Fig. 5.1.2–1).

Where the figure contains a thick portion, from which project one or more thinner parts whose boundaries in the neighbourhood of the junctions are parallel or nearly parallel lines, the division should be such as to include with the thick component a length of each projection equal to half its thickness at the junction.

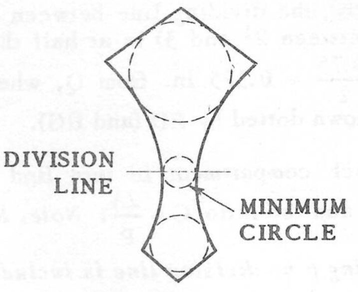


Figure 5.1.2–1

5.1.2.1 Procedure for First Approximation to J: Numerical example.

Consider the channel section shown in Fig. 5.1.2–2. Divide the section into five compartments numbered 1, 1^1, 2, 2^1 and 3 in accordance with the rule just given.

The dividing line between 1 and 2 (and hence between 1^1 and 2^1) is taken at a distance from P, where the fillet starts, equal to half the depth at P, viz $\frac{0.77}{2}$=0.385 in., as shown dotted at BC (and KL).

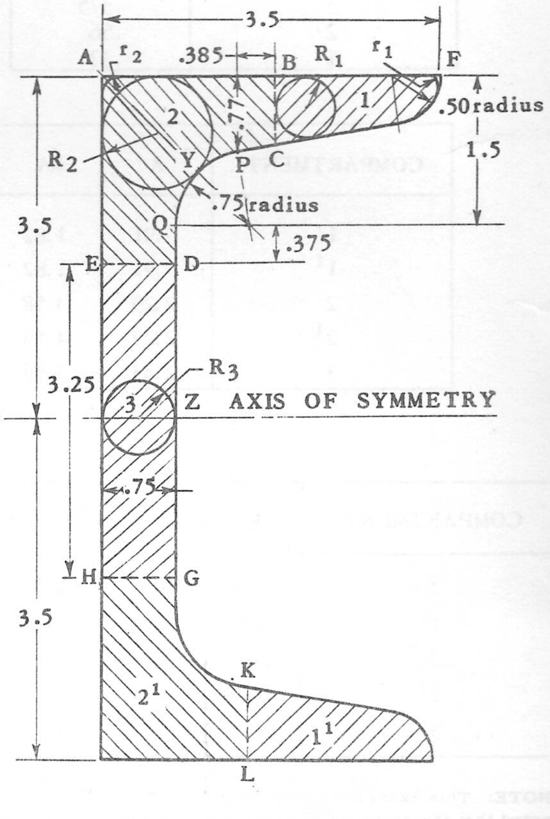


Figure 5.1.2–2

Similarly, the dividing line between 2 and 3 (and likewise between 2^1 and 3) is at half the web thickness viz $\frac{0.75}{2} = 0.375$ in. from Q, where the fillet ends, as shown dotted by ED (and HG).

For each compartment in turn find the area A, perimeter p and the ratio $C = \frac{2A}{p}$. *Note, however, that in determining p no division line is included.*

These values are given in Table 5.1.2–1 .

5.1.2.2 More Accurate Value of Torsion Constant

In order to find a more refined value of J, proceed as follows:–

(a) Draw the maximum inscribed circle of radius R in each compartment. The circles drawn in Compartments 1, 1^1, 2 and 2^1 touch the boundary at three points, as shown. The radii of the circles are:–

COMPARTMENT	R (IN.)
1	.32
2	.56
3	.375
2^1	.56
1^1	.32

(b) Round off the outwardly projecting corners (on the main boundary only) with a radius "r" which is a certain fraction of the radius R of the circle inscribed in any particular compartment.

This fraction depends on the angle β turned through by the tangent to the boundary at the corner in question: values of r/R are given on the accompanying graph (Fig. 5.1.2–3).

For a right-angled corner, $r/R = .50$. These values are:–

STATION	R (IN.)	β (deg.)	r/R	r (IN.)
F	.32	90	.50	.16
A	.56	90	.50	.28

(c) Draw in the corner radii. This gives a new figure having a component area and perimeter A_1 and p_1 respectively. Calculate A_1, p_1 and the ratio $C_1 = \frac{2A_1}{p_1}$ for each compartment.

(d) The torsion constant is then given by:–

$$J = \frac{A}{2} \lambda C_1^2 \quad \text{(see Table 5.1.2–2)}$$

λ in this expression is a coefficient, values of which are given in terms of $\frac{R}{C}$ in Fig. 5.1.2–4.

TABLE 5.1.2–1

COMPARTMENT	A	2A	p	$C = \frac{2A}{p}$	C^2	$\frac{A}{2}$	$\Delta J = \frac{A}{2} C^2$
1	.91	1.82	3.59	.506	.255	.455	.116
1^1	.91	1.82	3.59	.506	.255	.455	.116
2	2.29	4.58	5.67	.81	.655	1.145	.750
2^1	2.29	4.58	5.67	.81	.655	1.145	.750
3	2.43	4.86	6.50	.746	.558	1.215	.678

First approximation to torsion constant J = 2.410 in.^{4}

TABLE 5.1.2–2

COMPARTMENT	R	C	R/C	λ	A_1	$2A_1$	p_1	$\frac{2A_1}{p_1}$	C_1^2	λC_1^2	ΔJ
1	.320	.506	.63	.827	.87	1.74	3.55	.501	.25	.207	.090
1^1	.320	.506	.63	.827	.87	1.74	3.55	.501	.25	.207	.090
2	.560	.81	.69	.887	2.26	4.52	5.51	.820	.671	.595	.672
2^1	.560	.81	.69	.887	2.26	4.52	5.51	.820	.671	.595	.672
3	.375	.75	.50	.667	2.43	4.86	6.50	.746	.556	.371	.451

J = 1.975 in.^{4}

NOTE: This example is taken from "Strength of Materials" by Case, page 486. It should be noted that the above value of $\lambda = .827$ does not agree with his value of 0.845. The final J value is consequently at variance too.

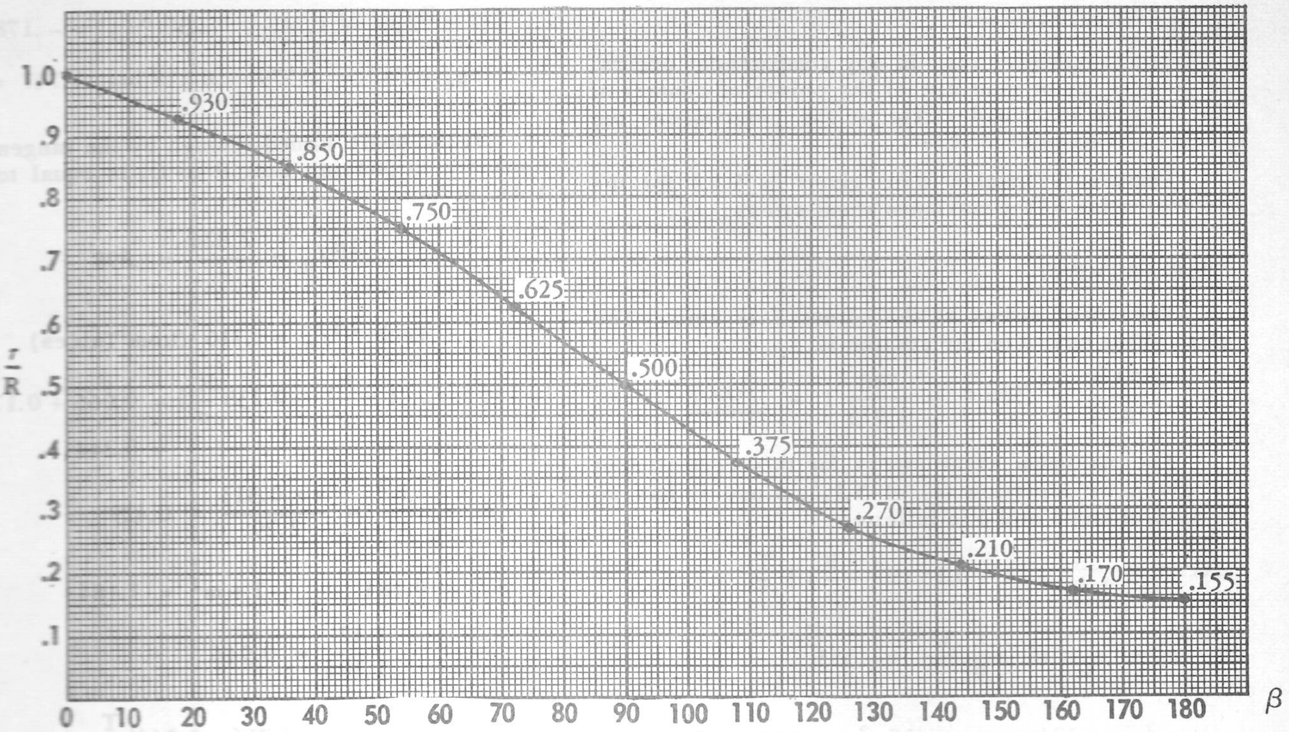


Figure 5.1.2–3 Plot of $\frac{r}{R}$ versus β

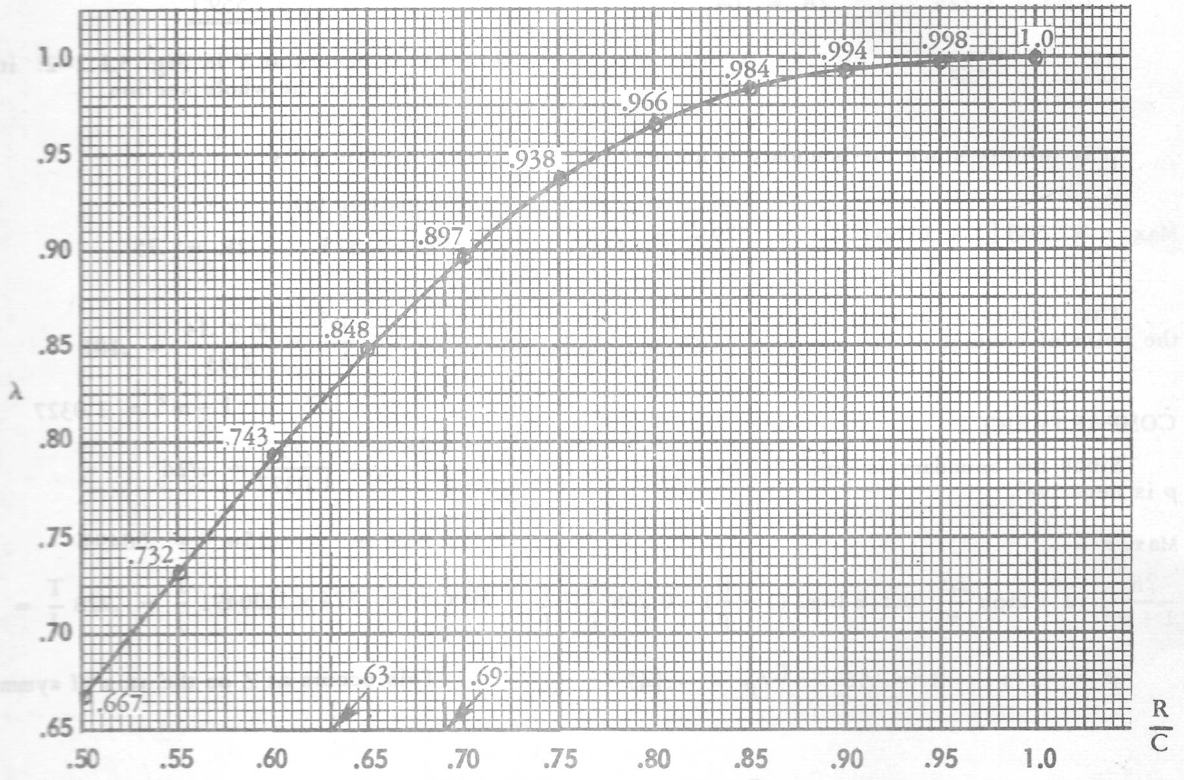


Figure 5.1.2–4 Plot of λ versus $^R/_C$

5.1.2.3 Shear Stresses

The maximum stress in any compartment occurs, in general, at or near one of the points of contact of the largest inscribed circle at the point where the radius of curvature is least.

Empirical formulas were given in 5.1.1 for the determination of the maximum shear stress.

COMPARTMENT 1

The shear stress at the corner F is zero. The mean shear stress on the section is

$$\frac{2A}{p}\frac{T}{J} = .506\frac{T}{J} = \frac{.506}{1.975}T = \underline{.2565\,T}$$

The maximum shear stress for this compartment is given by Equation (a) of 5.1.1, since it has no re-entrant angle and ρ is infinite.

$$\text{Max. } \tau = \frac{2R}{1+m^2}\left[1 + 0.15\left(m^2 - \frac{R}{\rho}\right)\right]\frac{T}{J}$$

$$m = \frac{\pi R^2}{A} = \frac{\pi \times .32^2}{.91} = .353$$

$m^2 = .124$ $\qquad .15m^2 = .0186$

$1 + m^2 = 1.124$ $\qquad 2R = .64$

$$\frac{2R}{1+m^2} = .568$$

ρ is infinite since the boundary is uncurved.

$$\text{Max. } \tau = .568\,[1 + .0186]\,\frac{T}{J} = .579\frac{T}{J} = \underline{.292T}$$

This occurs at or near point X in Fig. 5.1.2–2, the point of contact of the maximum inscribed circle.

COMPARTMENT 2

Since the boundary is concave, use Equation (b). ρ is negative.

Max. τ =

$$\frac{2R}{1+m^2}\left[1 + \tanh\frac{2\beta}{\pi}\left\{0.118\log_e\left(1 - \frac{R}{\rho}\right) - 0.238\frac{R}{\rho}\right\}\right]\frac{T}{J}$$

$\rho = -.75$ $\qquad R = .56$ $\qquad R/\rho = -.747$

$(1 - R/\rho) = 1.747$ $\qquad .238\,R/\rho = -.178$

$\log_e 1.747 = .5579$ $\qquad 0.118 \log_e 1.747 = 0.066$

β, the angle turned through by the tangent to the boundary in passing from P to Q, is equal to 81° = 1.412 radians.

$$2\beta/\pi = \frac{2.824}{\pi} = .898$$

$$\text{Tanh } 2\beta/\pi = .716 \text{ (from Tables)}$$

$$\left\{0.118\log_e\left(1 - \frac{R}{\rho}\right) - 0.238\frac{R}{\rho}\right\} = 0.066 + 0.178$$
$$= 0.244$$

$$m = \frac{\pi R^2}{A} = \frac{\pi \times 0.313}{2.29} = 0.429$$

$m^2 = 0.184$ $\qquad 1 + m^2 = 1.184$

$$\frac{2R}{1+m^2} = \frac{1.12}{1.184} = 0.944$$

$$\text{Max. } \tau = 0.944\,[1 + 0.716 \times 0.244]\,\frac{T}{J}$$
$$= 1.11\frac{T}{J} = \underline{.559\,T}$$

This occurs at Y in Fig. 5.1.2–2: it is the maximum stress on the whole section.

COMPARTMENT 3

Use Equation (a)

$R = .375$ $\qquad 2R = .75$

$$m = \frac{\pi R^2}{A} = \frac{\pi \times .14}{2.43} = .181$$

$m^2 = .0327$ $\qquad 1 + m^2 = 1.0327$

$$\frac{2R}{1+m^2} = \frac{.75}{1.0327} = .724$$

$\rho = \infty$ $\qquad .15m^2 = .00491$

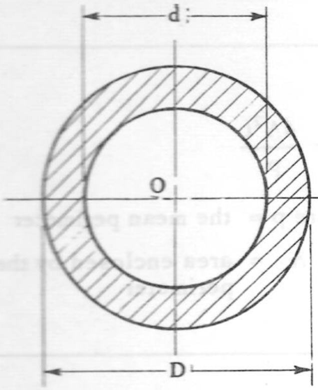
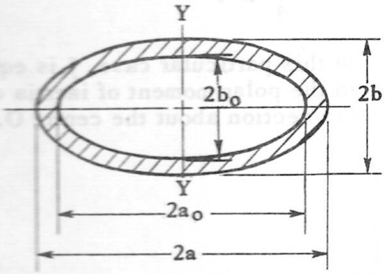
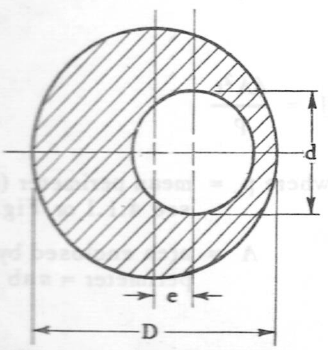
$$\text{Max. } \tau = .724 \times 1.00491\,\frac{T}{J} = .728\frac{T}{J} = \underline{.367\,T}$$

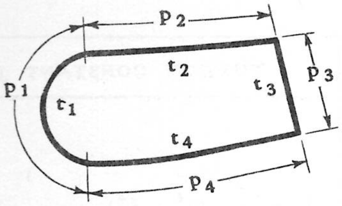
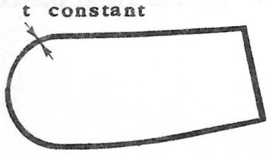
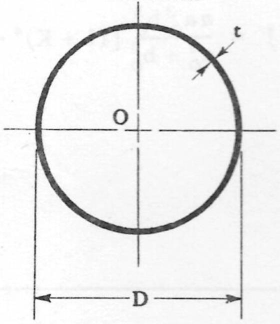
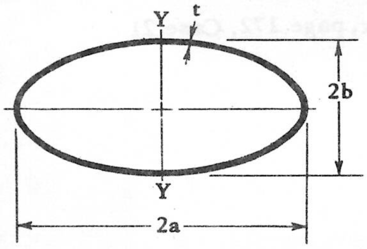
This occurs at Z on the axis of symmetry.

5.2 Hollow Sections: Thin walled and otherwise.

T = Torque (lb. in.)	G = Shear Modulus (p.s.i.)
J = Torsion Constant (in.^{4})	τ = Shear Stress (p.s.i.)
θ = Angle of Twist (Radians)	L = Length of Member (in.)

$$\frac{\theta G}{L} = \frac{T}{J}$$

SECTION	SHEAR STRESS	TORSION CONSTANT J
1 HOLLOW CONCENTRIC CIRCULAR TUBE 	Max. $\tau = \frac{16TD}{\pi(D^4 - d^4)}$ at outer boundary. For allowables see ANC–5, pages 29, 81 and 121.	$J = \frac{\pi}{32}(D^4 - d^4)$ In this particular case, J is equal to the polar moment of inertia of the section about the centre O.
2 HOLLOW ELLIPTICAL SECTION Outer and inner boundaries are similar ellipses: thickness not constant. 	Max. $\tau = \frac{2T}{\pi a_o b_o^2[(1+K)^4 - 1]}$ on outer surface at ends of minor axis YY. $K = \frac{a - a_o}{a_o} = \frac{b - b_o}{b_o}$	$J = \frac{\pi a_o{}^3 b_o^3}{a_o^2 + b_o^2}[(1+K)^4 - 1]$
3 ECCENTRIC HOLLOW SHAFT 	See Roark, page 172, Case 21	

SECTION	SHEAR STRESS	TORSION CONSTANT J
4 ANY THIN-WALLED ENCLOSED SECTION OF VARYING THICKNESS 	$\tau_1 = q/t_1$ $\tau_2 = q/t_2$ $\tau_3 = q/t_3$ $\tau_4 = q/t_4$ where $q = \dfrac{T}{2A}$ lb. per in. (Batho's formula)	$J = \dfrac{4A^2}{\Sigma P/t}$ where $\Sigma P/t = \dfrac{P_1}{t_1} + \dfrac{P_2}{t_2}$ + etc. and A = area enclosed by the mean perimeter
5 ANY THIN-WALLED ENCLOSED SECTION OF CONSTANT THICKNESS 	Average shear stress = q/t where $q = \dfrac{T}{2A}$ lb. per in. (Batho's formula)	$J = \dfrac{4A^2t}{p}$ where p = the mean perimeter and A = area enclosed by the mean perimeter
6 THIN-WALLED CIRCULAR TUBE D/t large t constant 	Max. $\tau = \dfrac{2T}{\pi D^2 t}$ For allowables see ANC–5, pages 29, 81 and 121.	$J = \dfrac{\pi}{4} D^3 t$ In this particular case, J is equal to the polar moment of inertia of the section about the centre O.
7 THIN-WALLED ELLIPTICAL SECTION OF CONSTANT THICKNESS 	Average shear stress = q/t where $q = \dfrac{T}{2A}$ lb. per in. (Batho's formula)	$J = \dfrac{4A^2t}{P}$ where p = mean perimeter (for which see 4.1.1 or Fig. 4.1.3–1) A = area enclosed by mean perimeter = πab

5.3 Thin-walled Open Sections

5.3.1 Introduction

When a thin-walled open section is subjected to pure (St. Venant) torsion as a result of the application of equal and opposite end torques T (Fig. 5.3–1), cross-sections of the tube warp in a longitudinal direction. In other words, points in the plane of the end sections do not remain in the same plane.

Shear stresses only are present in the walls when the tube is free to warp, that is, when there is no end constraint.

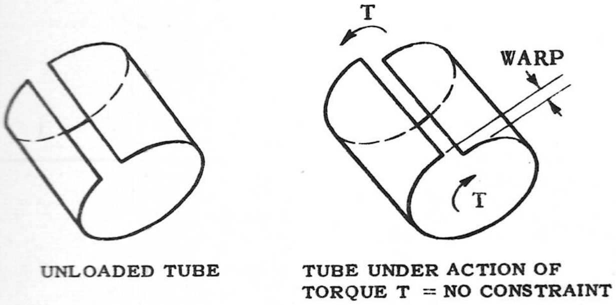


Fig. 5.3–1

It will be apparent that any member having a small thickness has practically the same torsional stiffness when bent into a channel, angle or other section as it has when flat. Thin open sections are consequently primarily unsuited as torque carrying members; in many cases, however, they have to take a certain amount of torque in addition to carrying out their primary function.

For a rectangular section (Case 1, 5.3.2) the distribution of shear stress is as shown in Fig. 5.3–2, the shear stresses acting in a direction parallel to the boundary, for small t/b ratios.

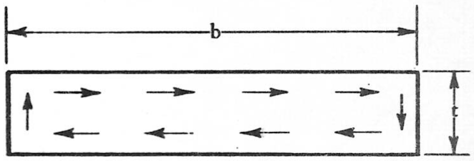


Fig. 5.3–2

Formulas for Cases 2 – 5 inclusive assume that the torsion constant J for the whole section is equal to the sum of the J values for the component rectangles of which the section is composed.

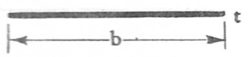
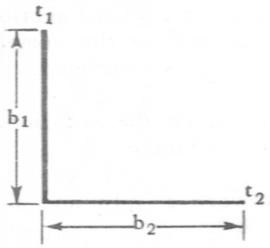
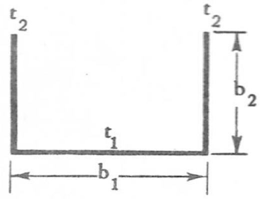
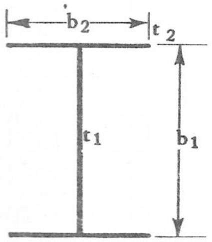
Similarly the J value for the remaining cases is based on the developed rectangle.

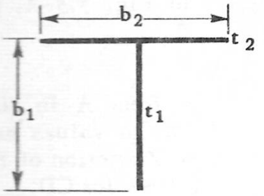
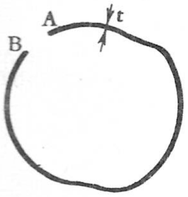
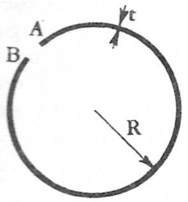
A comparison between a split and a solid tube appears in GEN 1090/809.

5.3.2 Formulas for Shear Stress and Torsion Constant

T = Torque (lb. in.)
J = Torsion Constant (in.^{4})
θ = Angle of Twist (Radians)
G = Shear Modulus (p.s.i.)
τ = Shear Stress (p.s.i.)
L = Length of Member (in.)

$$\frac{\theta G}{L} = \frac{T}{J}$$

SECTION	SHEAR STRESS	TORSION CONSTANT J
1 NARROW RECTANGLE OF CONSTANT THICKNESS $b/t > 20$ 	$\tau = \frac{3T}{bt^2}$	$J = \frac{1}{3} bt^3$
2 ANGLE t_1 and t_2 small compared with b_1 and b_2 	$\tau = \frac{3T}{b_1 t_1^2 + b_2 t_2^2}$ For Bulb Angle see R.Ae.S. Data Sheet 00.07.01	$J = \frac{1}{3} \Sigma bt^3$ $= \frac{1}{3} [b_1 t_1^3 + b_2 t_2^3]$
3 CHANNEL t_1 and t_2 small compared with b_1 and b_2 	$\tau = \frac{3T}{(b_1 t_1^2 + 2b_2 t_2^2)}$	$J = \frac{1}{3} \Sigma bt^3$ $= \frac{1}{3} [b_1 t_1^3 + 2b_2 t_2^3]$
4 I SECTION t_1 and t_2 small compared with b_1 and b_2 	$\tau = \frac{3T}{(b_1 t_1^2 + 2b_2 t_2^2)}$	$J = \frac{1}{3} \Sigma bt^3$ $= \frac{1}{3} [b_1 t_1^3 + 2b_2 t_2^3]$

SECTION	SHEAR STRESS	TORSION CONSTANT J
5 T SECTION t_1 and t_2 small compared with b_1 and b_2 	$\tau = \frac{3T}{(b_1t_1^2 + b_2t_2^2)}$	$J = \frac{1}{3} \Sigma bt^3$ $= \frac{1}{3} [b_1t_1^3 + b_2t_2^3]$
6 ANY THIN-WALLED OPEN TUBE t constant and small 	$\tau = \frac{3T}{st^2}$ s = length of periphery of cross-section from A to B	$J = \frac{1}{3} \Sigma_A^B t^3 ds$ $= \frac{1}{3} st^3$
7 THIN-WALLED OPEN CIRCULAR TUBE t constant and small 	$\tau = \frac{3T}{2\pi Rt^2}$ s = length of periphery of cross-section from A to B	$J = \frac{1}{3} st^3$ $= \frac{2}{3} \pi Rt^3$

5.3.3 Warp

For any open tube (Fig. 5.3–3) let

θ = angle of twist (radians),

L = length of tube (in.)

$\phi = \frac{\theta}{L}$ = angle of twist per unit length (radians per inch),

w = warp (in.)

$W = \frac{w}{\phi}$ = unit warp (in./radians per inch),

p_t = perpendicular distance from shear centre O to tangent at P, and

ds = element on periphery at P (Fig. 5.3–4).

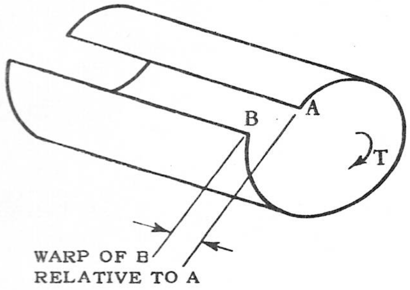


Fig. 5.3–3

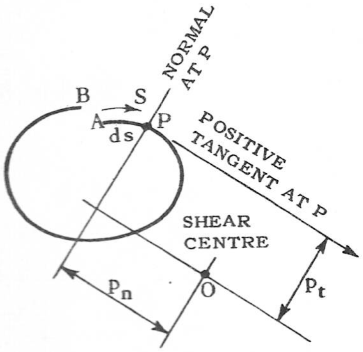


Fig. 5.3–4

Then from R.Ae.Soc. Data Sheet 00.07.00, but using the above notation,

$$W = -\int_A^B p_t \, ds$$

The minus sign signifies that the warp of B relative to A is into the paper for clockwise T.

SIGN OF p_t

When s is measured from A (Fig. 5.3–4), the tangent at any point P is considered positive in the direction of increasing s and p_t is positive if the positive tangent acts clockwise relative to the shear centre O. Consequently, in Fig. 5.3–4, p_t at P is positive.

Similarly, measuring s from A in the channel section of Fig. 5.3–5, all the p_t values are positive except in CD, whereas in the Z section of Fig. 5.3–6, p_t is positive for AB but negative for CD.

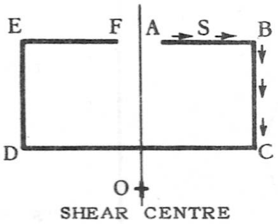


Fig. 5.3–5

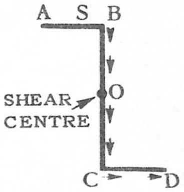


Fig. 5.3–6

5.3.3.1 **Numerical Example** – To find the unit warp W for the section 10CS–R–525 shown in Fig. 5.3–7 under the action of a contra-clockwise torque T (a contra-clockwise torque has been chosen in order to avoid the necessity for a minus sign in the expression for unit warp).

$$W = +\int_0^s p_t \, ds$$

The section is divided into convenient lengths ds and the p_t value for each section obtained.

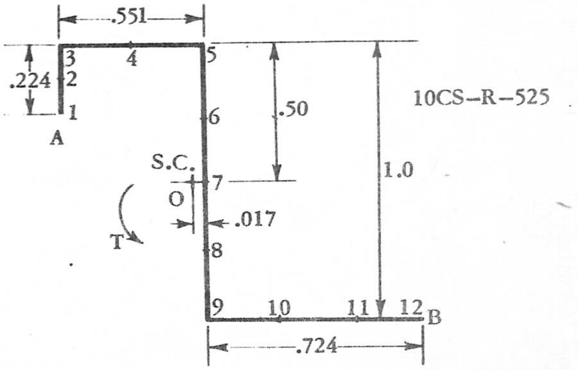


Fig. 5.3–7

STN	ds	p_t	$p_t ds \times 10^2$	$W \times 10^2$ Plotted in Fig. 5.3–8	s
1				0	
2	.10	.534	5.34	5.34	.10
3	.124	.534	6.62	11.96	.224
4	.25	.50	12.50	24.46	.474
5	.301	.50	15.05	39.51	.775
6	.25	.017	.43	39.94	1.025
7	.25	.017	.42	40.36	1.275
8	.25	.017	.43	40.79	1.525
9	.25	.017	.42	41.21	1.775
10	.25	–.50	–12.50	28.71	2.025
11	.25	–.50	–12.50	16.21	2.275
12	.224	–.50	–11.20	5.01	2.499

MEAN UNIT WARP

By plotting W against s in Fig. 5.3–8 the area under the curve is obtained. This is, $\Sigma_A^B W ds = .7244$.

$$\text{Mean unit warp } \overline{W} = \frac{\Sigma_A^B W ds}{\text{Length A to B}} = \frac{.7244}{2.499} = .29$$

This quantity is required for the calculation of the warping constant Γ (gamma).

5.3.4 Torsion Bending

When the cross-section is not free to warp, due to the constraint caused by building-in at one end, pure torsional stresses will be accompanied by tension or compression in the longitudinal direction of the beam.

These longitudinal stresses are proportional to the torsional warping displacement, now prevented.

The rate of change of angle of twist $\phi = \frac{d\theta}{dz}$ along the axis of the beam is not constant, a state of affairs which is consequently referred to as NON-UNIFORM TORSION.

The problem is normally tackled by splitting the torque into two parts, namely:–

T_1 = torque taken in shear
and T_2 = torque taken in torsion-bending, which for an I beam consists of differential bending of the flanges in opposite directions.

The relative proportion of T_1 and T_2 will depend on the dimensions of the cross-section and the length of the beam.

It is shown in GEN/1090/809.5.2 that the total torque T is given by the following expression:–

$$T = T_1 + T_2$$
$$= GJ\phi - E\Gamma \frac{d^2\phi}{dz^2}$$

$$\text{Where } \phi = \frac{d\theta}{dz}$$

Γ = warping (torsion-bending) constant of the section, values of which for common sections are given in 5.3.5.

This is the fundamental equation for the torsion of a thin-walled open section with end constraint.

5.3.5 Warping Constants

The warping (torsion bending) constant Γ (gamma) given in the fundamental equation above is equal to the sum of Γ_1 and Γ_2, where Γ_1 is the primary warping constant and Γ_2 is the secondary warping constant.

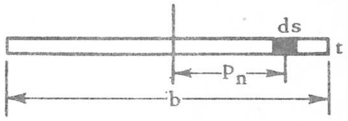
Γ_2 is normally very small compared with Γ_1 and can therefore be neglected except, for example, in the case of rectangular, angle and tee sections, for which $\Gamma_1 = 0$.

Values of Γ_1 (and Γ_2 where applicable) for common sections are given in the following tabular summary (5.3.5.1), and a method of calculation for Γ_1 in 5.3.5.2 for an unsymmetrical Zee section.

$$\Gamma_2 = \frac{1}{12}\int_o^s p_n^2 t^3 \, ds$$

where p_n is the perpendicular from the shear centre O to the normal to the median line at any point P in Fig. 5.3–4.

For a rectangular section:–



$$\Gamma_2 = \frac{1}{12}\int_o^s p_n^2 t^3 \, ds$$

$$= \frac{1}{12} t^2 \int_o^s p_n^2 \, t \, ds$$

$$\text{But } \int_o^s p_n^2 t \, ds = I = \frac{tb^3}{12}$$

$$\Gamma_2 = \frac{1}{12} t^2 \times \frac{tb^3}{12} = \frac{1}{144} b^3 t^3$$

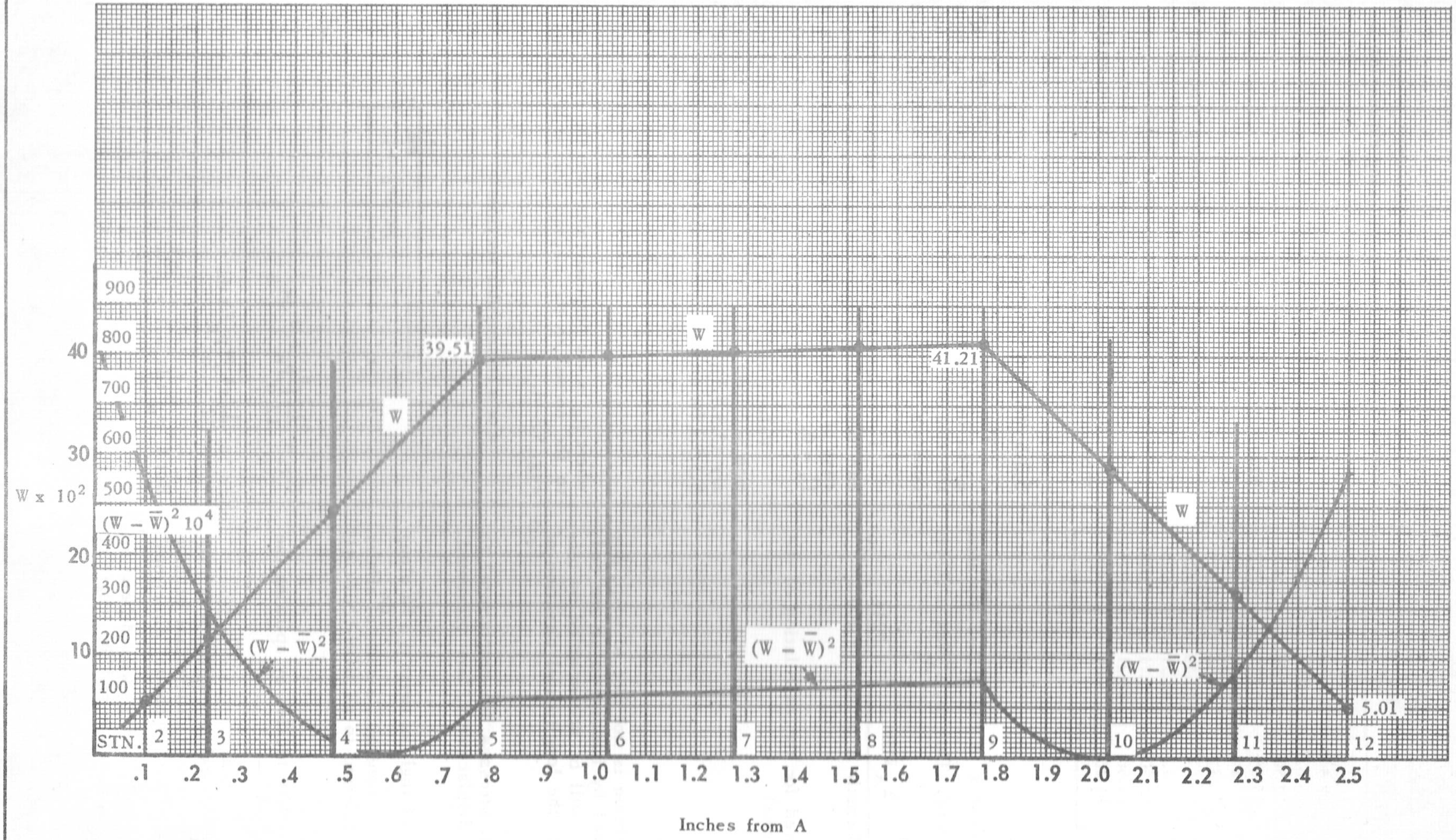
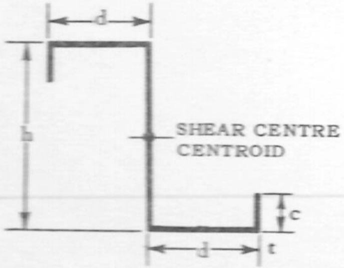
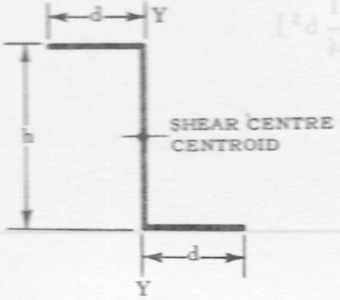
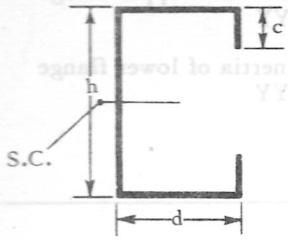
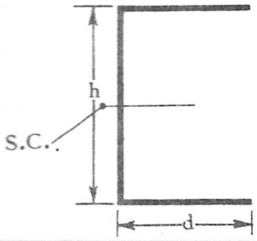
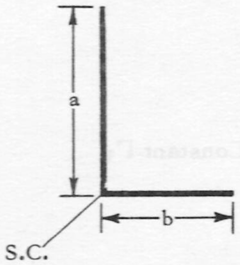
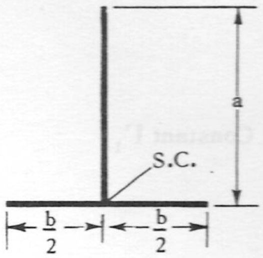
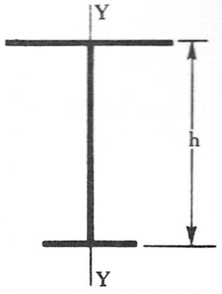
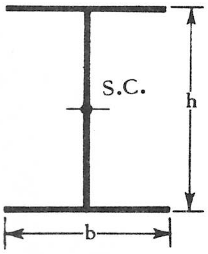


Fig. 5.3–8 Plot of W and $(W - \bar{W})^2$

5.3.5.1 Summary of Warping Constants

SECTION	
1 LIPPED Z-SECTION t constant  	See Fig. 5.3–9 for Primary Warping Constant Γ_1
2 Z-SECTION t constant  	See Fig. 5.3–9 for Primary Warping Constant Γ_1 Take c = 0
3 LIPPED CHANNEL t constant  	See Fig. 5.3–10
4 CHANNEL t constant  	See Fig. 5.3–10

SECTION	
5 ANGLE t constant  	$\Gamma_1 = 0$ $\Gamma_2 = \frac{t^3}{36}(a^3 + b^3)$
6 TEE t constant  	$\Gamma_1 = 0$ $\Gamma_2 = \frac{t^3}{36}[a^3 + \frac{1}{4} b^3]$
7 I SECTION t constant unequal flanges  	$\Gamma_1 = \frac{h^2 I_1 I_2}{I_1 + I_2}$ I_1 = moment of inertia of upper flange about axis YY I_2 = moment of inertia of lower flange about axis YY
8 I SECTION t constant equal flanges  	$\Gamma_1 = \frac{1}{24} t h^2 b^3$

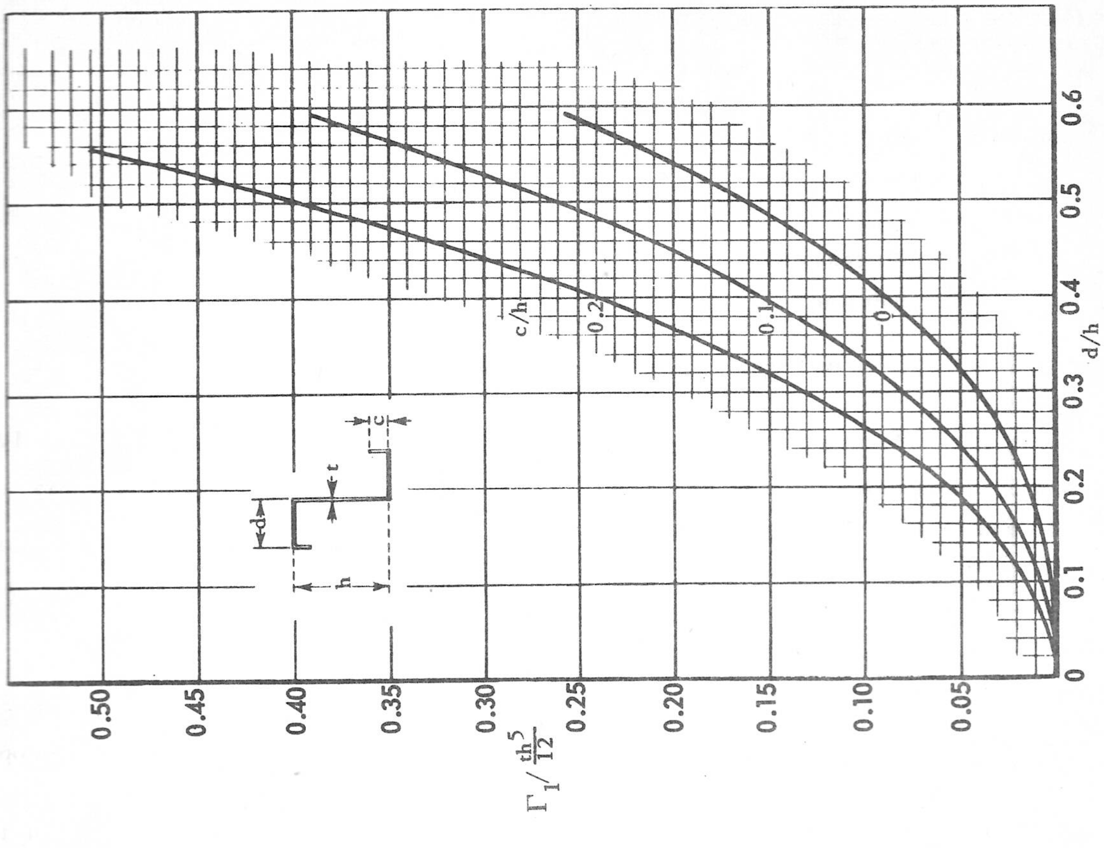


Fig. 5.3–9 Primary Warping Constant Γ_1 for Lipped Z-Sections

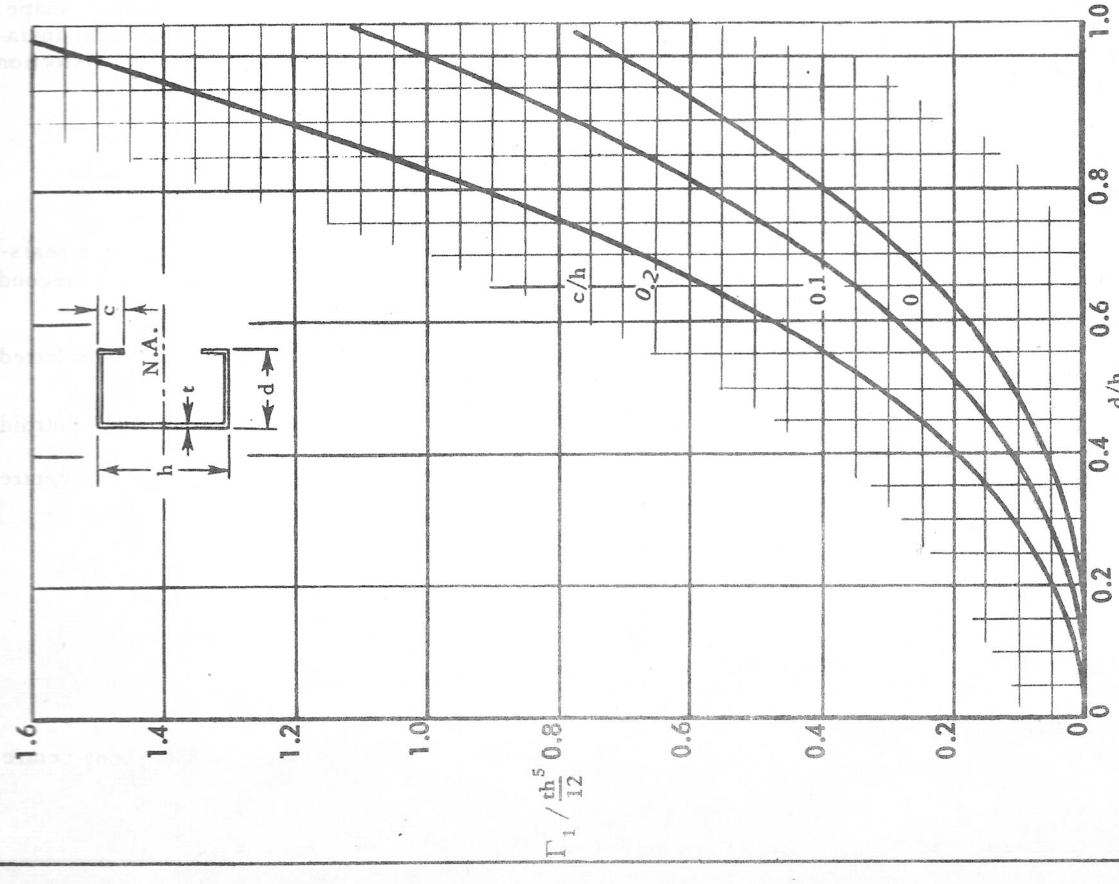


Fig. 5.3–10 Primary Warping Constant for a Lipped Channel Section (Ref. S.M. Tech. Memo 30)

5.3.5.2 Method of Calculating Warping Constant in the General Case – From "Buckling Strength of Metal Structures", Bleich, p.119,

Primary Warping Constant $\Gamma_1 = \int_0^s (W - \overline{W})^2 \, t ds$

$= t \int_0^s (W - \overline{W})^2 \, ds$

for constant t.

W = unit warp $= \int_0^s p_t \, ds$

$\overline{W}$ = mean unit warp

$= \frac{1}{s} \int_0^s W \, ds$

Having found W and $\overline{W}$ by the method of 5.3.3.1 tabulate values of $(W - \overline{W})$ and $(W - \overline{W})^2$ at each station. Plot $(W - \overline{W})^2$ against s and determine the area under this curve viz $\int_0^s (W - \overline{W})^2 \, ds$, as in the numerical example that follows.

NUMERICAL EXAMPLE
Z-Section as before

$\overline{W} = 29 \times 10^{-2}$ from 5.3.3.1 as are values of W

STN	$W \times 10^2$	$(W - \overline{W}) \times 10^2$	$(W - \overline{W})^2 \times 10^4$
1	0	− 29	841
2	5.34	− 23.66	560
3	11.96	− 17.04	290.4
4	24.46	− 4.54	20.6
5	39.51	+ 10.51	110
6	39.94	+ 10.94	119.5
7	40.36	+ 11.36	129
8	40.79	+ 11.79	139
9	41.21	+ 12.21	149
10	28.71	− .29	.09
11	16.21	− 12.79	163
12	5.01	− 23.99	575.5

(Plotted in Fig. 5.3–8)

$\frac{\Gamma_1}{t} = \Sigma_A^B (W - \overline{W}) \Delta s = 394.38 \times 10^4$
by graphical integration

$\frac{\Gamma_1}{t} = .0394$

$\Gamma_1 = .0394 \times .051 = 0.00201$

Check from R.Ae.S. 00.07.02 assuming that the section is as in Fig. 5.3–11.

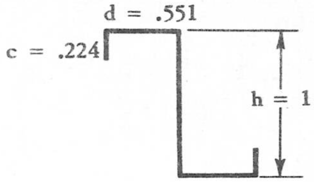


Fig. 5.3–11

$\frac{d}{h} = \frac{.551}{1.0} = .551$

$c = .224 \qquad \frac{c}{h} = .224$

$\frac{\Gamma_1}{t} = .50 \frac{h^5}{12}$ from curve $\qquad h = 1.0 \quad h^5 = 1.0$

$= .50 \times .0832 \qquad \frac{h^5}{12} = \frac{1}{12} = .0832$

$= .0416$

as compared with .0394 above.

5.3.6 Torsional Instability

5.3.6.1 Stringer Section Alone

The critical torsional instability stress of an open section stringer is given below.

This assumes that plane cross-sections of the stringer warp but do not change their geometric shape, known as Primary failure, as distinct from local instability (for which see 6.2), characterised by distortion of the section.

$$\sigma_{T1} = \frac{1}{I_P}\left[\overset{\text{TWISTING}}{GJ} + \overset{\text{BENDING}}{\left(\frac{\pi}{L}\right)^2 E_T \Gamma}\right]$$

See GEN/1090/809.8.1

The first term in the bracket represents the resistance of the stringer to pure twisting and the second term its resistance to bending.

I_P = Polar moment of inertia of the section referred to the shear centre $= I_{P_O} + A r_o^2$

I_{P_O} = Polar moment of inertia about the centroid

r_o = Polar distance from centroid to shear centre

A = area of section

G = shear modulus

J = St. Venant torsion constant

t = thickness of stringer

L = length of stringer

E_T = tangent modulus of elasticity

Γ = warping constant relative to the shear centre

NUMERICAL EXAMPLE

$A = .1274 \qquad r_o = .017$ (See GEN/1090/809/3.3)

$I_X = .0225 \qquad L = 20$

$I_Y = .01273$

Polar moment of inertia about centroid

$$I_{P_O} = I_X + I_Y = .0225 + .0127 = .0352$$

Polar moment of inertia about shear centre

$$I_P = I_{P_O} + Ar_O^2 = .0352 + .1274 \times .017^2 = .0352$$

$$J = 1/3 \times \text{developed length} \times t^3 = 1/3 \times 2.499 \times (.051)^3 \qquad .051^3 = 132 \times 10^{-6}$$

$$= 110 \times 10^{-6}$$

$$GJ = .385\,E\,J = 3.85 \times 10^6 \times 110 \times 10^{-6} = 423.5$$

$$E_T = E \text{ say } = 10 \times 10^6$$

$$\Gamma = .00206 = 20.6 \times 10^{-4} \qquad \text{(from 5.3.5.2)}$$

$$(\frac{\pi}{L})^2 = (\frac{\pi}{20})^2 = (.157)^2 = 2.47 \times 10^{-2}$$

$$(\frac{\pi}{L})^2 E_T\Gamma = 2.47 \times 10^{-2} \times 10 \times 10^6 \times 20.6 \times 10^{-4} = 509$$

$$\sigma_{T1} = \frac{1}{I_P}[GJ + (\frac{\pi}{L})^2 E_T\Gamma]$$

$$= \frac{1}{.0352}[423.5 + 509]$$

$$= \frac{932.5}{.0352} = \underline{26{,}500 \text{ p.s.i}}$$

5.3.6.2 Stringer Attached to Skin

The stiffness of the skin in its own plane and the anchorage of the skin at the sides of the panel are controlling factors in the location of the enforced axis of rotation.

If the stiffness of the skin in its own plane is assumed to be infinite, the axis of rotation is forced to lie in the plane of the skin, because rotation of the stringer-skin about any axis not in the plane of the skin would require a movement of the skin in its own plane.

It is customary to take the enforced axis of rotation at the intersection of the inside skin line and the principal axis of the stringer.

The torsional instability stress is then

$$\begin{array}{ccccc} & & \text{TWISTING} & \text{BENDING} & \text{SKIN BENDING} \\ \sigma_{T1} = & \frac{1}{I_P{}^1}[& GJ^1 & + (\frac{\pi}{L})^2 E_T\Gamma^1 + & (\frac{L}{\pi})^2 K\,] \end{array}$$

See GEN/1090/809.8.1

Many of these items have the same significance as in 5.3.6.1 except that they are now found with reference to the enforced axis of rotation, as signified by the dash. Items not specified earlier are:–

K = skin support stiffness. It represents the resistance of the skin to bending due to twist of the stiffener $= \dfrac{1}{K_1 + K_2}$

$K_1 = \dfrac{3d}{Et_{SK}^3}$ for non-buckled skin

$= \dfrac{d}{Et_{SK}^3}$ for buckled skin

d = stringer spacing = 4 in.

t_{SK} = skin thickness

$K_2 = \dfrac{4h + 12r}{Et^3}$

r = offset of rivet line from centre of rotation

h = depth of stringer

t = thickness of stringer

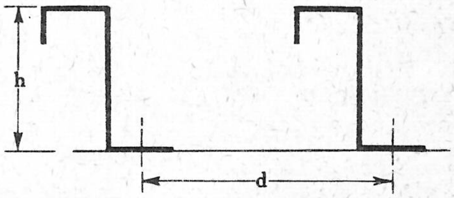


Fig. 5.3–12

NUMERICAL EXAMPLE

Same Z-section, attached to .036 in. skin. Take effskin width = 30t = 1.0 in. say.

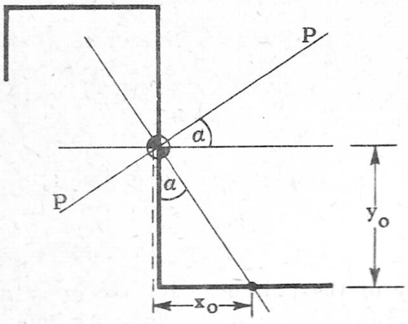


Fig. 5.3–13

(a) Warping Constant Γ^1

$$\Gamma^1 = \Gamma^1_{STRINGER} + \Gamma^1_{SKIN}$$

$$\Gamma^1_{STRINGER} = \Gamma + x_o^2 I_X + y_o^2 I_Y - 2x_o y_o I_{XY}$$

Where x_o and y_o are the co-ordinates of enforced axis relative to S.C. of stringer alone

$y_o = -.50$ See GEN/1090/809.1

$x_o = .017 + .50 \tan a$ $a = 34° 40'$

$= .017 + .3458$ $\tan a = .6915$

$= .3628$

$x_o^2 = .132$ $y_o^2 = .25$

$x_o^2 I_X = .132 \times .0225$ $y_o^2 I_Y = .25 \times .01273$

$= .00297$ $= .00318$

$2 x_o y_o I_{XY} = 2\times .3628 \times -.50 \times -.012995$

$= + .00472$

$\Gamma^1_{STRINGER} = .00206 + .00297 + .00318 - .00472$

$= \underline{.00349}$

Γ^1_{SKIN} = secondary warping constant of skin (primary warping constant is zero)

$= \frac{1}{144}\,(\text{effskin})^3\, t^3_{SKIN}$ from 5.3.5

$= \frac{1}{144} \times 1.0^3 \times .036^3$

= NEGLIGIBLE

$\Gamma'^1_{GROSS} = \underline{.00349} = 34.9 \times 10^{-4}$

(b) $J_{SK} = \frac{1}{3}\,\text{effskin} \times t^3_{SK}$

$= \frac{1}{3} \times 1.0 \times 46.5 \times 10^{-6}$

$= 15.5 \times 10^{-6}$

$J^1 = J + J_{SK}$

$= 110 \times 10^{-6} + 15.5 \times 10^{-6}$

$= 125.5 \times 10^{-6}$

$GJ^1 = 3.85 \times 10^6 \times 125.5 \times 10^{-6}$

$= 483$

(c) $I^1_{P_{STR}} = I_P + A r_o^2$ $r_o^2 = x_o^2 + y_o^2$

$= .0352 + .0487$ $= .132 + .25$

$= .0839$ $= .382$

$A r_o^2 = .1274 \times .382 = .0487$

$I^1_{P_{SKIN}} = \frac{1}{12}\,(\text{effskin})^3\, t_{SK}$ approx.

$= \frac{1}{12} \times 1.0 \times .036 = .003$

$I^1_{P_{GROSS}} = .0839 + .003 = .0869$

(d) Assume well-buckled skin

$K_1 = \frac{d}{E t^3_{SK}} = \frac{4}{10^7 \times 46.5 \times 10^{-6}} = .0086$

$K_2 = \frac{4h + 12r}{E t^3_{SK}} = \frac{4 \times 1.0}{10^7 \times 46.5 \times 10^{-6}}$

since $r = 0$

$= .0086$

$K_1 + K_2 = .0172$

$K = \frac{1}{.0172} = 58.2$

(e) $(\frac{\pi}{L})^2 = 2.47 \times 10^{-2}$

$(\frac{L}{\pi})^2 = \frac{10^2}{2.47} = 40.4$

$$\sigma_{T1} = \frac{1}{I_p{}^1}\left[GJ^1 + \left(\frac{\pi}{L}\right)^2 E_T\Gamma^1 + \left(\frac{L}{\pi}\right)^2 K\right]$$

$$= \frac{1}{.0869}\left[483 + 24.7\times 10^4 \times 34.9\times 10^{-4} + 40.4\times 58.2\right]$$

$$= \frac{1}{.0869}\left[483 + 861 + 2351\right]$$

$$= \frac{3695}{.0869} = 42{,}600 \text{ p.s.i}$$

5.4 Thin Multi-walled Sections (concentrated T, constant J)

In order to explain a method of finding the shear stresses due to torque in thin multi-walled sections, consider the two cell wing-beam shown in Fig. 5.4–1 under the action of a torque T, assumed concentrated at the ends.

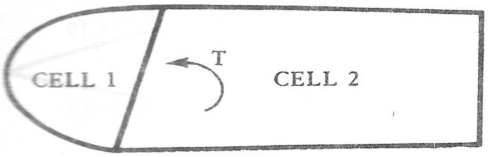


Fig. 5.4–1

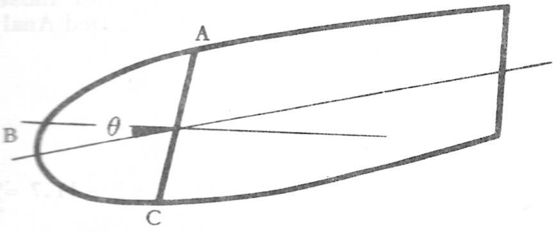


Fig. 5.4–2

Let θ be the angle of twist of the whole beam due to the action of the torque T. If there is no distortion of the section it will be clear that the boundary wall AC will rotate through some angle θ (Fig. 5.4–2) and consequently the angle of twist of Cell 1 will be the same as that of Cell 2.

It can be shown (see GEN/1090/810) that

$$\frac{\theta G}{L} = \frac{\Sigma \text{ shear flux} \times p/t}{2A}$$

Thus for the two cell beam under consideration it is necessary to determine the value of the right-hand side of this equation for both Cell 1 and for Cell 2 in terms of the unknown shear fluxes and then equate the two, since each is equal to $\frac{\theta G}{L}$.

A second relationship results from the fact that the total torque T on the section must equal the sum of the individual torques in each cell (see GEN/1090/810). That is,

$$T = T_1 + T_2$$
$$= 2A_1q_1 + 2A_2q_2$$

where q_1 and q_2 are the shear fluxes in Cells 1 and 2 respectively.

The method will be demonstrated with reference to a 3 cell beam, the solution of which is exactly the same as the above in principle but of necessity a little more complicated.

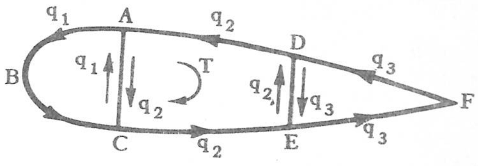
NUMERICAL EXAMPLE

To determine the distribution of shear stress, due to a concentrated torque T = 10.000 lb. in., on the section shown in Fig. 5.4–3 given that:–

Enclosed areas are $A_1 = 89.3 \text{ in.}^2$

$A_2 = 250 \text{ in.}^2$

$A_3 = 125 \text{ in.}^2$



T = 10000

Fig. 5.4–3

Other given data is as follows:–

MEMBER	ABC	AD	CE	DFE	AC	DE
LENGTH p (in.)	25.72	25	25	51.93	10	10
THICKNESS t (in.)	.05	.04	.02	.01	.06	.03

Let the reactive shear fluxes in the members be as indicated in Fig. 5.4–3. The following information is tabulated:–

MEMBER	ABC	AD	CE	DFE	AC	DE
Shear flux	q_1	q_2	q_2	q_3	$q_1 - q_2$	$q_2 - q_3$
perimeter p	25.72	25	25	51.93	10	10
thickness t	.05	.04	.02	.01	.06	.03
p/t	514.4	625	1250	5193	166.7	333
Shear flux × p/t	$514.4 q_1$	$625 q_2$	$1250 q_2$	$5193 q_3$	$166.7\ (q_1 - q_2)$	$333\ (q_2 - q_3)$

Cell 1 – Take contra-clockwise reactive shear flux as positive.

$$\frac{\theta G}{L} = \frac{\Sigma \text{ shear flux} \times p/t}{2A_1} = \frac{514.4\, q_1 + 166.7(q_1 - q_2)}{178.6}$$

$$= 3.83\, q_1 - 0.932\, q_2 \qquad (1)$$

Cell 2

$$\frac{\theta G}{L} = \frac{625\, q_2 + 1250\, q_2 - 166.7\,(q_1 - q_2) + 333(q_2 - q_3)}{500}$$

$$= -\,0.333\, q_1 + 4.75\, q_2 - 0.66\, q_3 \qquad (2)$$

Cell 3

$$\frac{\theta G}{L} = \frac{5193\, q_3 - 333\,(q_2 - q_3)}{250}$$

$$= 22.2\, q_3 - 1.332\, q_2 \qquad (3)$$

By solving these three simultaneous equations the following results are obtained:

$q_1 = 5.38\, q_3$

$q_1 = 1.322\, q_2$

$q_2 = 4.06\, q_3$

From the second relationship

$T = 2A_1 q_1 + 2A_2 q_2 + 2A_3 q_3$

$10{,}000 = 178.6\, q_1 + 500\, q_2 + 250\, q_3$

Substituting in terms of q_3,

$10{,}000 = 960\, q_3 + 2030\, q_3 + 250\, q_3 = 3240\, q_3$

whence $q_3 = 3.10$ lb.per in.

$q_2 = 12.52$ lb. per in.

$q_1 = 16.60$ lb. per in.

$(q_1 - q_2) = 4.08$ lb. per in.

$(q_2 - q_3) = 9.42$ lb. per in.

As shown in Fig. 5.4–4

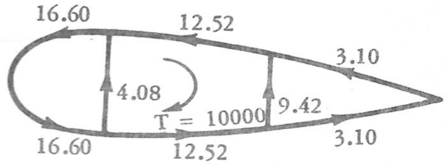


Fig. 5.4–4

These results are in agreement with those in J.Ae.Sc., January 1945, p.103, "A Simplified Analysis of Hollow Beams" by Eryk Kosko.

Check on $\frac{\theta G}{L}$

Cell 1. $\frac{\theta G}{L} = 3.83\, q_1 - 0.932\, q_2 = 63.5 - 11.7 = \underline{51.8}$

Cell 2. $\frac{\theta G}{L} = -\,0.333\, q_1 + 4.75\, q_2 - 0.66\, q_3$

$= -\,5.54 + 59.7 - 2.05 = \underline{52}$

Cell 3. $\frac{\theta G}{L} = 22.2\, q_3 - 1.332\, q_2 = 68.8 - 16.7 = \underline{52.1}$

Check on T

$T = 2A_1 q_1 + 2A_2 q_2 + 2A_3 q_3$

$= 178.6 \times 16.6 + 500 \times 12.52 + 250 \times 3.10$

$= 2970 + 6260 + 776 = \underline{10006}$ as against 10000.

COLUMNS

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6 COLUMNS

Introduction and Reference Table

Column problems usually involve the determination of (a) the buckling load and/or (b) the distribution of bending moment.

The Column Reference Table below indicates the Section that deals with one or other of these subdivisions (a) and (b).

COLUMN REFERENCE TABLE

PROBLEM		(a) BUCKLING LOAD	(b) BENDING MOMENT
Single Span Column No Lateral Load	I Constant	See 6.1.1 for Euler formulas for pinned and fixed ends. See 6.1.3 for Column Stress curves for pinned and fixed ends.	
	I Varying	See 6.3.1 for standard cases and 6.3.2 for general method	
Continuous Column No Lateral Load	I Constant	See 6.4 for Charts	
Straight Beam-Column	I Constant	Buckling stress, corresponding to buckling load of 6.1, must not be exceeded by combined stress at any station.	See 6.5.2 for summary of standard cases, 6.5.3 for Perry's approximation and 6.5.5 for Polar diagrams
	I Varying		See 6.5.6 for general method
Column with small initial Bow	I Constant		See 6.6 for summary of standard cases.

6.1 Primary Instability in Compression

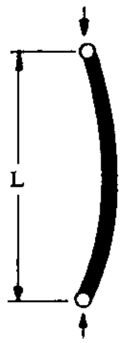
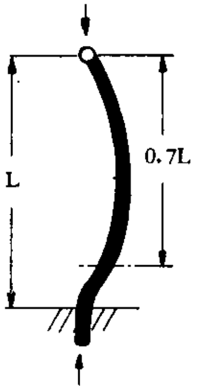
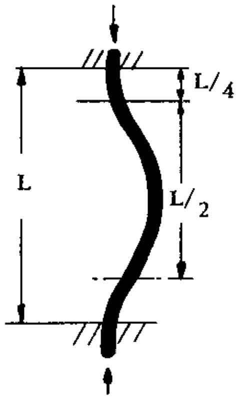
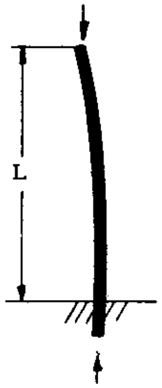
6.1.1 Long Columns

Euler's formula for the primary failure by lateral bending (buckling) of long columns, that is, columns loaded in the elastic range is:-

$$F_c = C\frac{\pi^2 E}{(L/\rho)^2} = \frac{\pi^2 E}{(L/\rho\sqrt{c})^2}$$

$$= \frac{\pi^2 E}{(L'/\rho)^2} \qquad \text{ANC-5, Equation 1.381}$$

Where F_c = allowable compressive stress
E = Modulus of Elasticity
L = length of column
ρ = radius of gyration of cross section
C = fixity coefficient, the value of which depends on the degree of end fixity or constraint. Values of C are given below: they are obtained in GEN/1090/811.1
L/ρ = slenderness ratio
L'/ρ = effective slenderness ratio
L' = effective column length $= \frac{L}{\sqrt{c}}$

PINNED AT EACH END	PINNED AT ONE END FIXED AT OTHER END	FIXED AT BOTH ENDS	FREE AT ONE END FIXED AT OTHER END (LIKE A MAST)
 $C = 1.0$	 $C = \frac{1}{(0.7)^2} = 2.047$	 $C = 4.0$	 $C = 1/4$

ABC in Fig. 6.1.2 – 1 is a typical Euler curve of F_c plotted against the effective slenderness ratio L'/ρ for an aluminium alloy column having $E = 10 \times 10^6$ p.s.i. (Values used in plotting this curve are given in GEN/1090/811.2)

6.1.2 Short Columns

The Euler formula of 6.1.1 cannot be used at the lower end of the L'/ρ scale (in the short column range) because primary bending failure occurs at stresses below those obtained thereby.

This is due to unavoidable eccentricities and to the reduction in slope of the stress-strain curve above the proportional limit of the material, so that, instead of E, it is necessary to use an effective value E' in the Euler formula.

For short columns this becomes:–

$$F_c = C\,\frac{\pi^2 E'}{(L'/\rho)^2} \qquad \text{ANC 5, Equation 1.381 a.}$$

For short columns it is more convenient, however, to employ a transition curve, tangential to the Euler curve at some point B in Fig. 6.1.2–1 and having the value F_{co}, the column yield stress (upper limit of column stress for primary failure), at $L'/\rho = 0$.

This transition curve is either a parabola, as shown in Fig. 6.1.2–1, or a straight line, depending on the material, the general expression for F_c over the short column range, that is, to the left of B, being:–

$$F_c = F_{co}\left\{1 - K\left[\frac{L'/\rho}{\pi\sqrt{E}/F_{co}}\right]^n\right\} \qquad \text{ANC5 Equation 1.382}$$

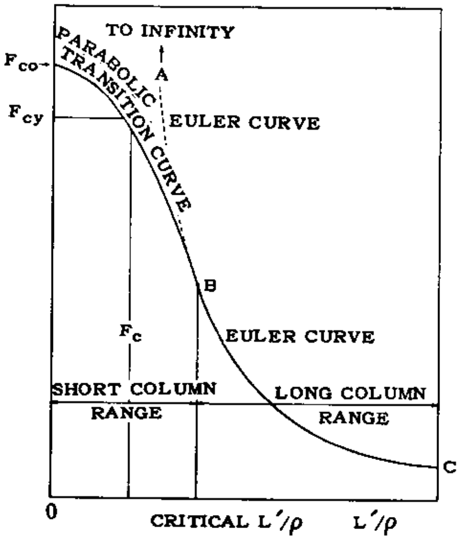


Fig. 6.1.2–1

Here F_{co} is defined above, K is a coefficient and the index n can be 2.0 (Johnson Parabola), 1.5 (Parabola) or 1.0 (straight line), as the case may be.

Expressions for F_c and for the critical value of L'/ρ corresponding to point B, to the left of which columns are "short", are given for n = 2.0, 1.5 and 1.0 in the tabular summary that follows and are proved in GEN/1090/811.3.

F_c = Allowable compressive stress.
F_{co} = Column yield stress (upper limit of column stress for primary failure).
F_{cy} = Compressive yield stress.
} as shown in Fig. 6.1.2 – 1

n	SHORT COLUMN FORMULA	ANC–5 EQUATION	USED ON COLUMN CURVE FOR THESE MATERIALS		F_{cy} k.s.i.	F_{co} k.s.i.	CRITICAL L'/ρ, BELOW WHICH VALUE COLUMN IS SHORT
2.0 PARABOLA (JOHNSON)	$F_{co}\left\{1-\frac{F_{co}(L'/\rho)^2}{4\pi^2 E}\right\}$	1.383	Aluminum alloy 75S–T6	Fig. 6.1.3–3	70	$1.075\,F_{cy} = 75.3$	$1.414\,\pi\sqrt{\frac{E}{F_{co}}} =$ 51
			Plain carbon steel 1025	Fig. 6.1.3–4		36	124
			Heat treated steel F_{tu}=125 ksi	Fig. 6.1.3–5		100	75.6
			Heat treated steel F_{tu}=150 ksi	Fig. 6.1.3–6		135	65.0
			Heat treated steel F_{tu}=180 ksi	Fig. 6.1.3–7		165	58.9
			Magnesium alloy sheet FS – 1h	Fig. 6.1.3–9	(0.025 – 0.064) = 25 (0.065 – 0.260) = 24	$1.05\,F_{cy}$ = 26.25 = 25. 2	70
1.5 PARABOLA	$F_{co}\left\{1-0.3027\left[\frac{L'/\rho}{\pi\sqrt{E}/F_{co}}\right]^{1.5}\right\}$	1.384	Alloy steel 4130	Fig. 6.1.3–8		79.5	$1.528\,\pi\sqrt{\frac{E}{F_{co}}} =$ 91.5
1.0 STRAIGHT LINE	$F_{co}\left\{1-0.385\,\frac{L'/\rho}{\pi\sqrt{E}/F_{co}}\right\}$	1.385	Aluminum alloy 24S–T3	Fig. 6.1.3 –1	41.5	$F_{cy}\left[1+\frac{F_{cy}}{200}\right]$ 50	$1.732\,\pi\sqrt{\frac{E}{F_{co}}}$ 77
			Aluminum alloy 61S–T6	Fig. 6.1.3–2	35.6	42	84

See GEN/1090/811.3 for proofs of short column formulas for F_c and critical L'/ρ.
Long column formula in each case is $F_c = \frac{\pi^2 E}{(L'/\rho)^2}$

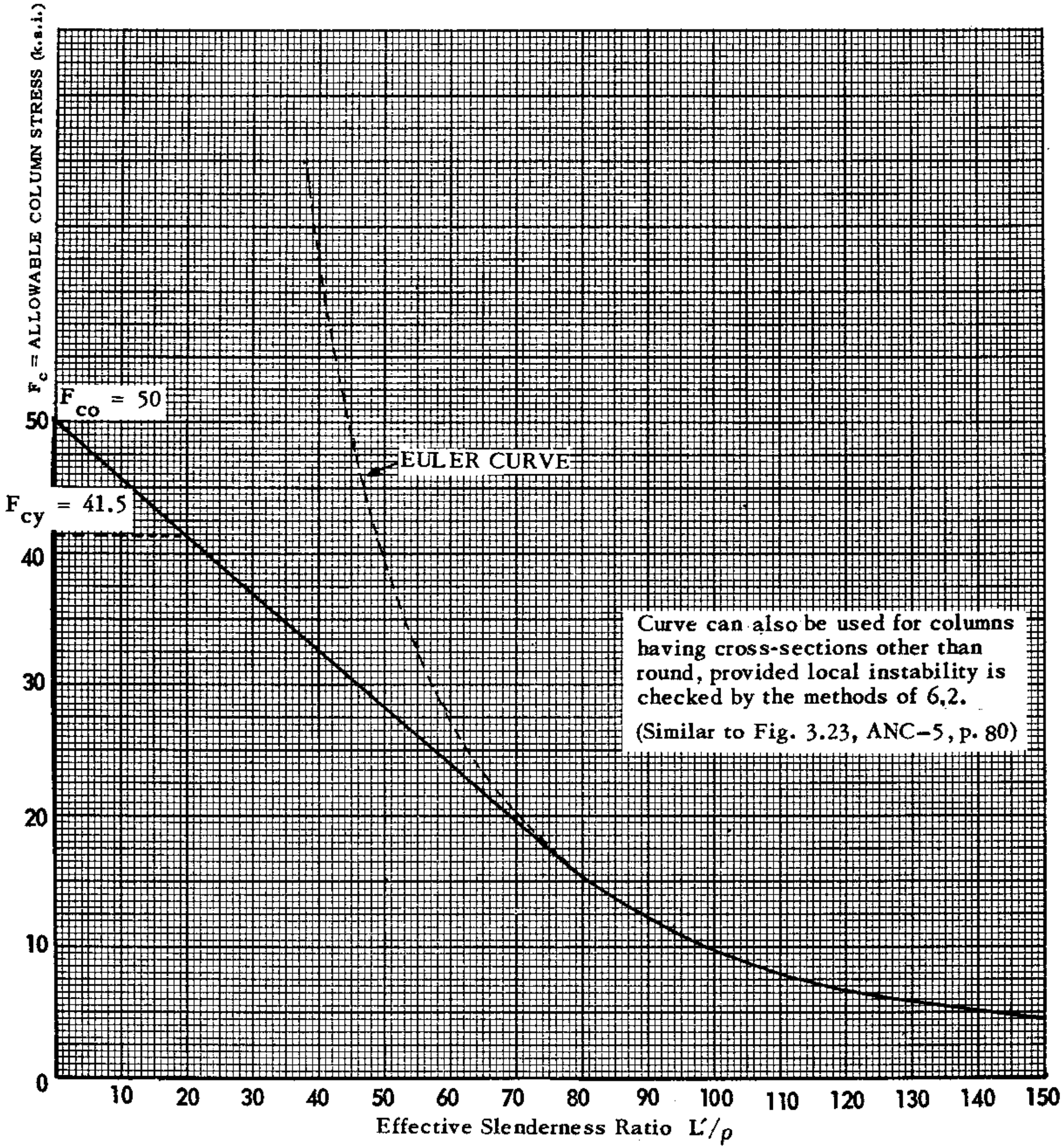
For Magnesium alloy extruded shapes use the expression $F_c = K\,\frac{(F_{cy})^n}{(L'/\rho)^m}$ as in ANC – 5, Table 4.21 (a), p.117.

6.1.2 (cont'd)

Tabular Summary of Formulas for Short Columns

6.1.3 Column Stress Curves

The values of F_{cy} and F_{co} used in obtaining these curves are given in the tabular summary of 6.1.2. Detailed calculations appear in GEN/1090/811.4.



Curve can also be used for columns having cross-sections other than round, provided local instability is checked by the methods of 6.2.

(Similar to Fig. 3.23, ANC–5, p. 80)

Fig. 6.1.3–1 Allowable Column Stress for 24 S–T3

6.1.3 Column Stress Curves (cont'd)

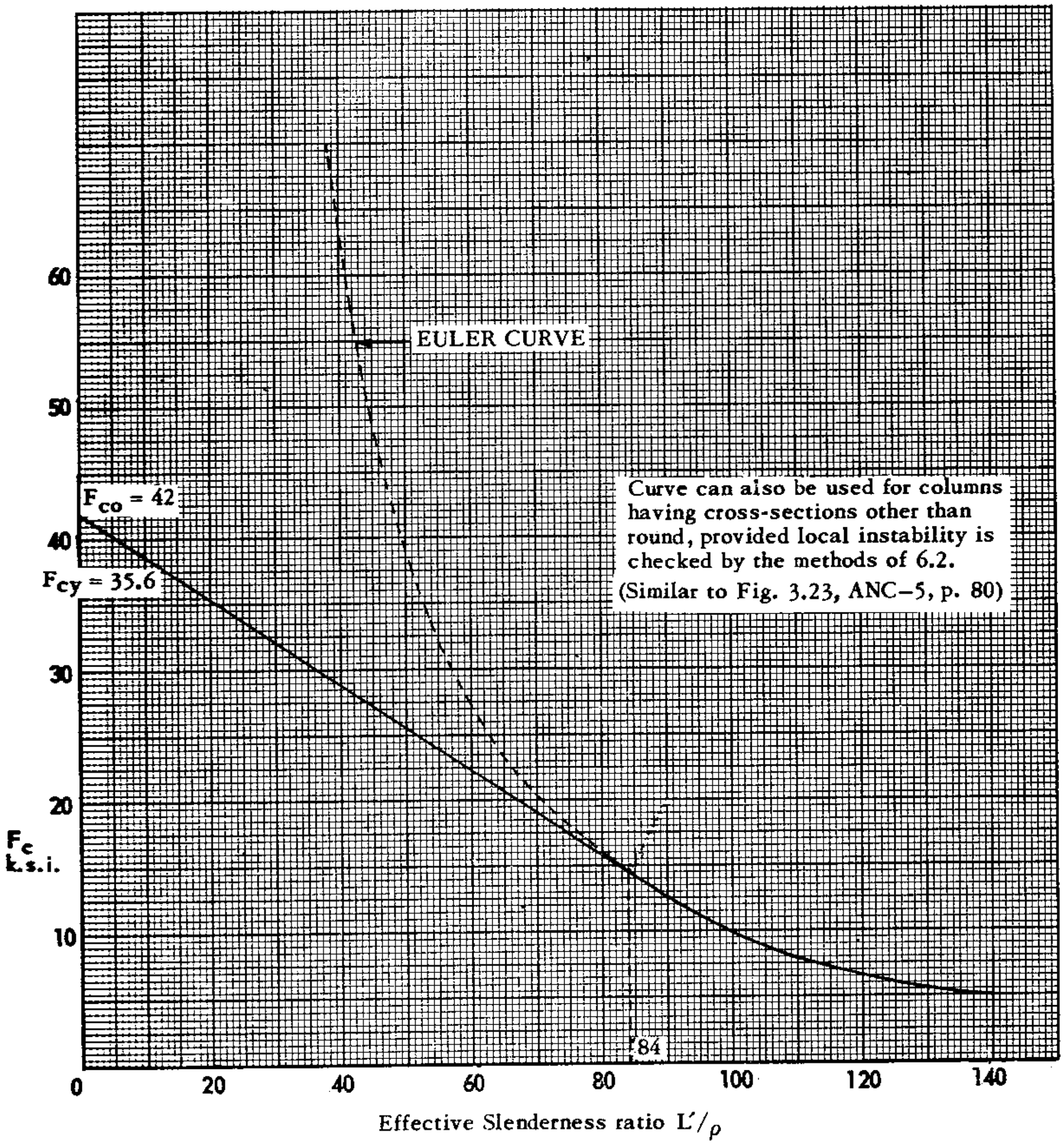


Fig. 6.1.3–2 Allowable Column Stress for 61S–T6

6.1.3 Column Stress Curves (cont'd)

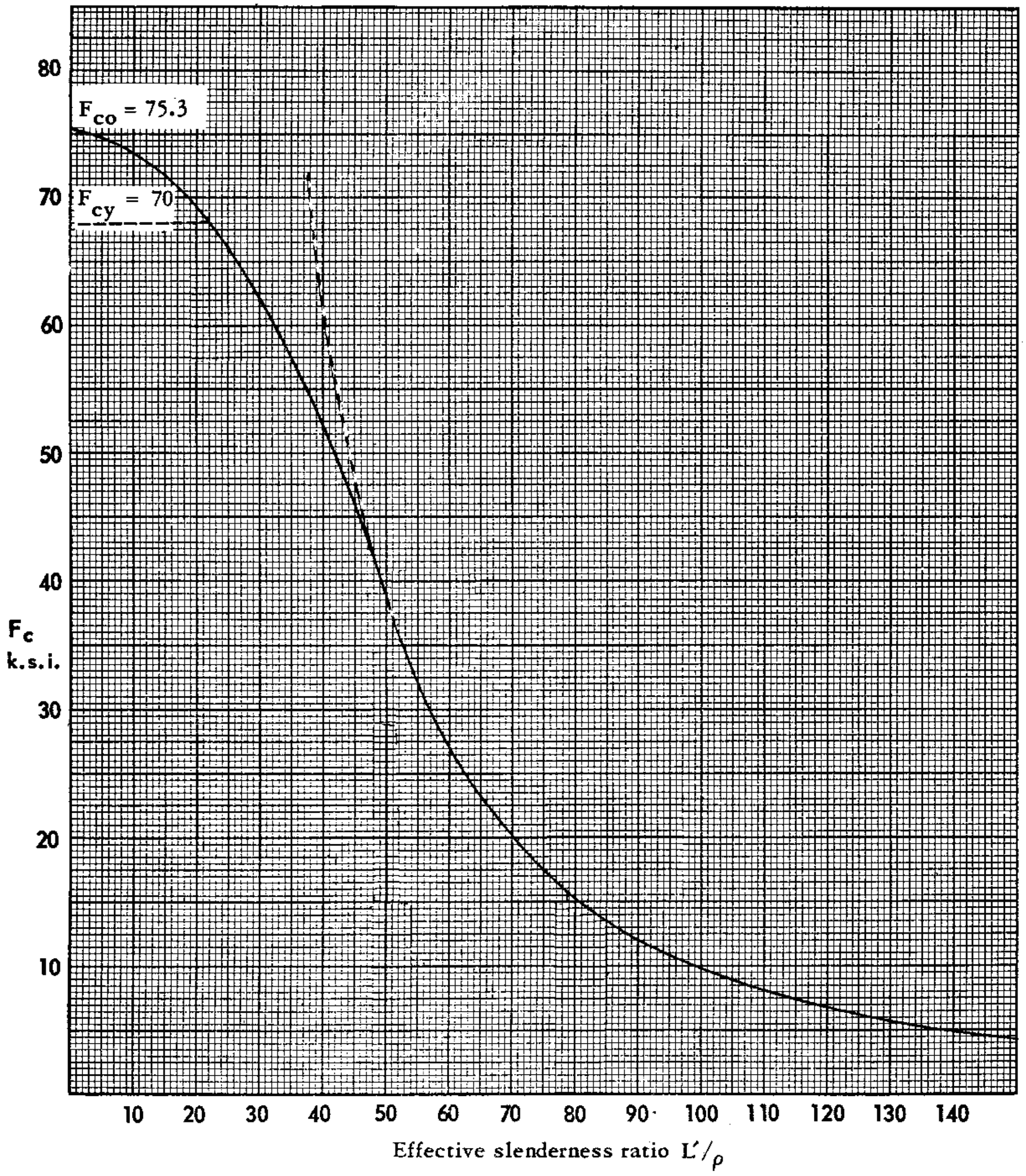
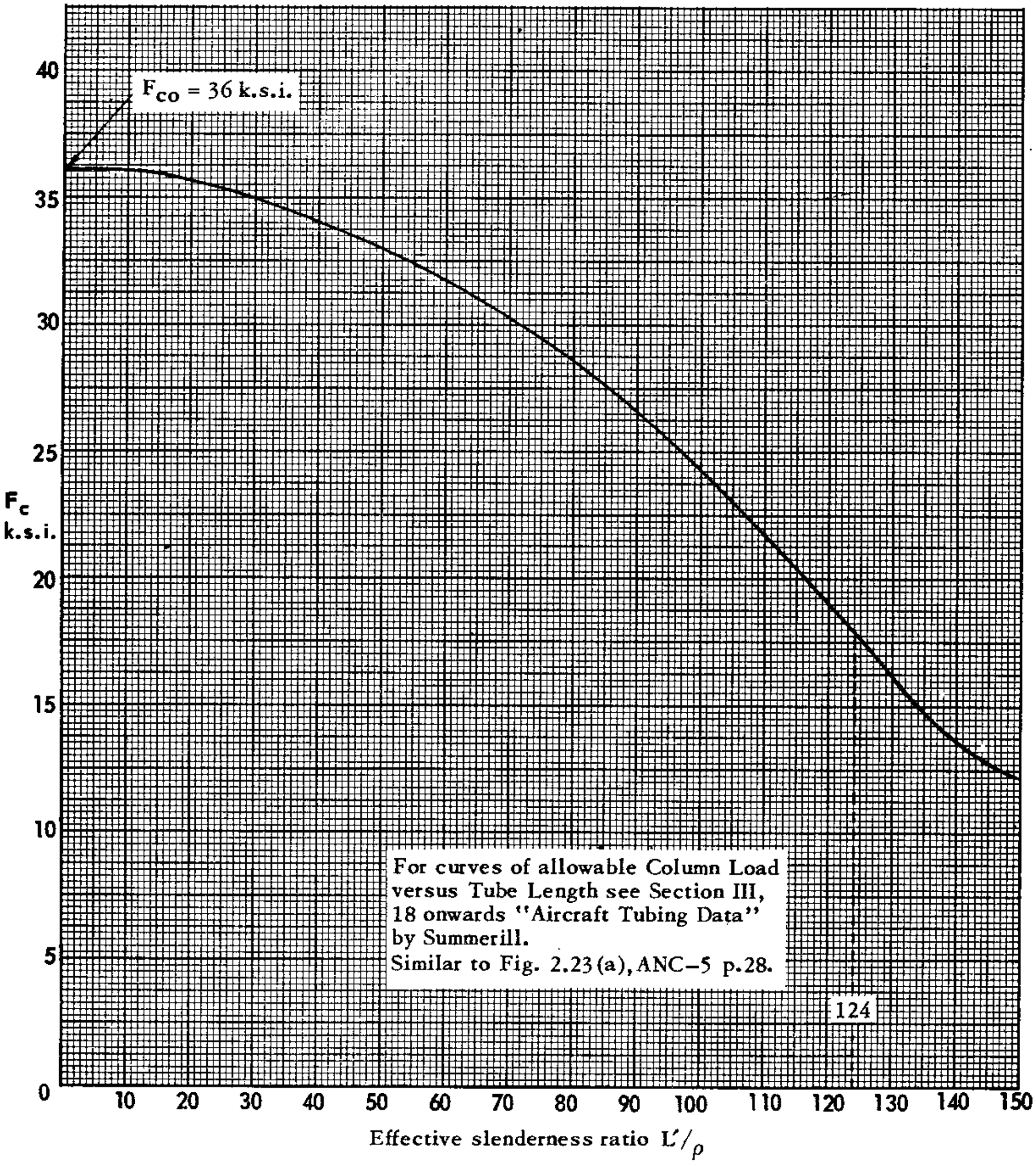


Fig 6.1.3–3 Fig. 6.1.3–3 Allowable Column Stress for 75S–T6

6.1.3 Column Stress Curves (cont'd)



F_{co} = 36 k.s.i.

For curves of allowable Column Load versus Tube Length see Section III, 18 onwards "Aircraft Tubing Data" by Summerill.
Similar to Fig. 2.23 (a), ANC-5 p.28.

Fig. 6.1.3-4 Allowable Column Stress for Plain Carbon Steel 1025 Round Tubing

6.1.3 Column Stress Curves (cont'd)

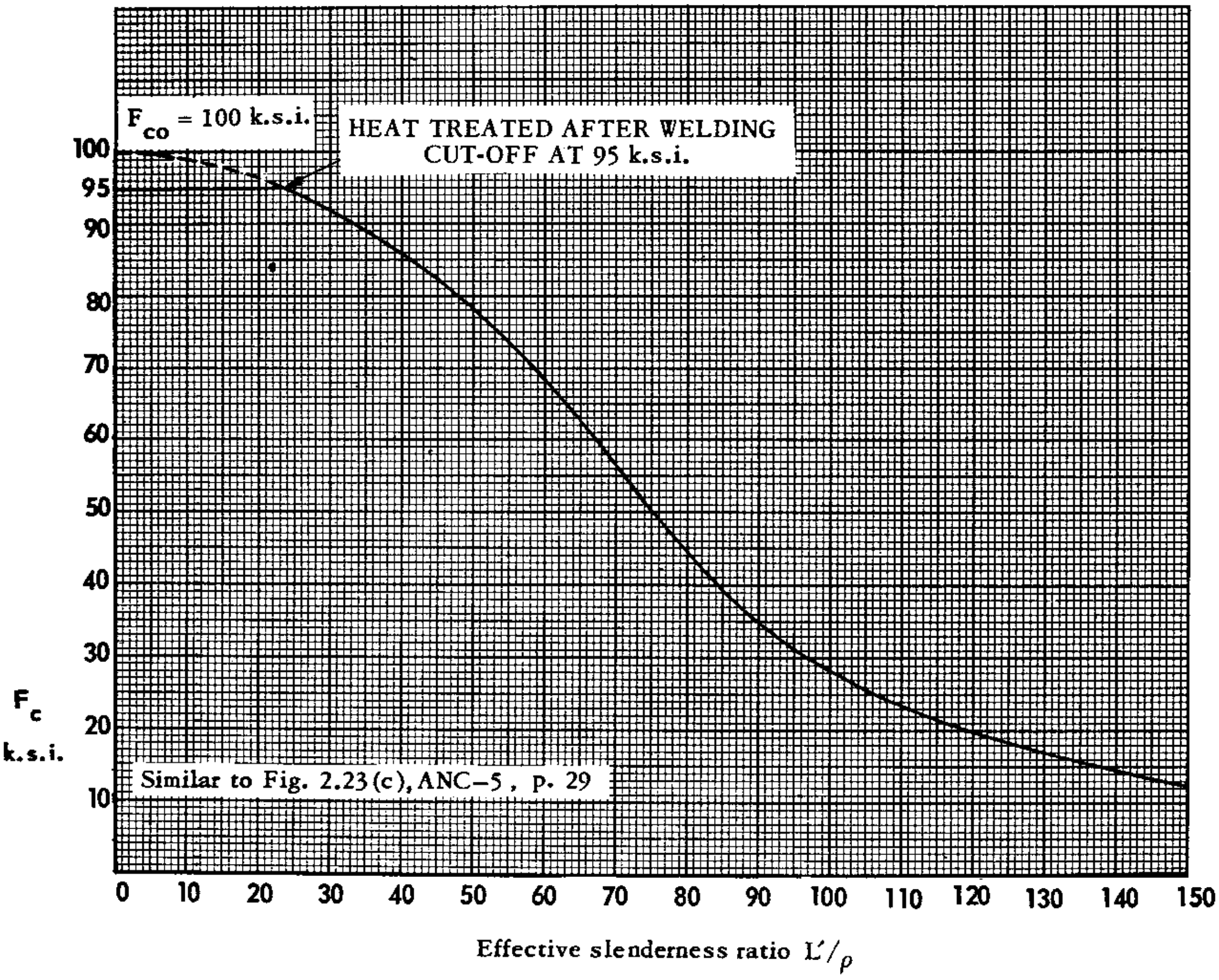


Fig. 6.1.3–5 Allowable Column Stress Curve for Heat-Treated Alloy Steel Round Tubing F_{tu} = 125 k.s.i.

6.1.3 Column Stress Curves (cont'd)

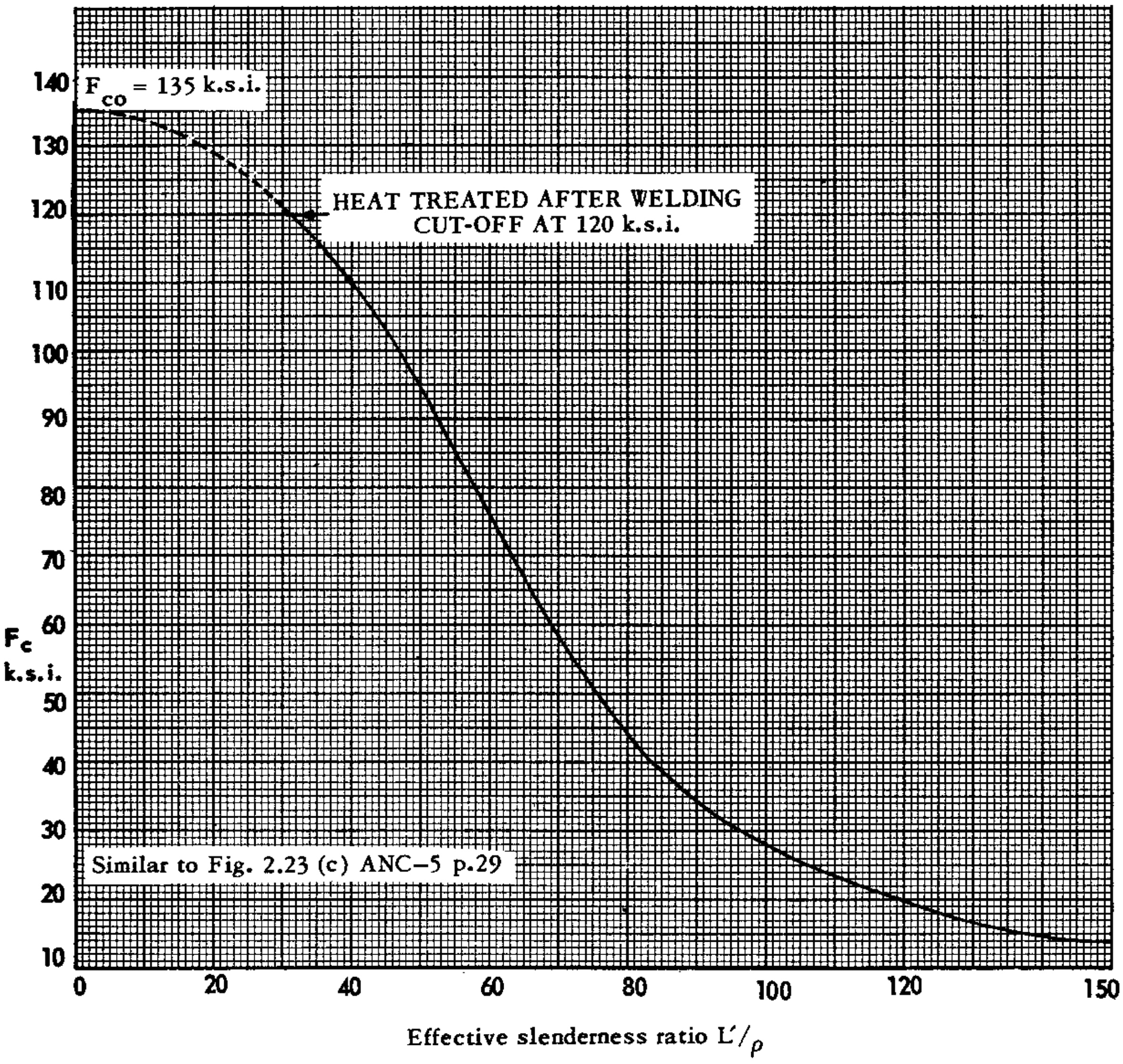


Fig. 6.1.3–6 Heat Treated Alloy Steel Round Tubing F_{tu} = 150 k.s.i.

6.1.3 Column Stress Curves (cont'd)

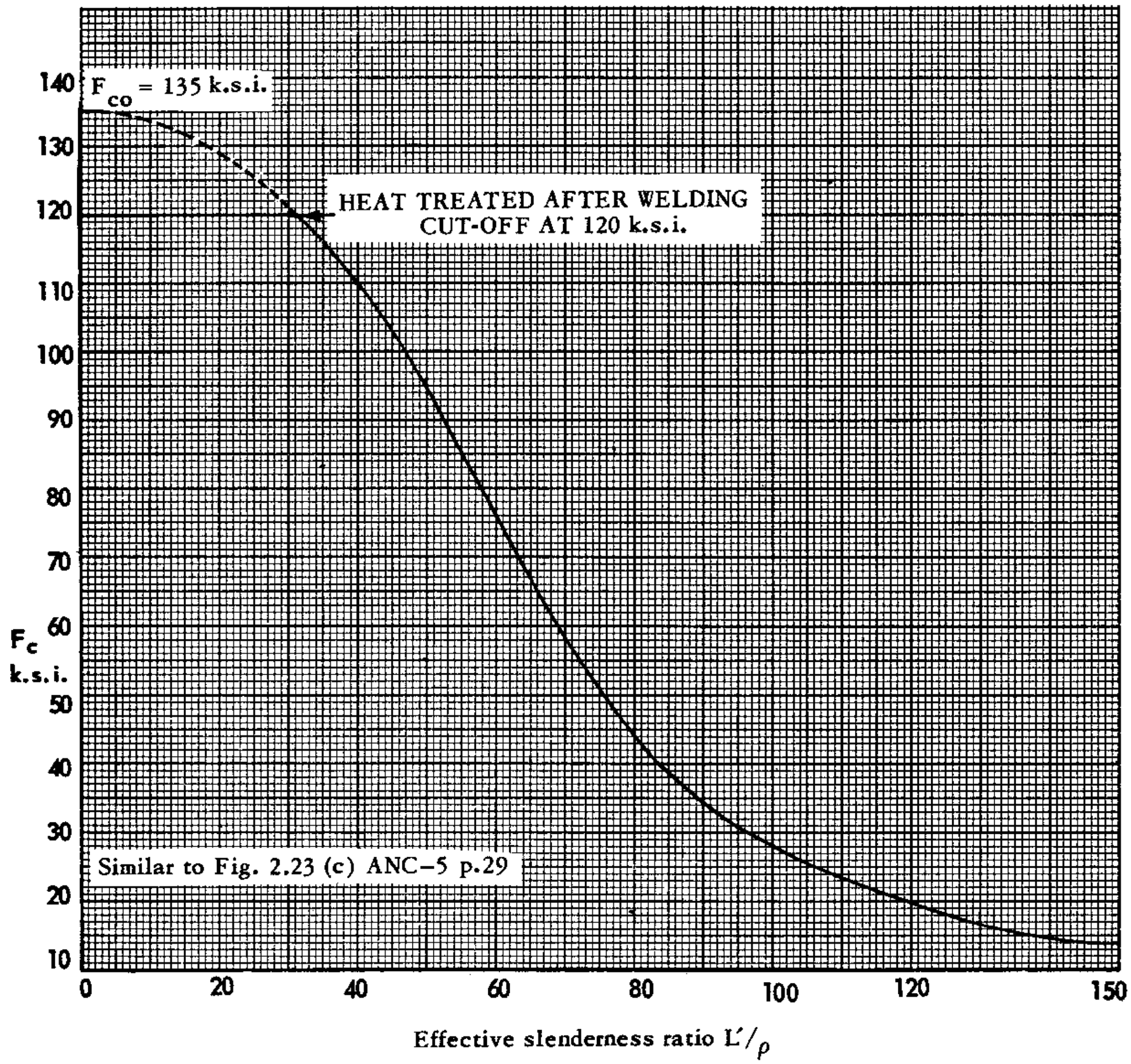


Fig. 6.1.3–6 Heat Treated Alloy Steel Round Tubing F_{tu} = 150 k.s.i.

6.1.3 Column Stress Curves (cont'd)

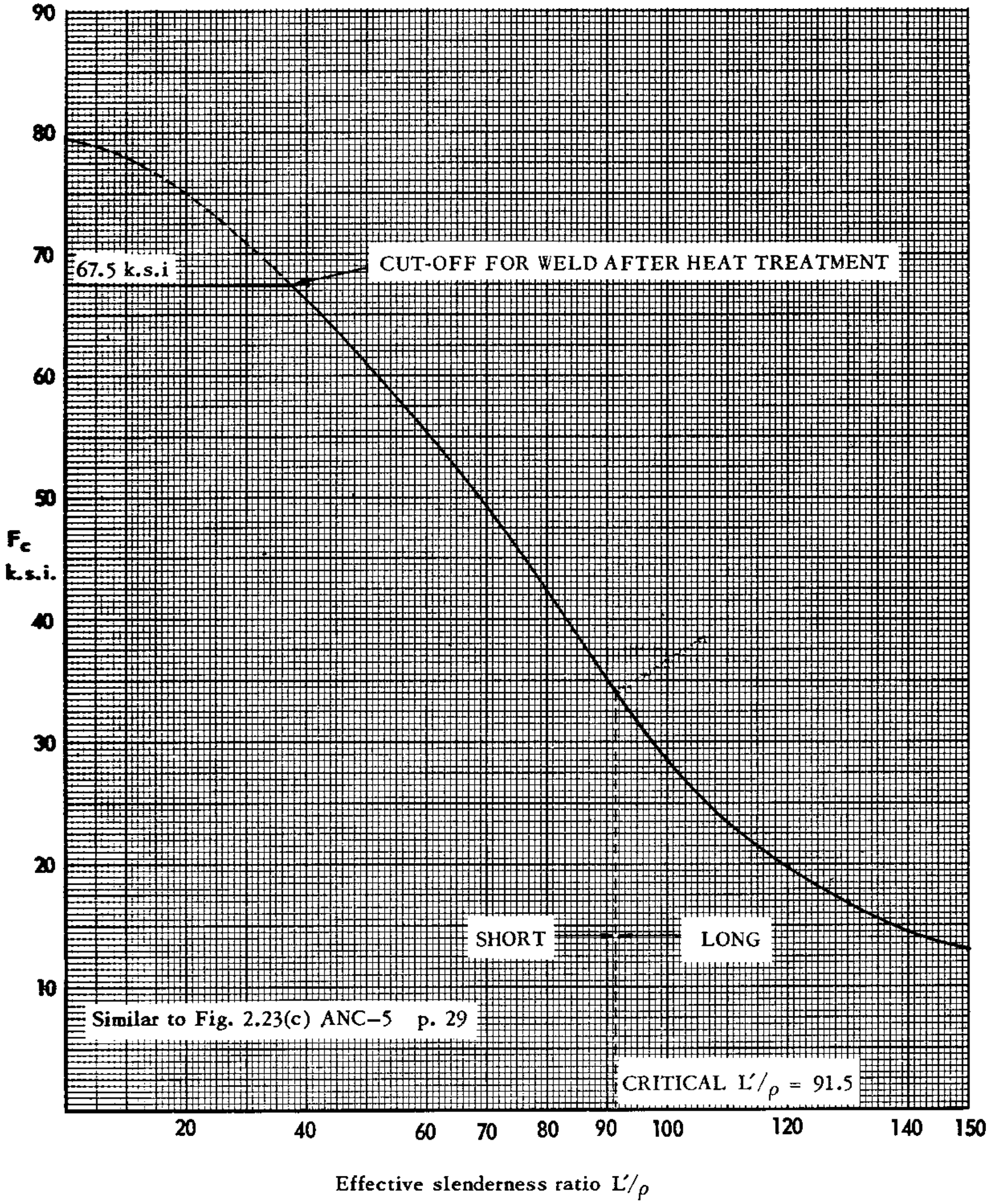


Fig. 6.1.3–8 Alloy Steel 4130 Round Tubing

6.1.3 Column Stress Curves (cont'd)

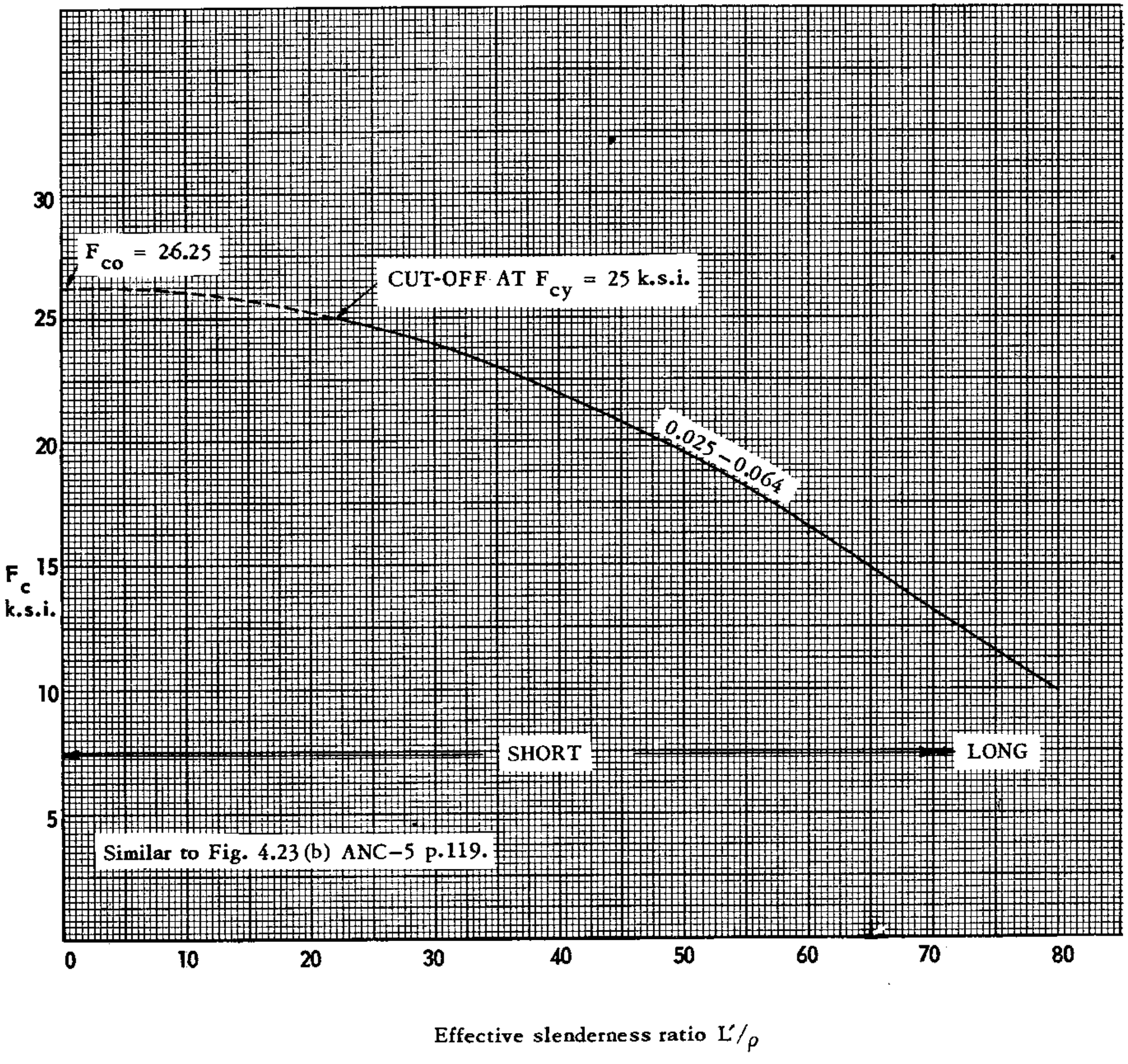


Fig. 6.1.3–9 Magnesium Alloy Sheet FS–1h

6.1.3 Column Stress Curves (cont'd)

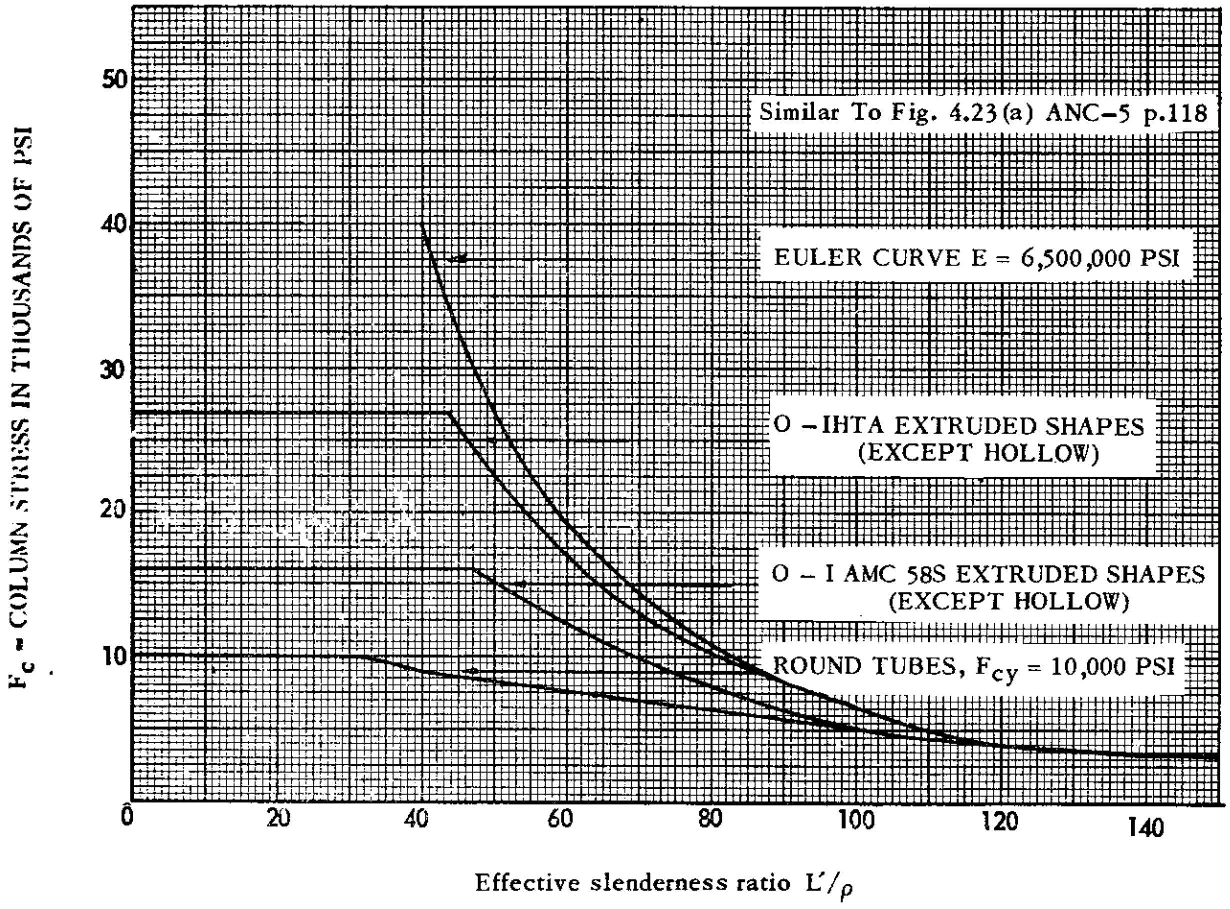
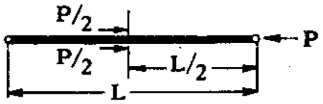
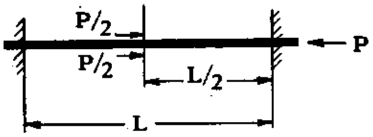
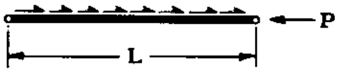


Fig. 6.1 3–10 Magnesium Alloy Columns

6.1.4 Miscellaneous Formulas

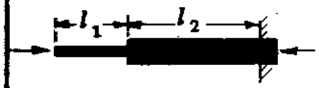
	$P_{cr} = 1.89 \frac{\pi^2 EI}{L^2}$
	$P_{cr} = 7.58 \frac{\pi^2 EI}{L^2}$
	$P_{cr} = 1.87 \frac{\pi^2 EI}{L^2}$
	$P_{cr} = 7.24 \frac{\pi^2 EI}{L^2}$
	$P_{cr} = 3.17 \frac{\pi^2 EI}{L^2}$
	$P_{cr} = 7.68 \frac{\pi^2 EI}{L^2}$

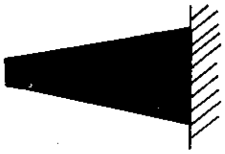
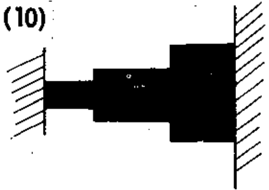
Note that the above information, while believed to be correct, has not been substantiated and should therefore be used with caution.

6.3 BUCKLING LOAD OF COLUMNS OF VARYING CROSS SECTION

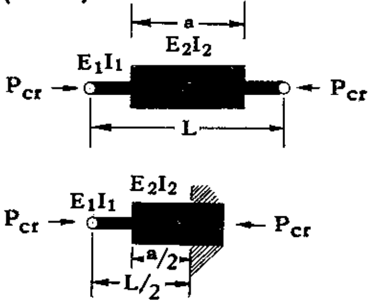
TABLE OF CONTENTS

6.3.1 Long Columns of Varying Section

(1) PINNED – PINNED 	See Fig. 6.3.1–1
(2) PINNED – PINNED 	See Fig. 6.3.1–2
(3) PINNED – PINNED 	See Fig. 6.3.1–3
(4) PINNED – FIXED 	See Fig. 6.3.1–1
(5) FREE – FIXED  (LIKE A MAST)	$\tan \mu_1 l_1 \tan \mu_2 l_2 = \frac{\mu_1}{\mu_2}$ See Numerical example in GEN/1090/811.5

(6) PINNED – PINNED 	See Roark, p. 298, Table XV
(7) FIXED – FIXED 	See Roark, p. 298, Table XV
(8) PINNED – PINNED  ELLIPTICAL TAPER	See Fig. 6.3.1–4
(9) 	See J. Ae. Sc., April 54, and J. App. Mech. Dec.'51, 415.
(10) 	J. App. Mech. Dec.'51, 415

6.3.1 Long Columns of Varying Section (cont'd)



Buckling Load $P_{cr} = m \dfrac{E_2 I_2}{L^2}$

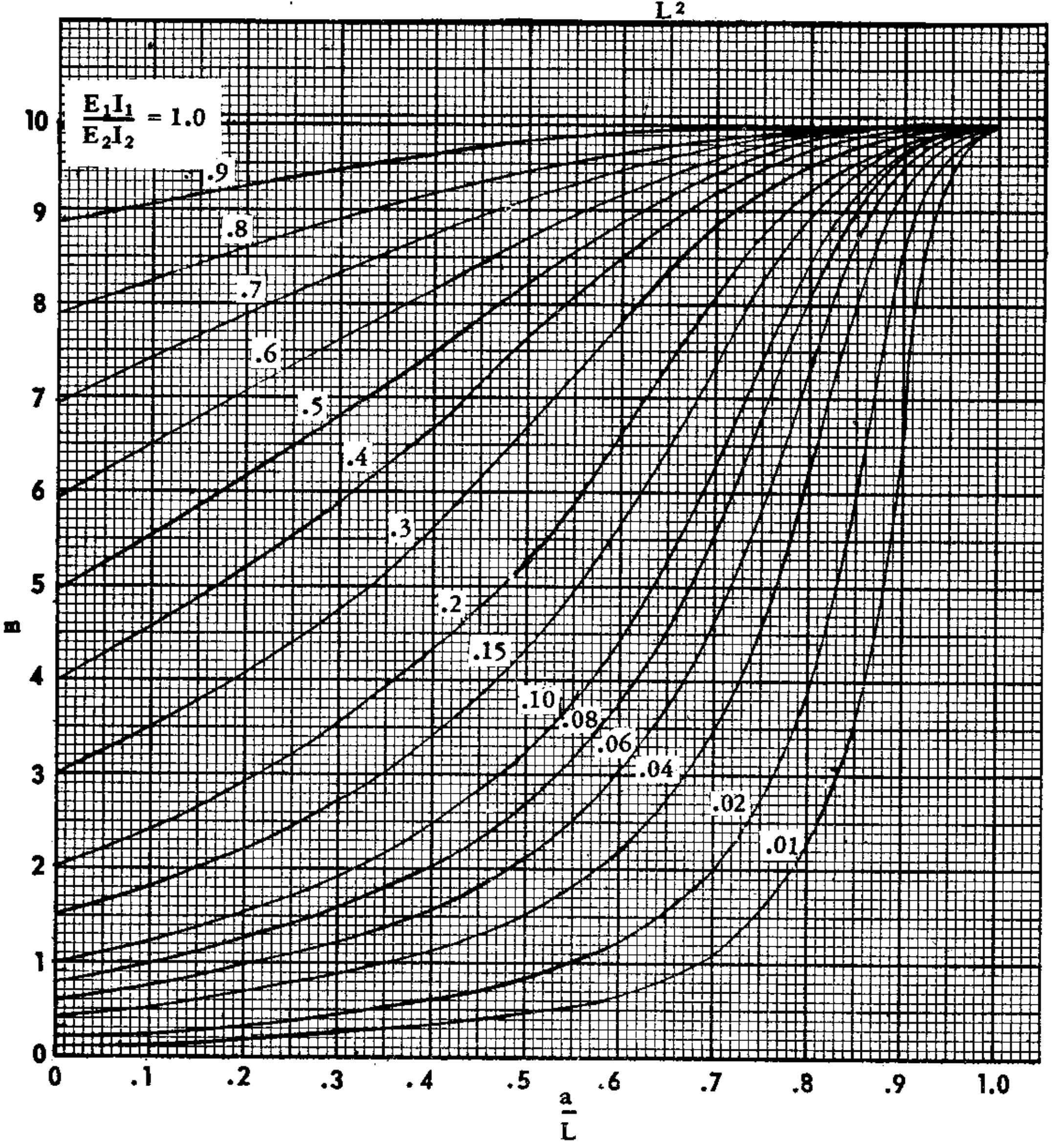


Fig. 6.3.1–1 Buckling Load for Column of Varying Section

6.3.1 Long Columns of Varying Section (cont'd)

Buckling Load $P_{cr} = m \dfrac{E_1I_1}{L^2}$

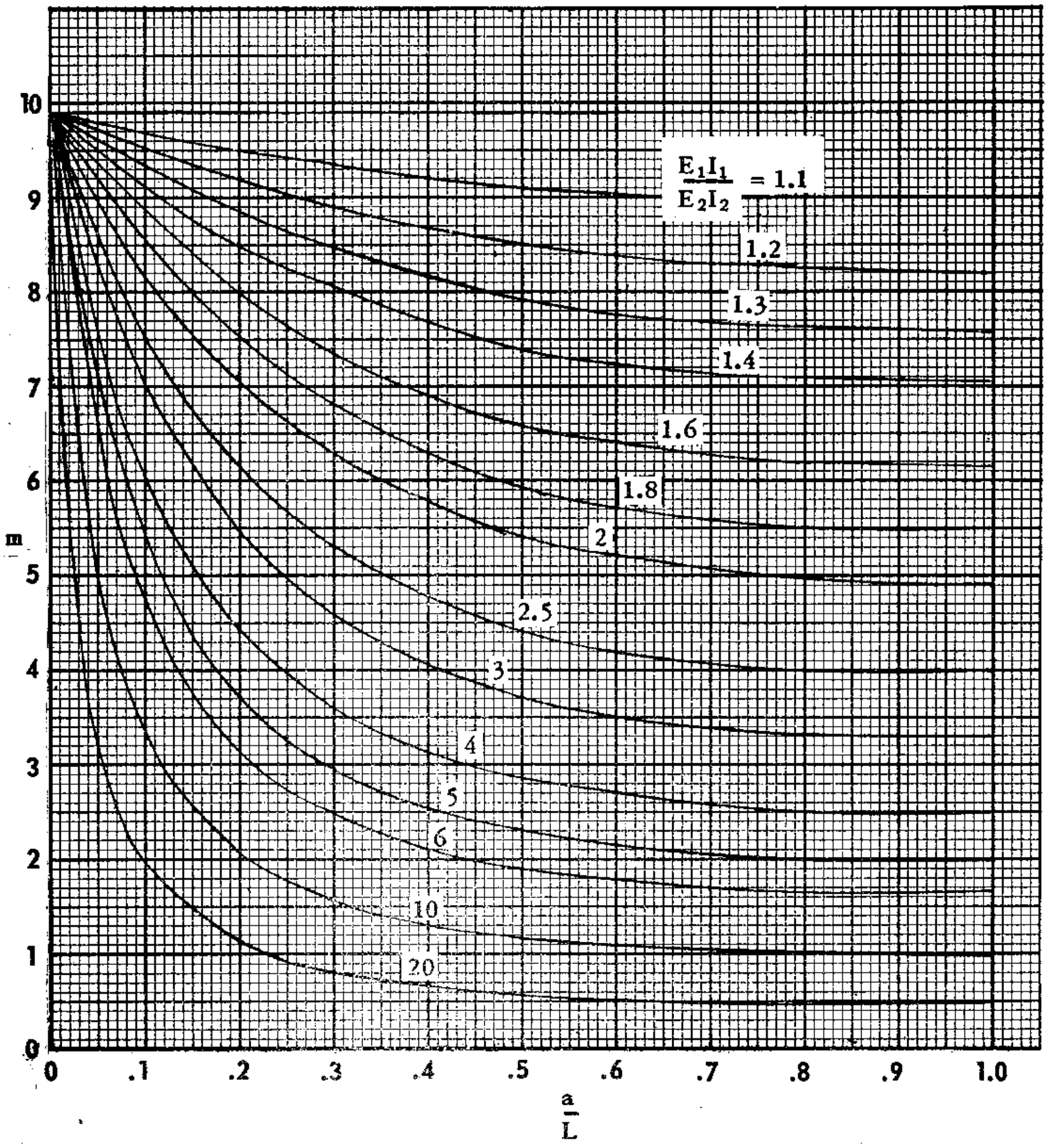
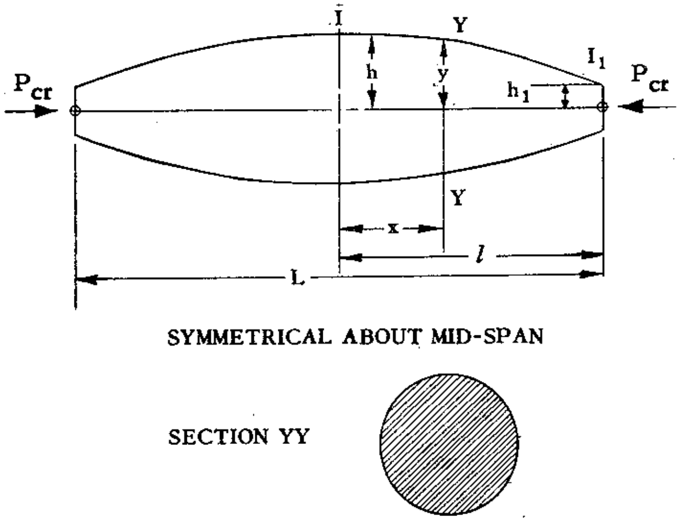


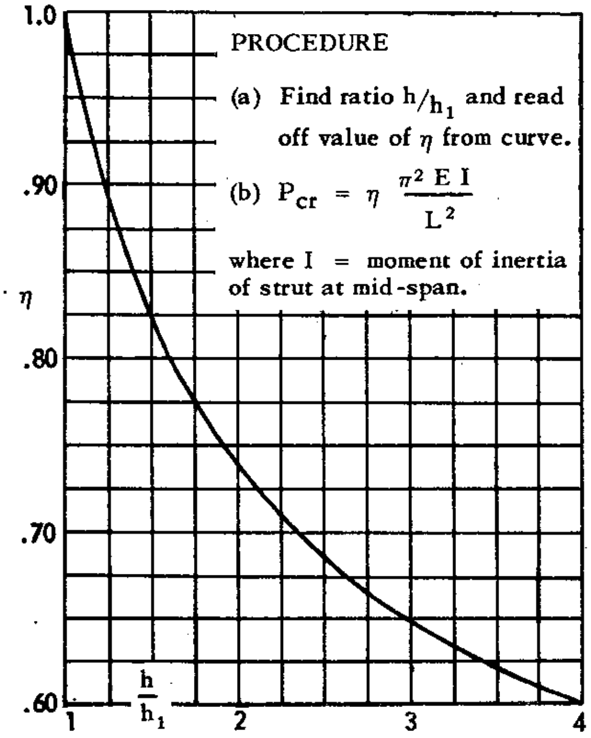
Fig. 6.3.1–2 Buckling Load for Column of Varying Section

6.3.1 Long Columns of Varying Section (cont'd)
Pinned-Pinned Strut with Elliptical Taper



Equation to profile is $y^2 = h^2 - (h^2 - h_1^2)\left(\frac{x}{l}\right)^2$

See GEN/1090/812.1 for derivation.



PROCEDURE

(a) Find ratio h/h_1 and read off value of η from curve.

(b) $P_{cr} = \eta \frac{\pi^2 E I}{L^2}$

where I = moment of inertia of strut at mid-span.

Fig. 6.3.1-4

Pinned-Pinned Strut with Elliptical Taper

6.3.2 Approximate General Method for Pinned Ends

Procedure

(1) Divide the column length into 20 equal increments

(2) Compute I at the mid-point of each increment

(3) Evaluate $\frac{k_1}{EI}$ (or $\frac{k_1}{I}$ if E is constant), the value of k_1 for each increment being obtained from Column 2 of the table below.

Note that these values of k_1 *are the same for any pin-ended column and are independant of* I *and the length of the column.* If the column is symmetrical about its mid-length, it is only necessary to consider 10 increments and double the value of $\Sigma\frac{k_1}{EI}$ so obtained.

INCREMENT	1	2	3	4	5
k_1	.020	.177	.476	.887	1.370
INCREMENT	20	19	18	17	16
INCREMENT	6	7	8	9	10
k_1	1.879	2.362	2.773	3.072	3.229
INCREMENT	15	14	13	12	11

(4) Critical Buckling Load

$$P_{cr} = \frac{320 \times (0.9)}{L^2 \Sigma_o^L \frac{k_1}{EI}}$$

$$= \frac{288\ E}{L^2 \Sigma_o^L \frac{k_1}{I}}$$ for constant E.

6.3.2.1 Numerical Example

Pinned - pinned column with elliptical taper, shown in Fig. 6.3.2–1.

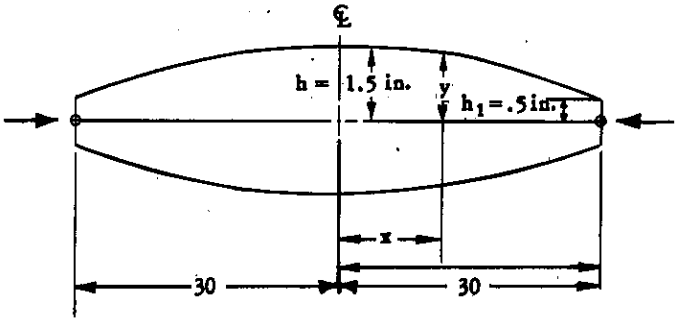


Fig. 6.3.2–1

From GEN/1090/812.1, the equation to an elliptical profile is:

$$y^2 = h^2 - (h^2 - h_1^2)\ \left(\frac{x}{l}\right)^2$$

Values of y and $I = \frac{\pi}{4} y^4$ are given in the table below and plotted in Fig. 6.3.2–2.

$$(h^2 - h_1^2) = (2.25 - .25) = 2.0$$

x	0	5	10	15	20	25	30
(x/l)	0	.167	.33	.50	.66	.83	1.0
$(x/l)^2$	0	.0278	.11	.25	.442	.687	1.0
$(h^2 - h_1^2)(x/l)^2$	0	.0556	.22	.50	.884	1.374	2.0
y^2	2.25	2.194	2.03	1.75	1.366	.876	.25
y	1.5	1.481	1.426	1.322	1.170	.937	.50
y^4	5.07	4.80	4.12	3.06	1.865	.765	.0625
I	3.98	3.77	3.24	2.40	1.464	.601	.049

The half column length is now divided into ten equal increments and the value of I at the centre of each increment obtained from Fig. 6.3.2–2 and tabulated.

6.3.2.1 Numerical Example (cont'd)

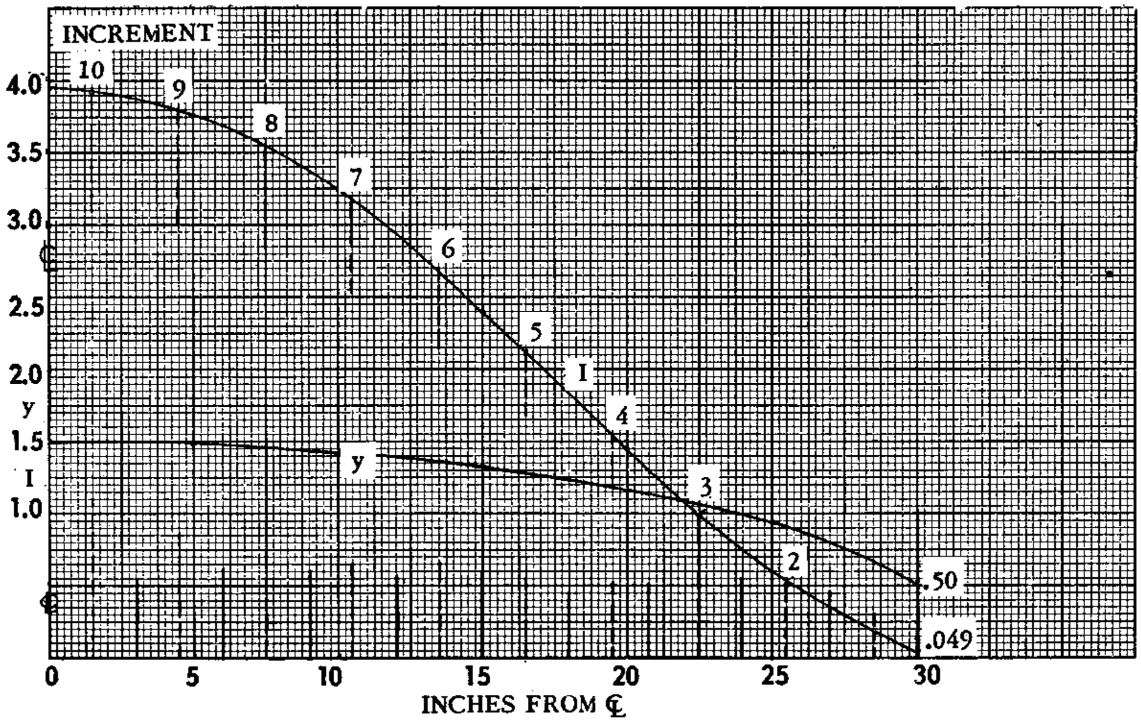


Fig. 6.3.2–2 Plot of y and I

INCREMENT	k_1	I	k_1/I
1	.02	.20	.100
2	.177	.50	.354
3	.476	1.0	.476
4	.887	1.56	.568
5	1.370	2.12	.646
6	1.879	2.66	.706
7	2.362	3.16	.747
8	2.773	3.56	.778
9	3.072	3.78	.812
10	3.229	3.93	.822

$$\Sigma \frac{k_1}{I} = 6.009$$

$$\Sigma_O^L \frac{k_1}{I} = 2 \times 6.009 = 12.018$$

$$P_{cr} = \frac{288E}{L^2 \Sigma_O^L \frac{k_1}{I}} = \frac{288 \times 10^7}{3600 \times 12.018}$$

= $\underline{67000 \text{ lb.}}$ (approximately)

Check from Fig. 6.3.1–4

For $\frac{h}{h_1} = 3.0$, $\eta = .649$

$$P_{cr} = .649 \frac{\pi^2 EI}{L^2}$$

$$= .649 \frac{9.85 \times 10^7 \times 3.98}{3600}$$

$$= .649 \times 109{,}000$$

$$= \underline{70800 \text{ lb.}}$$

6.4 BUCKLING LOAD OF CONTINUOUS COLUMNS

TABLE OF CONTENTS

This section is concerned with the determination of the buckling load of columns, continuous over two or three spans, carrying axial load but no lateral load.

6.4.1 Two Span

6.4.1.1 Two Span, Pinned at Ends. E, I and P constant. (Fig. 6.4.1–1).

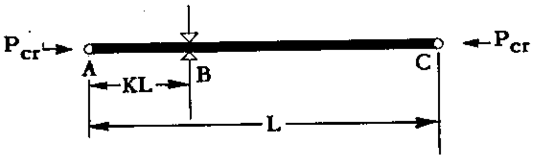


Fig. 6.4.1–1

Buckling load $P_{cr} = \beta^2 \dfrac{EI}{L^2}$

where L = overall length of column

β = a coefficient, plotted in Fig. 6.4.1–2 against K

K = ratio $\dfrac{\text{length AB}}{\text{length AC}}$

Fig. 6.4.1–2 is similar to Curve (1), Fig. 2, J. App. Mech., Dec. 1942, p. A–190. The method of obtaining $P_{cr} = 33.5 \dfrac{EI}{L^2}$ and hence $\beta = 5.8$ for the particular case in which one span is twice the other ($K = \dfrac{1}{3}$) is given in the numerical example of GEN/1090/811.6.

Clearly when $K = \dfrac{1}{2}$ the set up is the same as for the second column mode (b) given in GEN/1090/811.6 and consequently $P_{cr} = 4\pi^2 \dfrac{EI}{L^2}$

and $\beta = 4\pi^2$

$\beta = 2\pi = 6.28$, as in Fig. 6.4.1–2.

Also when K = 0 or K = 1.0, the problem reduces to the standard case of a column with one end pinned and the other end fixed.

$$P_{cr} = 2.047\,\pi^2 \frac{EI}{L^2} \text{ from 6.1.1}$$

$$\beta^2 = 2.047\,\pi^2$$

$$\beta = 4.5$$

6.4.1 Two Span (cont'd)

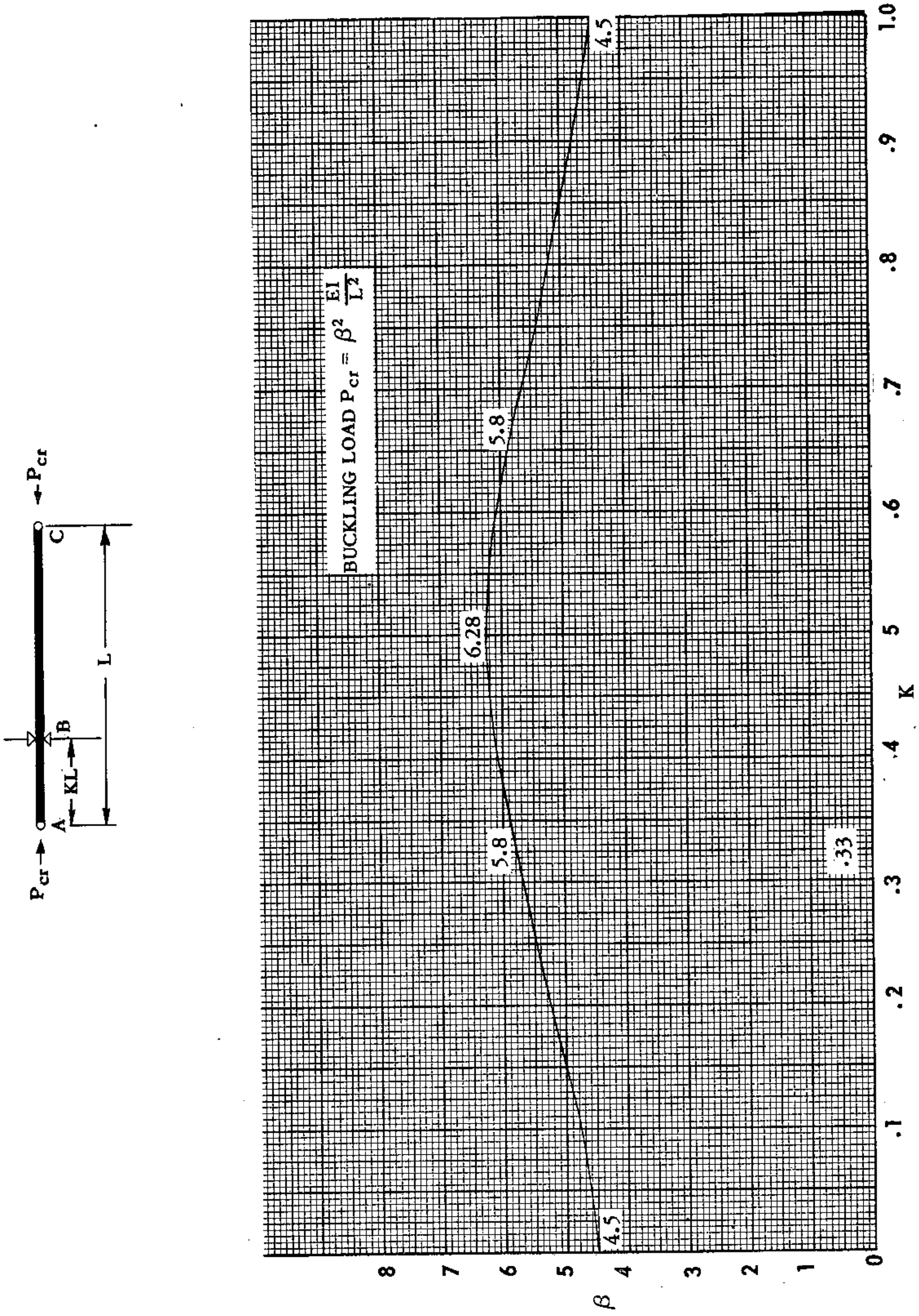


Fig. 6.4.1-2 Buckling Load for Two Span Continuous Column Pinned at Ends
E, I and P Constant

6.4.1.2 Two Span: Pinned at Ends: E.I. and P not constant.

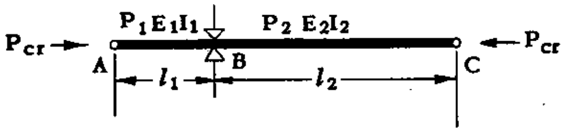


Fig. 6.4.1–3

The following method is from "Buckling of Struts and Frameworks", Ratzersdorfer, p. 280 (in German).

Let

P_1 = applied end load in span AB

P_2 = applied end load in span BC

I_1 = moment of inertia of cross section of span AB

I_2 = moment of inertia of cross sectio. of span BC

E_1 = modulus of elasticity of span AB

E_2 = modulus of elasticity of span BC

l_1 = span AB such that $\frac{l_1}{E_1I_1}$ is less than $\frac{l_2}{E_2I_2}$

l_2 = span BC

Equivalent pin-jointed length of span AB, $\bar{l}_1 = \frac{l_1}{C_1}$

Equivalent pin-jointed length of span BC, $\bar{l}_2 = \frac{l_2}{C_2}$

where C_1 and C_2 are obtained from Fig. 6.4.1–4.,

knowing the ratio $\frac{l_1}{l_2} \cdot \frac{E_2I_2}{E_1I_1}$ for a particular value

of $\lambda = \frac{P_2}{P_1} \cdot \frac{E_2I_2}{E_1I_1}$

The critical loads are:

$$P_{1\,cr} = \frac{\pi^2 E_1 I_1}{\bar{l}_1^{\,2}}$$

$$P_{2\,cr} = \frac{\pi^2 E_2 I_2}{\bar{l}_2^{\,2}}$$

ILLUSTRATIVE EXAMPLE

$E_1 = E_2,\ I_1 = I_2,\ l_1 = 0.5\ l_2$ and $P_1 = P_2$

$$\frac{l_1}{l_2}\ \frac{E_2I_2}{E_1I_1} = 0.5$$

$$\lambda = \frac{P_2}{P_1}\ \frac{E_2I_2}{E_1I_1} = 1.0$$

From Fig. 6.4.1–4 $C_1 = 0.613$
$C_2 = 1.23$

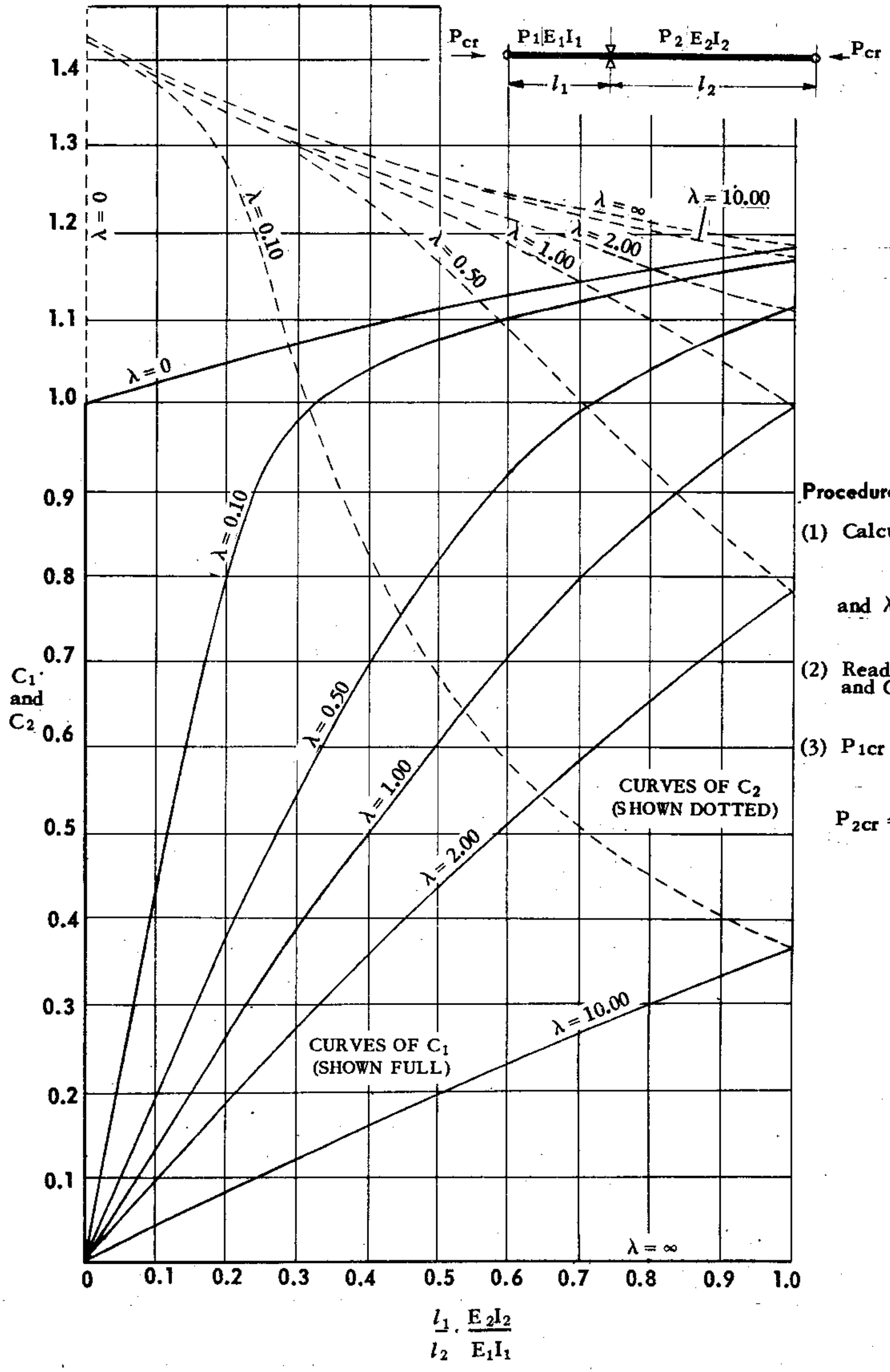
$$\bar{l}_1 = \frac{l_1}{C_1} = 1.63\ l_1 \qquad \bar{l}_1^{\,2} = 2.645$$

$$\bar{l}_2 = \frac{l_2}{C_2} = \frac{l_2}{1.23} = 0.812\ l_2$$

$$P_{1\,cr} = \frac{\pi^2\, EI}{\bar{l}_1^{\,2}} = \frac{\pi^2\, EI}{2.645\, l_1^{\,2}} = 3.72\ \frac{EI}{l_1^{\,2}} = 33.5\ \frac{EI}{L^2}$$

This is the same result as quoted in 6.4.1.1.

6.4.1.2 Two Span (cont'd)



Procedure

(1) Calculate $\frac{l_1}{l_2} \cdot \frac{E_2I_2}{E_1I_1}$

and $\lambda = \frac{P_2}{P_1} \; \frac{E_2I_2}{E_1I_1}$

(2) Read off C_1 from the full curves and C_2 from the dotted curves

(3) $P_{1cr} = \frac{\pi_2 E_1I_1}{\bar{l}_1^2}$ where $\bar{l}_1 = \frac{l_1}{C_1}$

$P_{2cr} = \frac{\pi_2 E_2I_2}{\bar{l}_2^2}$ where $\bar{l}_2 = \frac{l_2}{C_2}$

Fig. 6.4.1–4 Buckling Load for Two Span Continuous Column, Pinned at Ends. E, I and P not constant

6.4.1.3 Two Span – Pinned at A: Fixed at C (Fig. 6.4.1–5) E, I and P constant

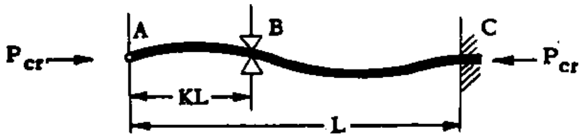


Fig. 6.4.1–5

Buckling load $P_{cr} = \beta^2 \frac{EI}{L^2}$

Clearly when K = 0, the column is fixed – fixed

$$\text{and } P_{cr} = 4\pi^2 \frac{EI}{L^2} \text{ (from 6.4.1.1 and 6.1.1)}$$

$$\beta^2 = 4\pi^2$$

$$\beta = 2\pi = 6.28$$

When K = L the column is pinned at A and fixed at C without an intermediate support and

$$P_{cr} = 2.047\,\pi^2 \frac{EI}{L^2} \qquad \text{from 6.1.1}$$

$$\beta^2 = 2.047\,\pi^2$$

$$\beta = 4.5.$$

Values of β are plotted against K in Fig. 6.4.1–6.

6.4.1.4 Two Span – Fixed at Ends A and C (Fig. 6.4.1–7) E, I and P constant

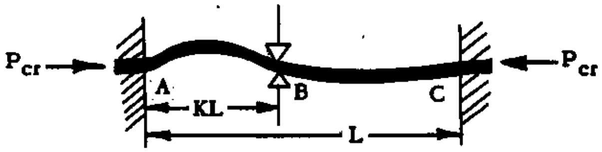


Fig. 6.4.1–7

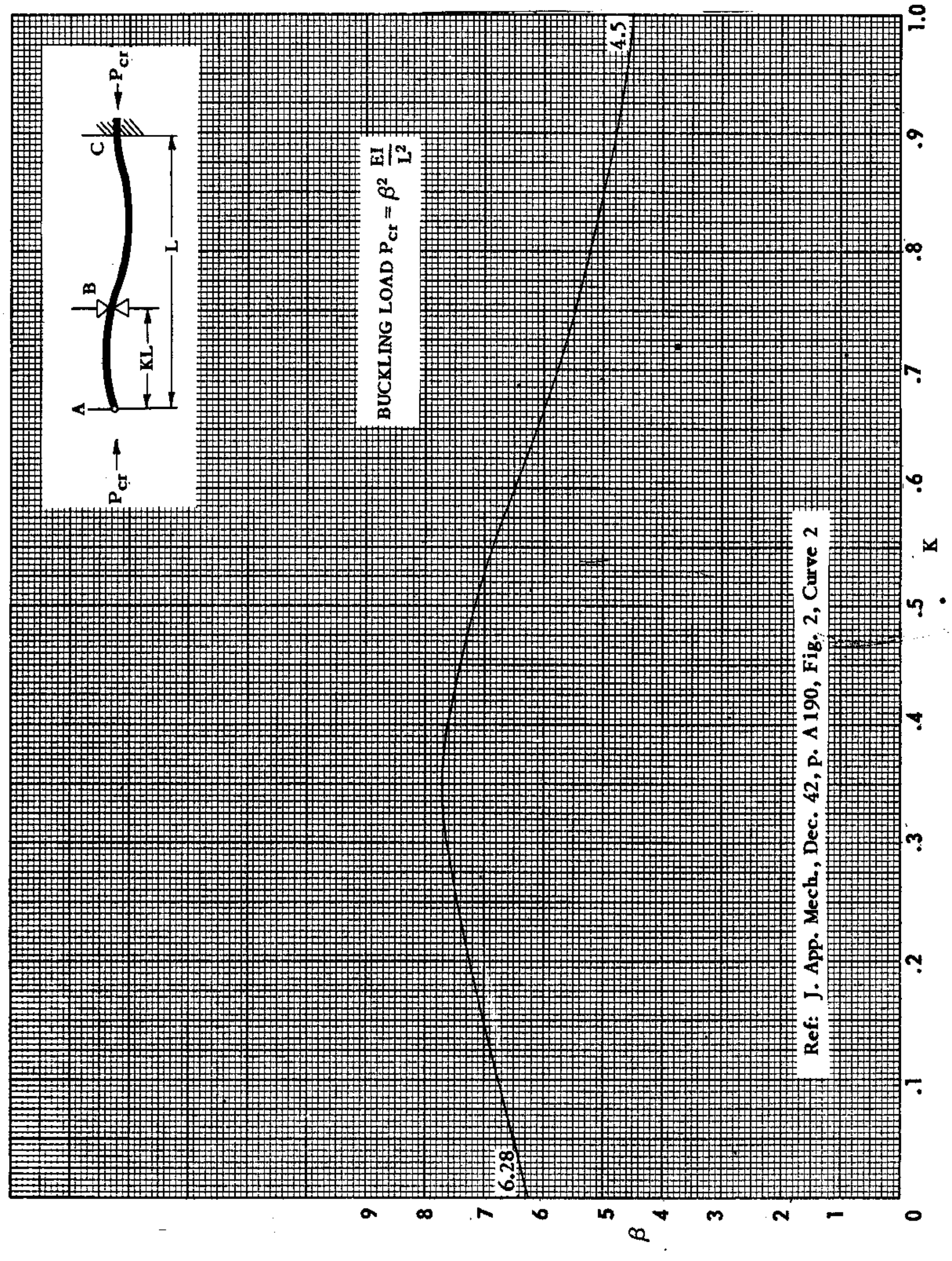
Buckling load $P_{cr} = \beta^2 \frac{EI}{L^2}$

When K = 0 or K = 1.0 the column is a single span one with fixed ends

$$P_{cr} = 4\pi^2 \frac{EI}{L^2} \quad \text{and } \beta = 6.28$$

β is plotted against K in Fig. 6.4.1–8.

6.4.1.3 Two Span (cont'd)

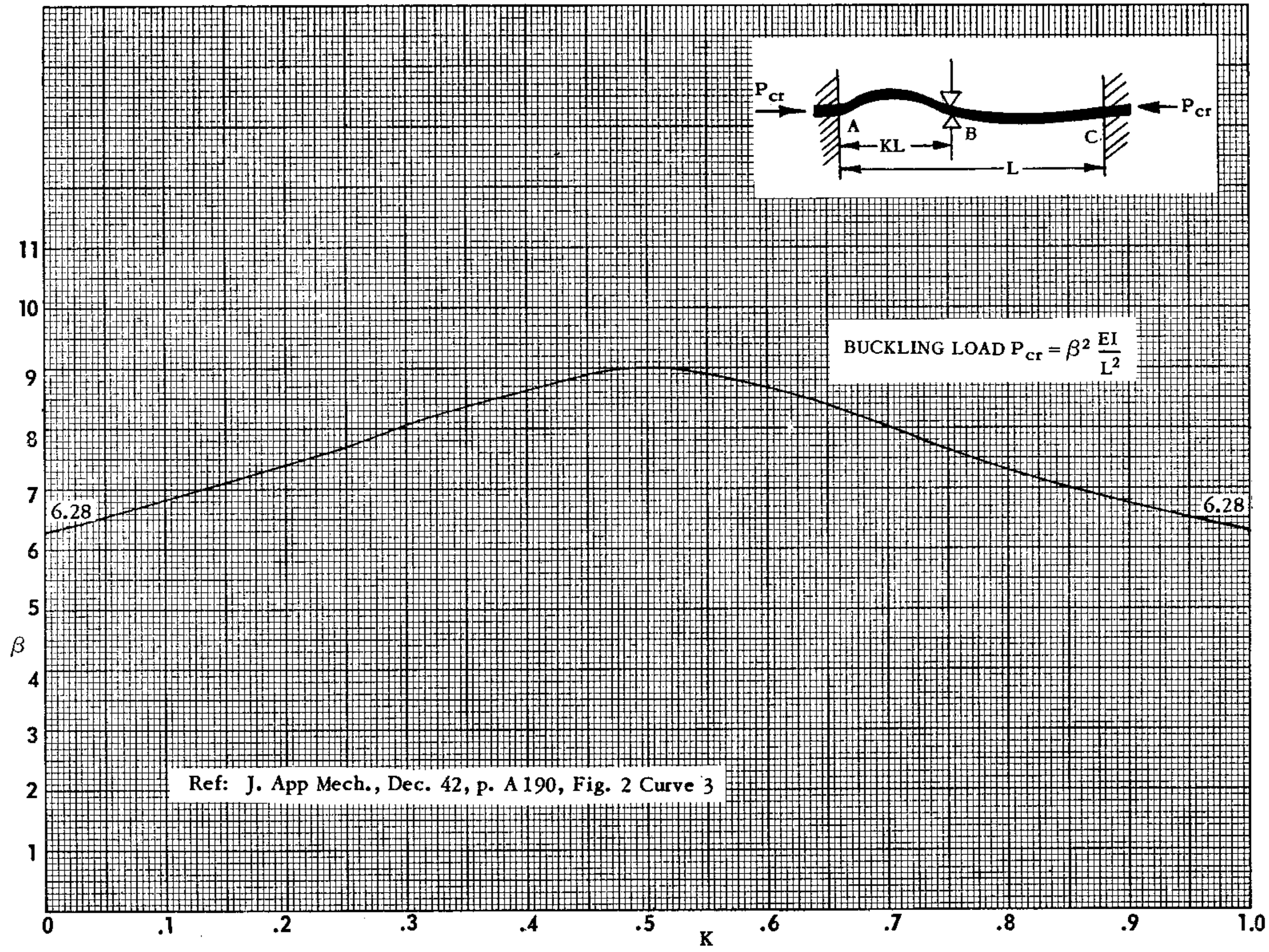


BUCKLING LOAD $P_{cr} = \beta^2 \frac{EI}{L^2}$

Ref: J. App. Mech., Dec. 42, p. A 190, Fig. 2, Curve 2

Fig. 6.4.1-6 Buckling Load for Two Span Continuous Column – Pinned at A Fixed at C
E, I and P constant

6.4.1.3 Two Span (cont'd)



BUCKLING LOAD $P_{cr} = \beta^2 \frac{EI}{L^2}$

Ref: J. App Mech., Dec. 42, p. A 190, Fig. 2 Curve 3

Fig. 6.4.1–8 Buckling Load for Two Span Continuous Column: Fixed at A and C
E, I and P constant

6.4.2 Three Span

6.4.2.1 Three Span, Pinned at Ends E, I and P constant

See Figs. 3 and 4 in J. App. Mech., Dec. 1942 p. A–190.

6.4.2.2 Three Span, Pinned at Ends E, I and P not constant

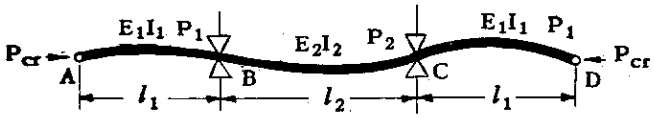


Fig. 6.4.2–1

The following method is taken from "Buckling of Struts and Frameworks", Ratzersdorfer, p. 280 (in German).

Let P_1 = applied end load in spans AB and CD

P_2 = applied end load in span BC

I_1 = moment of inertia of cross section of spans AB and CD

I_2 = moment of inertia of cross section of span BC

E_1 = modulus of elasticity of spans AB and CD

E_2 = modulus of elasticity of span BC

l_1 = span AB = span CD

l_2 = span BC

Equivalent pin-jointed length of span AB, $\bar{l}_1 = \frac{l_1}{C_1}$

Equivalent pin-jointed length of span BC, $\bar{l}_2 = \frac{l_2}{C_2}$

where C_1 and C_2 are obtained from Fig. 6.4.2–2, knowing the ratio $\frac{l_1}{l_2} \cdot \frac{E_2I_2}{E_1I_1}$ for a particular value of

$$\lambda = \frac{P_2}{P_1} \frac{E_2I_2}{E_1I_1} .$$

The critical loads are:

$$P_{1cr} = \frac{\pi^2 E_1I_1}{\bar{l}_1^{\,2}}$$

$$P_{2cr} = \frac{\pi^2 E_2I_2}{\bar{l}_2^{\,2}}$$

ILLUSTRATIVE EXAMPLE

Continuous beam of constant E, I and P in which $l_1 = l_2 = 10$ in. and AD = 30 in.

From Fig. 6.4.2–2,

$$\frac{l_1}{l_2}\frac{E_2I_2}{E_1I_1} = 1.0$$

$\lambda = 1.0$ since P is constant

and $C_1 = 1.0$

$$\bar{l}_1 = l_1 = 10$$

$$P_{cr} = \frac{\pi^2 EI}{l^2} = \underline{0.985\ EI.}$$

From Fig. 3, Curve (1), J. App. Mech., Dec. 1942 p. A–190 for $j = 1/3$, $\beta = 9.42$ and $\beta^2 = 88.7$.

$$P_{cr} = \beta^2 \frac{EI}{L^2}$$

$$= 88.7 \frac{EI}{30^2} = \underline{0.985\ EI} \text{ as above.}$$

6.4.2 Three Span (cont'd)

Procedure

(1) Calculate $\frac{l_1}{l_2}$ $\frac{E_2I_2}{E_1I_1}$ and $\lambda = \frac{P_2}{P_1}$ $\frac{E_2I_2}{E_1I_1}$

(2) Read off C_1 from the full curves and C_2 from the dotted curves

(3) $P_{1cr} = \frac{\pi^2 E_1 I_1}{\bar{l}_1^{\,2}}$ where $\bar{l}_1 = \frac{l_1}{C_1}$

$P_{2cr} = \frac{\pi^2 E_2 I_2}{\bar{l}_2^{\,2}}$ where $\bar{l}_2 = \frac{l_2}{C_2}$

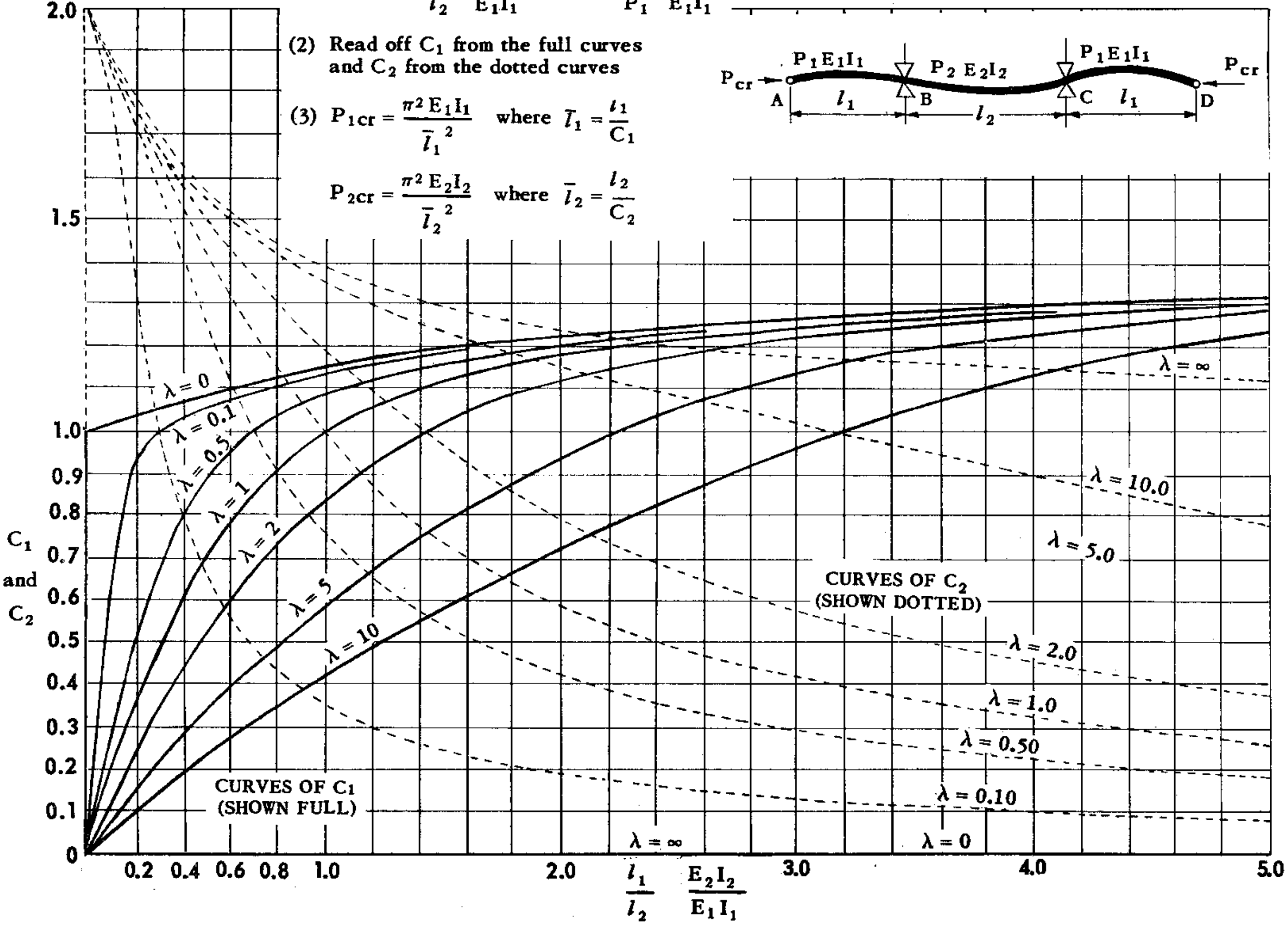


Fig. 6.4.2.2 Buckling Load for Three Span Continuous Column, Pinned at Ends
E, I and P not constant

6.5

STRAIGHT BEAM-COLUMNS

TABLE OF CONTENTS

6.5.1 Introduction

Methods are given herein for finding the distribution of bending moment over the length of a beam-column, that is, a member subjected to end load (compression or tension) and lateral load. Note that the formulas quoted are only valid for values of P less than the buckling load under compression alone (no lateral load).

Only straight beam-columns are dealt with in this particular section: for those with slight initial bow, see 6.6.

When the beam-column reduces to a standard case, a quick solution is given directly by the tabular summaries of 6.5.2.

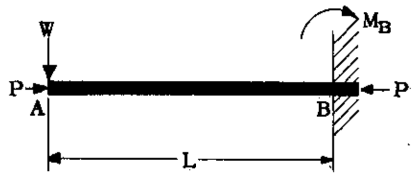
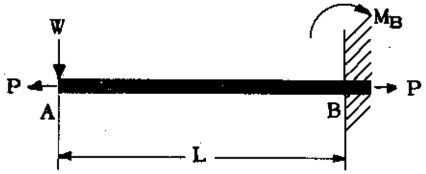
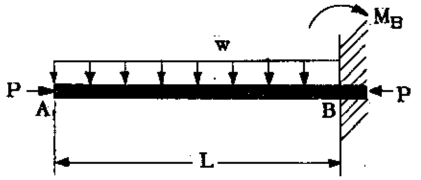
Knowing the end moments, the distribution of bending moment can in many cases be found by constructing a Polar Diagram (6.5.5), except where the beam column has varying I and a complex loading system, under which circumstances the general treatment of 6.5.6 can be used.

The Perry Approximation of 6.5.4 is useful for a quick estimate of bending moments, even when the loading is complex.

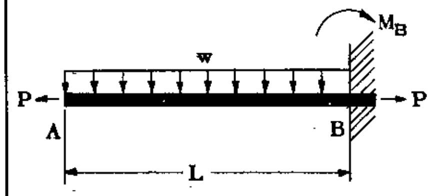
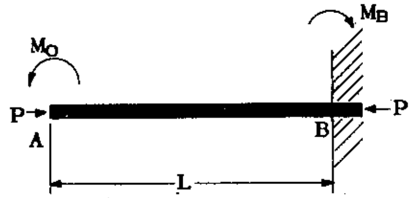
Stiffness and carry-over charts, to include the effect of end load, are given in 6.5.7.

6.5.2 Tabular Summaries: Constant I Beam-Columns

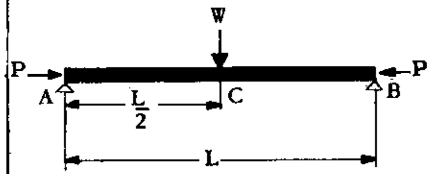
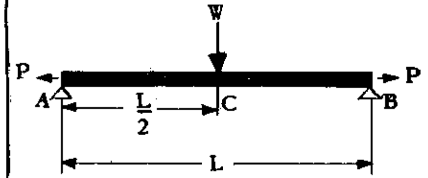
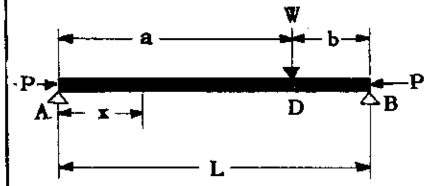
Common cases of Beam-Columns are given in the sections that follow.

CANTILEVER BEAM-COLUMN	BENDING MOMENT (M) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION (y) [POSITIVE UPWARD] AND SLOPE (θ) [POSITIVE WHEN UPWARD TO THE RIGHT]
(1) AXIAL COMPRESSION + CONCENTRATED LATERAL LOAD AT TIP. 	$M_B = -Wj \tan U$ This is the max. B.M.	Max. $y = -\frac{W}{P}(j \tan U - L)$ at A $\theta = \frac{W}{P} \frac{(1 - \cos U)}{\cos U}$ at A
(1A) AXIAL TENSION + CONCENTRATED LATERAL LOAD AT TIP. 	$M_B = -Wj \tanh U$ This is the max. B.M.	Max. $y = -\frac{W}{P}(L - j \tanh U)$ at A
(2) AXIAL COMPRESSION + UNIFORMLY DISTRIBUTED LATERAL LOAD. 	$M_B = -wj[j(1 - \sec U) + L \tan U]$ This is the max. B.M.	Max. $y = -\frac{wj}{P}\left[j(1+\frac{1}{2}U^2 - \sec U) + L(\tan U - U)\right]$ at A $\theta = \frac{w}{P}\left[\frac{L}{\cos U} - j\frac{(1 - \cos 2U)}{\sin 2U}\right]$ at A

6.5.2.1 Cantilever Beam-Columns: I Constant $\quad j^2 = \frac{EI}{P} \quad U = \frac{L}{j}$

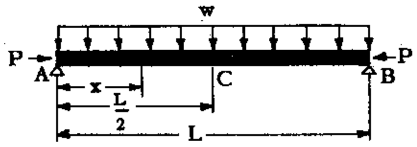
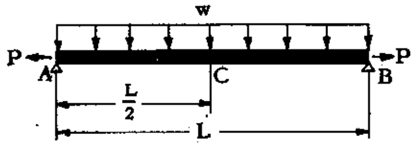
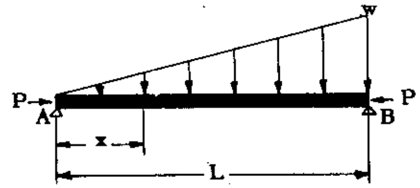
CANTILEVER BEAM-COLUMN	BENDING MOMENT (M) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION (y) [POSITIVE UPWARD] AND SLOPE (θ) [POSITIVE WHEN UPWARD TO THE RIGHT]
(2A) AXIAL TENSION + UNIFORMLY DISTRIBUTED LATERAL LOAD.  	$M_B = -wj\,[L \tanh U - j\,(1 - \operatorname{sech} U)]$ This is the max. B.M.	Max. $y = -\frac{wj}{P}\left[j\,(1 - \frac{1}{2}U^2 - \operatorname{sech} U) - L\,(\tanh U - U)\right]$ at A
(3) AXIAL COMPRESSION + END COUPLE AT TIP.  	$M_B = -M_o \sec U$	

6.5.2.1 Cantilever Beam-Columns: I Constant (cont'd) $\qquad j^2 = \frac{EI}{P} \qquad U = \frac{L}{j}$

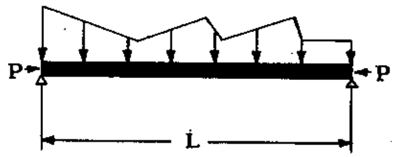
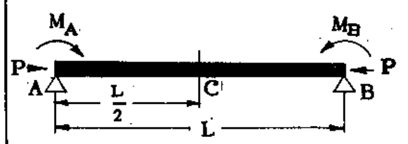
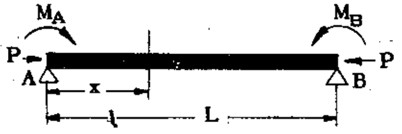
SIMPLY SUPPORTED BEAM-COLUMN	BENDING MOMENT (M) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION (y) [POSITIVE UPWARD] AND SLOPE (θ) [POSITIVE WHEN UPWARD TO THE RIGHT]
(1A) AXIAL COMPRESSION + CONCENTRATED LATERAL LOAD AT CENTRE  	$M_A = M_B = 0$ Max. $M = \frac{1}{2} Wj \tan\frac{U}{2}$ at C See also Perry Approximation in 6.5.3	Max. $y = -\frac{1}{2}\frac{Wj}{P}(\tan\frac{U}{2} - \frac{U}{2})$ at C $\theta = -\frac{W}{2P}\frac{(1-\cos\frac{U}{2})}{\cos\frac{U}{2}}$ at A
(1B) AXIAL TENSION + CONCENTRATED LATERAL LOAD AT CENTRE  	$M_A = M_B = 0$ Max. $M = \frac{1}{2} Wj \tanh\frac{U}{2}$ at C	Max. $y = -\frac{W}{P}(\frac{L}{4} - \frac{j}{2}\tanh\frac{U}{2})$ at C
(2) AXIAL COMPRESSION + CONCENTRATED LATERAL LOAD ANYWHERE ON SPAN  	$M_A = M_B = O$ (A to D) $M = \frac{Wj \sin\frac{b}{j}\sin\frac{x}{j}}{\sin U}$ (D to B) $M = \frac{Wj \sin\frac{a}{j}\sin\frac{(L-x)}{j}}{\sin U}$ Max. M occurs at $x = \frac{1}{2}\pi j$ if $\frac{1}{2}\pi j < a$ $x = (L - \frac{1}{2}\pi j)$ if $(L - \frac{1}{2}\pi j) > a$ $x = a$ if $\frac{1}{2}\pi j > a$ and $(L - \frac{1}{2}\pi j) < a$	(A to C) $y = \frac{Wj}{P}\left[\frac{\sin\frac{b}{j}\sin\frac{x}{j}}{\sin U} - \frac{bx}{Lj}\right]$ (C to B) $y = \frac{Wj}{P}\left[\frac{\sin\frac{a}{j}\sin\frac{L-x}{j}}{\sin U} - \frac{a(L-x)}{Lj}\right]$ $\theta = -\frac{W}{P}\left[\frac{b}{L} + \frac{\sin\frac{a}{j}}{\tan U} - \cos\frac{a}{j}\right]$ at A $\theta = \frac{W}{P}\left[\frac{a}{L} - \frac{\sin\frac{a}{j}}{\sin U}\right]$ at B

6.5.2.2 Simply Supported Beam-Columns: I Constant

$j^2 = \frac{EI}{P}$ $U = \frac{L}{j}$

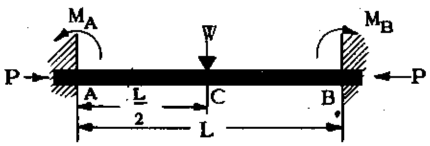
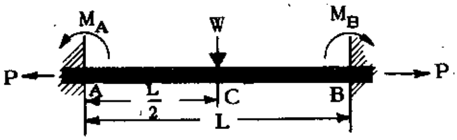
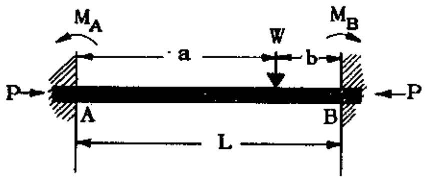
SIMPLY SUPPORTED BEAM-COLUMN	BENDING MOMENT (M) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION (y) [POSITIVE UPWARD] AND SLOPE (θ) [POSITIVE WHEN UPWARD TO THE RIGHT]
(3A) AXIAL COMPRESSION + UNIFORMLY DISTRIBUTED LATERAL LOAD 	$M_A = M_B = 0$ $M = A \cos\frac{x}{j} + B \sin\frac{x}{j} - wj^2$ where $A = wj^2$ $B = wj^2 \tan\frac{U}{2}$ Max. $M = wj^2 (\sec\frac{U}{2} - 1)$ at C See also Perry approximation in 6.5.3	Max. $y = -\frac{wj^2}{P} (\sec\frac{U}{2} - 1 - \frac{1}{8} U^2)$ at C $\theta = -\frac{wj}{P} \left[\frac{1 - \cos U}{\sin U} - \frac{U}{2}\right]$ at A
(3B) AXIAL TENSION + UNIFORMLY DISTRIBUTED LATERAL LOAD 	$M_A = M_B = 0$ Max. $M = wj^2 (1 - \text{sech}\,\frac{U}{2})$ at C	Max. $y = -\frac{w}{P}\left[\frac{L^2}{8} - j^2 (1 - \text{sech}\,\frac{U}{2})\right]$ at C
(4) AXIAL COMPRESSION + TRIANGULAR LATERAL LOAD  $W = \frac{1}{2} wL$	$M_A = M_B = 0$ $M = wj^2 \left[\frac{\sin\frac{x}{j}}{\sin U} - \frac{x}{L}\right]$ Max. M occurs at $x = j \text{ arc} \cos (\frac{\sin U}{U})$ See Worked Example in 6.5.4	$y = -\frac{w}{P}\left(\frac{x^3}{6L} + \frac{j^2 \sin\frac{x}{j}}{\sin U} - \frac{jx}{U} - \frac{1}{6} Lx\right)$ $\theta = -\frac{w}{P}\left(\frac{j}{\sin U} - \frac{j}{U} - \frac{L}{6}\right)$ at A $\theta = \frac{w}{P}\left(\frac{j}{U} - \frac{j}{\tan U} - \frac{L}{3}\right)$ at B

6.5.2.2 Simply Supported Beam-Columns: I Constant (cont'd) $j^2 = \frac{EI}{P}$ $U = \frac{L}{j}$

SIMPLY SUPPORTED BEAM-COLUMN	BENDING MOMENT (M) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION (y) [POSITIVE UPWARD] AND SLOPE (θ) [POSITIVE WHEN UPWARD TO THE RIGHT]
(5) AXIAL COMPRESSION + IRREGULAR LATERAL LOAD  	Use Perry Approximation of 6.5.3 in the absence of a better method	
(6) AXIAL COMPRESSION + EQUAL COUPLE AT EACH END. NO LATERAL LOAD.  	Max. $M = M_A \sec \frac{U}{2}$ at C This is Southwell's secant formula, proved in GEN/1090/811.9. B.M. at any station can be found by polar diagram (6.5.5)	Max. $y = -\frac{M_A}{P}\left(\frac{1-\cos\frac{U}{2}}{\cos\frac{U}{2}}\right)$ at C $\theta = -\frac{M_A}{Pj}\tan\frac{U}{2}$ at A
(7) AXIAL COMPRESSION + UNEQUAL COUPLES AT ENDS. NO LATERAL LOAD.  $M_B > M_A$	$M = \frac{(M_B - M_A \cos U)}{\sin U}\sin\frac{x}{j} + M_A \cos\frac{x}{j}$ Max. M occurs at $x = j \arctan\frac{(M_B - M_A \cos U)}{M_A \sin U}$ B.M. at any station can be found by polar diagram (6.5.5)	$y = \frac{1}{P}\left[M_A + (M_B - M_A)\frac{x}{L} - \frac{(M_B - M_A \cos U)}{\sin U}\sin\frac{x}{j} - M_A \cos\frac{x}{j}\right]$ $\theta = \frac{1}{P}\left[\frac{M_B - M_A}{L} - \frac{M_B - M_A \cos U}{j \sin U}\right]$ at A $\theta = \frac{1}{P}\left[\frac{M_B - M_A}{L} - \frac{M_B - M_A \cos U}{j \sin U}\cos U + \frac{M_A}{j}\sin U\right]$ at B

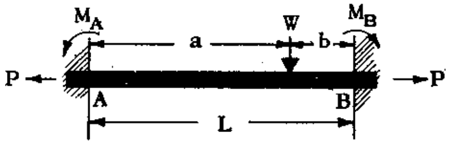
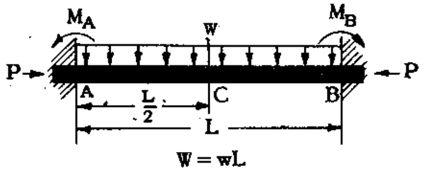
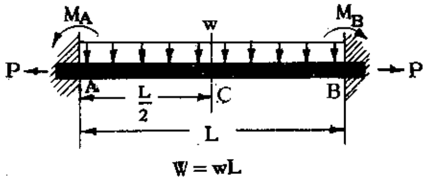
6.5.2.2 Simply Supported Beam-Columns: I Constant (cont'd)

$j^2 = \frac{EI}{P}$ $\qquad U = \frac{L}{j}$

FIXED-END BEAM COLUMN	BENDING MOMENT (M) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION (y) [POSITIVE UPWARD AND SLOPE (θ) [POSITIVE WHEN UPWARD TO THE RIGHT]
(1A) AXIAL COMPRESSION + CONCENTRATED LATERAL LOAD AT CENTRE 	$M_A = -\frac{WL}{8} C_A \qquad M_B = -\frac{WL}{8} C_B$ where coefficients C_A and C_B are given in Fig. 6.5.2.3 – 1 for $\frac{a}{L} = \frac{b}{L} = 0.50$ $M_C = \frac{WL}{8} \left(\frac{\tan \frac{U}{4}}{\frac{U}{4}} \right)$ Use polar diagram of 6.5.5 for finding intermediate B.M. See also Perry approximation in 6.5.3	$\text{Max. } y = -\frac{WL^3}{192\,EI} \gamma$ where $\gamma = 3 \left[\frac{\tan \frac{U}{4} - \frac{U}{4}}{\left(\frac{U}{4}\right)^3} \right]$
(1B) AXIAL TENSION + CONCENTRATED LATERAL LOAD AT CENTRE 	$M_A = M_B = \frac{1}{2} Wj \left(\frac{\cosh \frac{U}{2} - 1}{\sin h \frac{U}{2}} \right)$ Max. + M $= \frac{1}{2} Wj \left(\frac{1 - \cos h \frac{U}{2}}{\sin h \frac{U}{2} \cos h \frac{U}{2}} + \tan h \frac{U}{2} \right)$ at C Use polar diagram of 6.5.5 for finding intermediate B.M.	$\text{Max. } y = -\frac{Wj}{2P} \left[\frac{U}{2} - \tan h \frac{U}{2} - \frac{(1 - \cos h \frac{U}{2})^2}{\sinh \frac{U}{2} \cos h \frac{U}{2}} \right]$
(2A) AXIAL COMPRESSION + CONCENTRATED LATERAL LOAD ANYWHERE ON SPAN 	See Fig. 6.5.2.3–1 for values of M_A and M_B. Use polar diagram of 6.5.5 for finding intermediate B.M.	

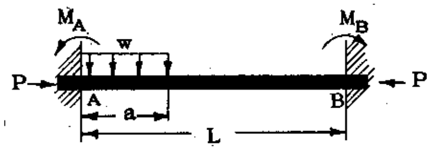
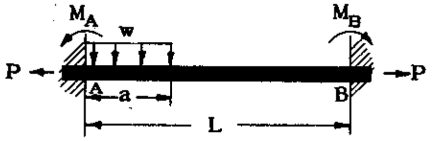
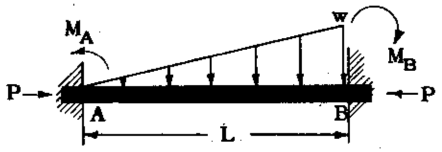
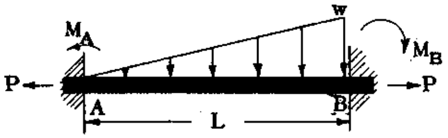
6.5.2.3 Fixed End Beam-Columns: I Constant

$j^2 = \frac{EI}{P} \qquad U = \frac{L}{j}$

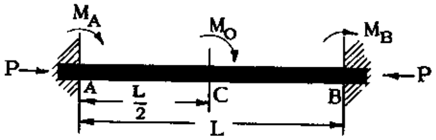
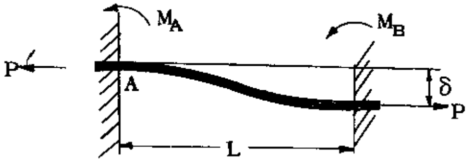
FIXED-END BEAM COLUMN	BENDING MOMENT (M) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION (y) [POSITIVE UPWARD] AND SLOPE (θ) [POSITIVE WHEN UPWARD TO THE RIGHT]
(2B) AXIAL TENSION + CONCENTRATED LATERAL LOAD ANYWHERE ON SPAN 	See Fig. 6.5.2.3–1 for values of M_A and M_B Use polar diagram of 6.5.5 for finding intermediate B.M.	
(3A) AXIAL COMPRESSION + UNIFORMLY DISTRIBUTED LATERAL LOAD 	$M_A = M_B = -\frac{wL^2}{C}$ where coefficient C is given in Fig. 6.5.2.3–5 $M_C = wj^2 (\frac{U}{2} \operatorname{cosec} \frac{U}{2} - 1)$ Proof in GEN/1090/811.7.4 See also Perry Approximation in 6.5.3	$\text{Max. } y = -\frac{1}{384} \frac{WL^3}{EI} \gamma$ at C $\text{where } \gamma = 3 \left[\frac{\tan \frac{U}{4} - \frac{U}{4}}{(\frac{U}{4})^3} \right]$
(3B) AXIAL TENSION + UNIFORMLY DISTRIBUTED LATERAL LOAD 	$M_A = M_B = wj^2 (\frac{U}{2} \cot h \frac{U}{2} - 1)$ $\text{Max. } + M = wj^2 (1 - \frac{U}{2} \operatorname{cosec} h \frac{U}{2})$ at C	$\text{Max. } y = -\frac{wj^2}{8P} \left[\frac{4U(1 - \cos h \frac{U}{2})}{\sin h \frac{U}{2}} + U^2 \right]$ at C

6.5.2.3 Fixed End Beam-Columns: I Constant (cont'd)

$j^2 = \frac{EI}{P}$ $\qquad U = \frac{L}{j}$

FIXED-END BEAM COLUMN	BENDING MOMENT (M) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	
(4A) AXIAL COMPRESSION + UNIFORMLY DISTRIBUTED LATERAL OVER PART OF SPAN  	See Fig. 6.5.2.3–2 for values of M_A and M_B. Use polar diagram of 6.5.5 for finding intermediate B.M.	
(4B) AXIAL TENSION + UNIFORMLY DISTRIBUTED LATERAL LOAD OVER PART OF SPAN  		
(5A) AXIAL COMPRESSION + TRIANGULAR LATERAL LOADING  	See Fig. 6.5.2.3–3 for values of M_A and M_B	
(5B) AXIAL TENSION + TRIANGULAR LATERAL LOADING  	See Fig. 6.5.2.3–4 for values of M_A and M_B	

6.5.2.3 Fixed End Beam-Columns: I Constant (cont'd) $j^2 = \frac{EI}{P}$ $U = \frac{L}{j}$

FIXED-END BEAM COLUMN	BENDING MOMENT (M) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	DEFLECTION (y) [POSITIVE UPWARD] AND SLOPE (θ) [POSITIVE WHEN UPWARD TO THE RIGHT]
(6) AXIAL COMPRESSION: NO LATERAL LOAD BUT MOMENT APPLIED AT CENTRE  	$M_A = + \frac{M_O}{4} \frac{\alpha''}{\beta''}$ $M_B = - \frac{M_O}{4} \frac{\alpha''}{\beta''}$ where $\alpha'' = 6 \left[\frac{\frac{U}{2} \operatorname{cosec} \frac{U}{2} - 1}{(\frac{U}{2})^2} \right]$ and $\beta'' = 3 \left[\frac{1 - \frac{U}{2} \cot \frac{U}{2}}{(\frac{U}{2})^2} \right]$ (Ref. Lockheed Report No. 5048)	$y = 0$ at C . See Lockheed Report Extract in GEN/1090/811
(7A) AXIAL COMPRESSION: NO LATERAL LOAD. SUPPORT B SINKS  	$M_A = M_B = - \frac{6EI}{L^2} \delta . \frac{1}{C}$ C = a coefficient, values of which are given in Fig. 6.5.2.3–6 For proof, see GEN/1090/811.7.6	
(7B) AXIAL TENSION: NO LATERAL LOAD SUPPORT B SINKS  	$M_A = M_B = - \frac{6EI}{L^2} \delta . \frac{1}{C}$ C = a coefficient, values of which are given in Fig. 6.5.2.3–6 For proof, see GEN/1090/811.7.6	

6.5.2.3 Fixed End Beam-Columns: I Constant (cont'd)

$j^2 = \frac{EI}{P}$ $U = \frac{L}{j}$

6.5.2.3 Fixed End Beam-Columns: I Constant (cont'd)

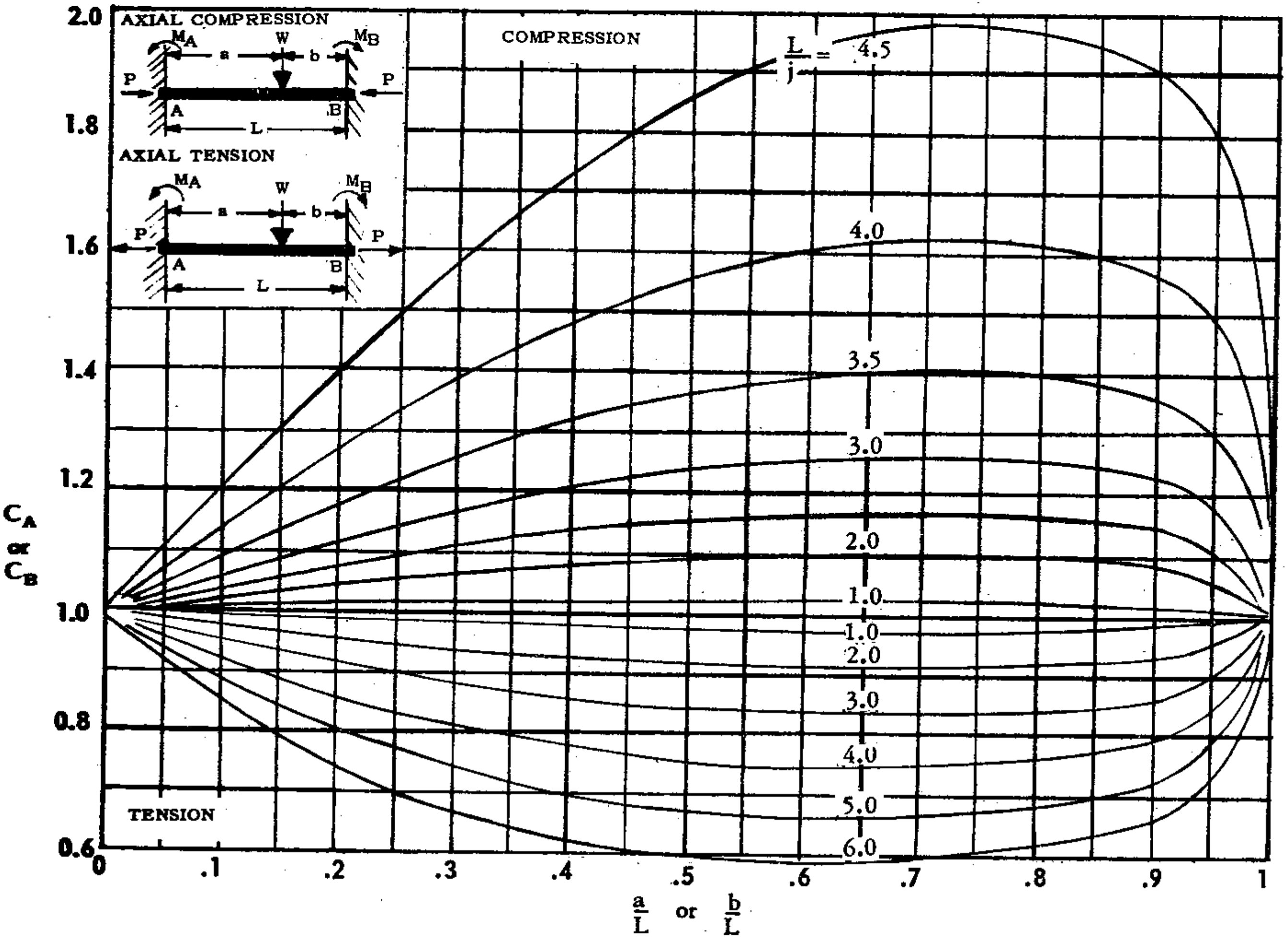
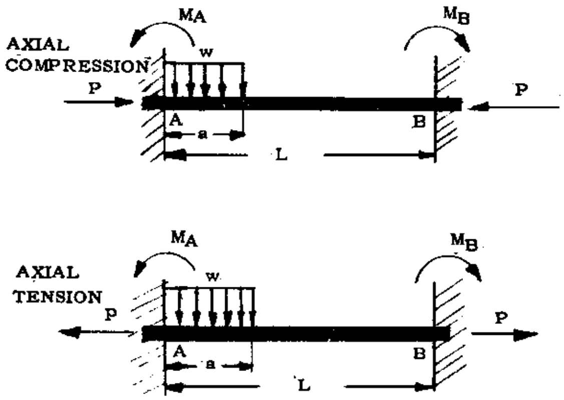


Fig. 6.5.2.3–1 Fixed End Moment Coefficients for Beam-Columns carrying Concentrated Lateral Load anywhere on Span [From N.A.C.A., T.N. 534]

Procedure

To find M_A	To find M_B
(i) Calculate $\frac{a}{L}$ and read off value of C_A from the **appropriate** curve of $\frac{L}{j}$, tension or compression, where $j^2 = \frac{EI}{P}$.	(i) Calculate $\frac{b}{L}$ and read off value of C_B from the appropriate curve of $\frac{L}{j}$, tension or compression.
(ii) $M_A = -\frac{Wab}{L^2} \cdot bC_A$	(ii) $M_B = -\frac{Wab}{L^2} \cdot aC_B$
For W at mid-span $\frac{a}{L} = .50$, $M_A = -\frac{WL}{8} C_A$	For W at mid-span $\frac{b}{L} = .50$, $M_B = -\frac{WL}{8} C_B$

6.5.2.3 Fixed End Beam-Columns: I Constant (cont'd)



Procedure

To find M_A

(i) Calculate $\frac{a}{L}$ and read off value of C_A from the appropriate curve of $\frac{L}{j}$, tension or compression, where $j^2 = \frac{EI}{P}$

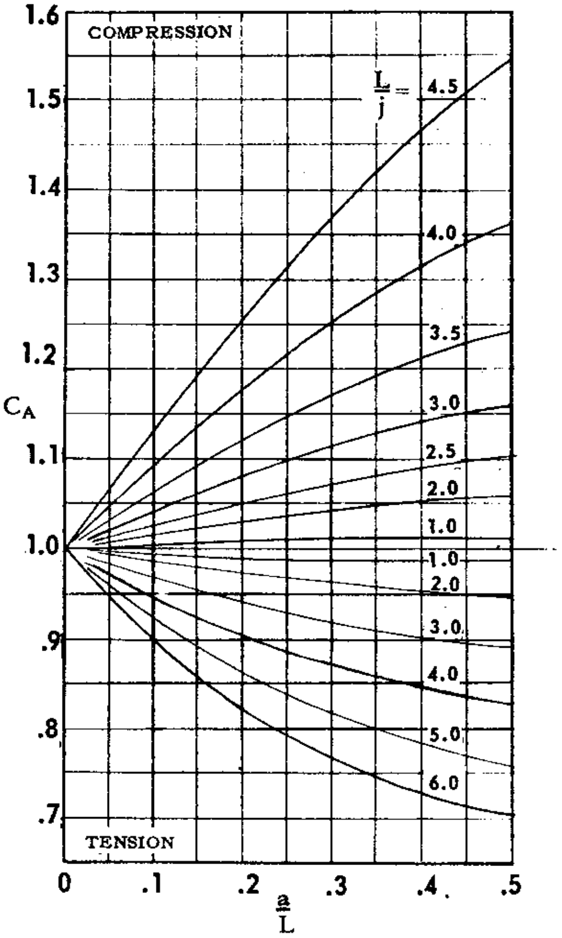
(ii) $M_A = -\frac{wa^2}{12}\left[6 - \frac{8a}{L} + 3\left(\frac{a}{L}\right)^2\right] C_A$

To find M_B

(i) Calculate $\frac{a}{L}$ and read off value of C_B from the appropriate curve of $\frac{L}{j}$, tension or compression.

(ii) $M_B = -\frac{wa^2}{12}\left[4\frac{a}{L} - 3\left(\frac{a}{L}\right)^2\right] C_B$

When $\frac{a}{L}$ is > 0.5, find the fixed end moments for a uniformly distributed load over the entire span (Case 3A or 3B) and subtract the fixed end moments that would be caused by a uniform load over the part of the span that is not loaded.



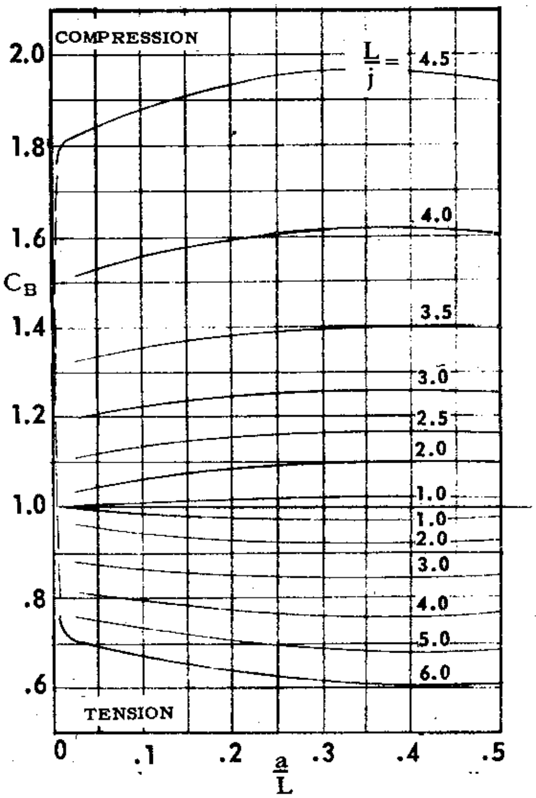


Fig. 6.5.2.3–2 Fixed End Moment Coefficients for Beam-Columns carrying Uniformly Distributed Lateral Load over part of Span [From N.A.C.A., T.N. 534]

6.5.2.3 Fixed End Beam-Columns: I Constant (cont'd)

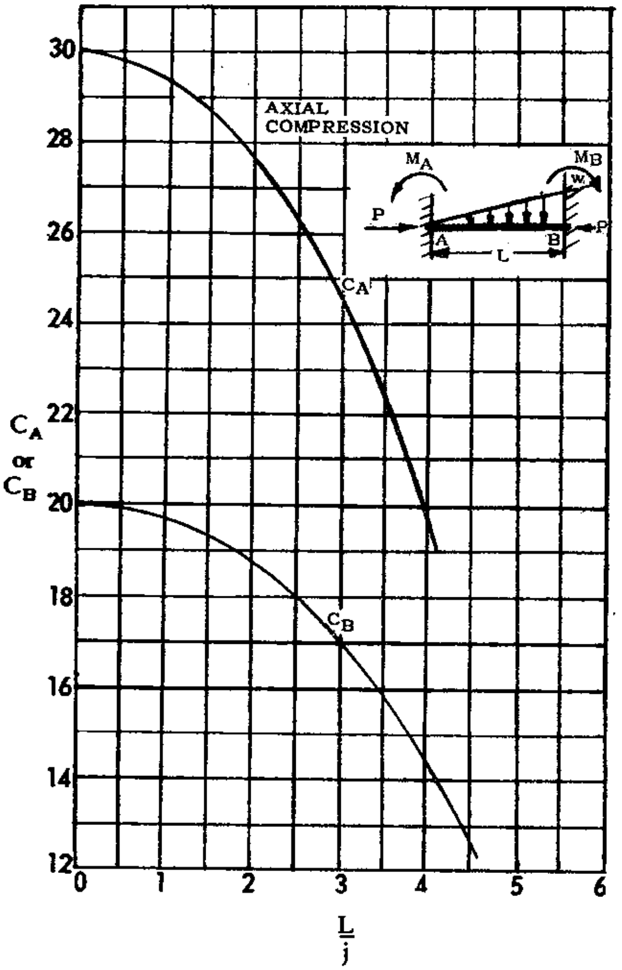


Fig. 6.5.2.3–3 Fixed End Moment Coefficients for Compression Beam Column Carrying Triangular Lateral Loading [From N.A.C.A., T.N. 534]

Procedure

To find M_A

(i) Calculate $\frac{L}{j}$ where $j^2 = \frac{EI}{P}$ and read off the value of C_A.

(ii) $M_A = -\frac{wL^2}{C_A} = -\frac{2WL}{C_A}$ since $W = \frac{1}{2}wL$

To find M_B

(i) Calculate $\frac{L}{j}$ and read off the value of C_B

(ii) $M_B = -\frac{wL^2}{C_B} = -\frac{2WL}{C_B}$

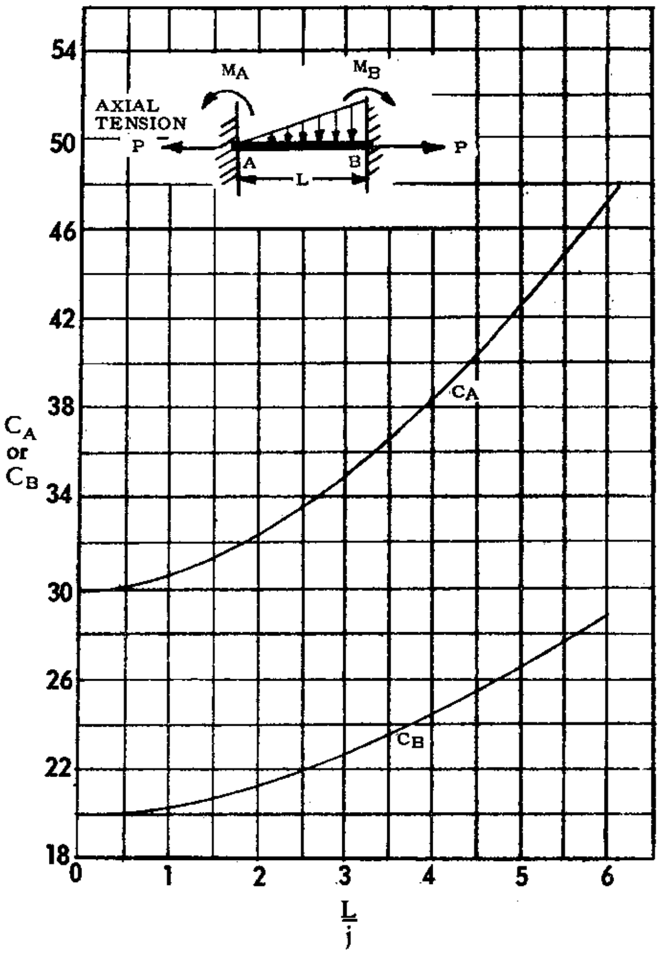


Fig. 6.5.2.3–4 Fixed End Moment Coefficients for Tension Beam Column Carrying Triangular Lateral Loading [From N.A.C.A., T.N. 534]

Procedure

To find M_A

(i) Calculate $\frac{L}{j}$ where $j^2 = \frac{EI}{P}$ and read off the value of C_A.

(ii) $M_A = \frac{wL^2}{C_A} = -\frac{2WL}{C_A}$ since $W = \frac{1}{2}wL$

To find M_B

(i) Calculate $\frac{L}{j}$ and read off the value of C_B.

(ii) $M_B = -\frac{wL^2}{C_B} = -\frac{2WL}{C_B}$

6.5.2.3 Fixed End Beam-Columns: I Constant (cont'd)

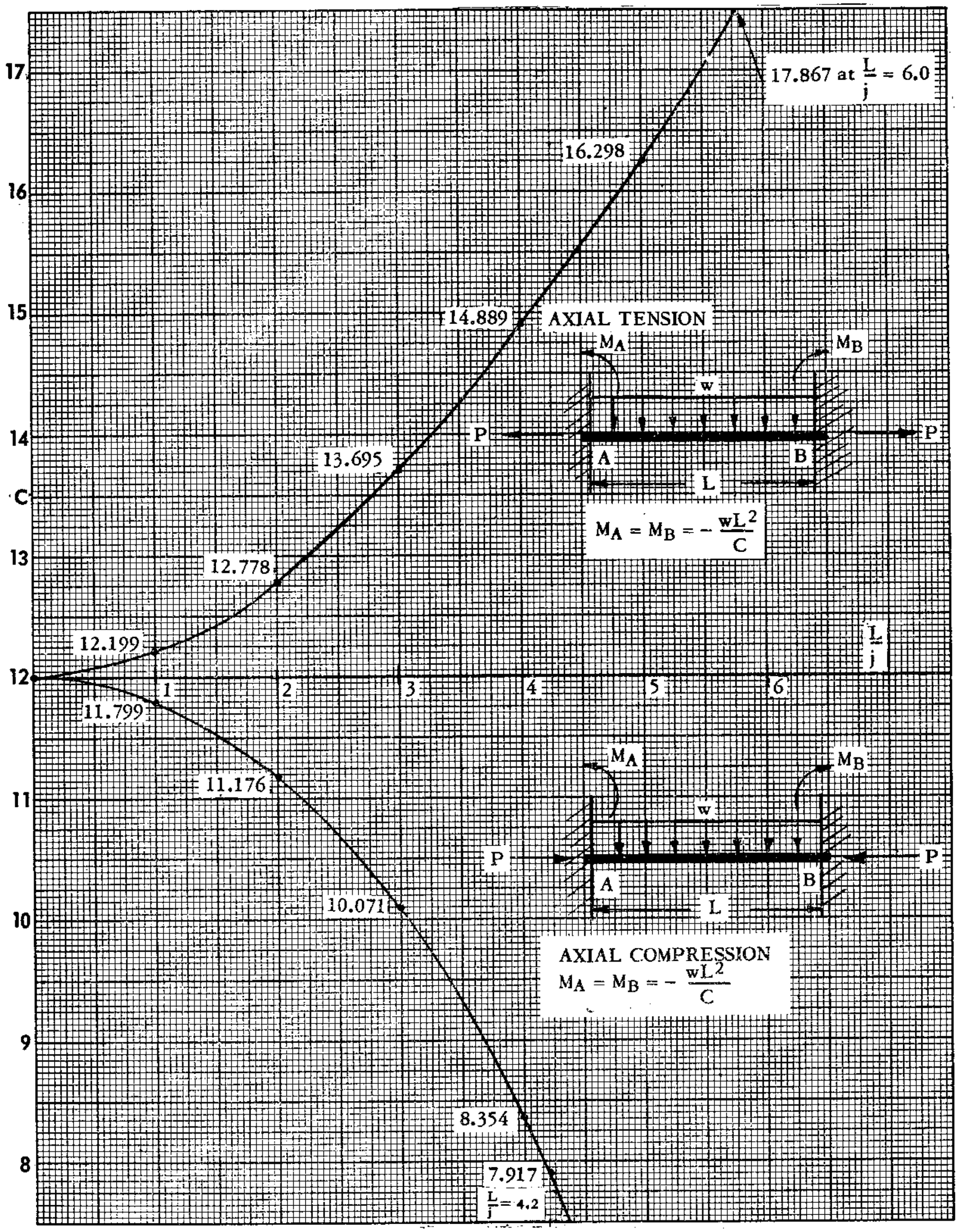
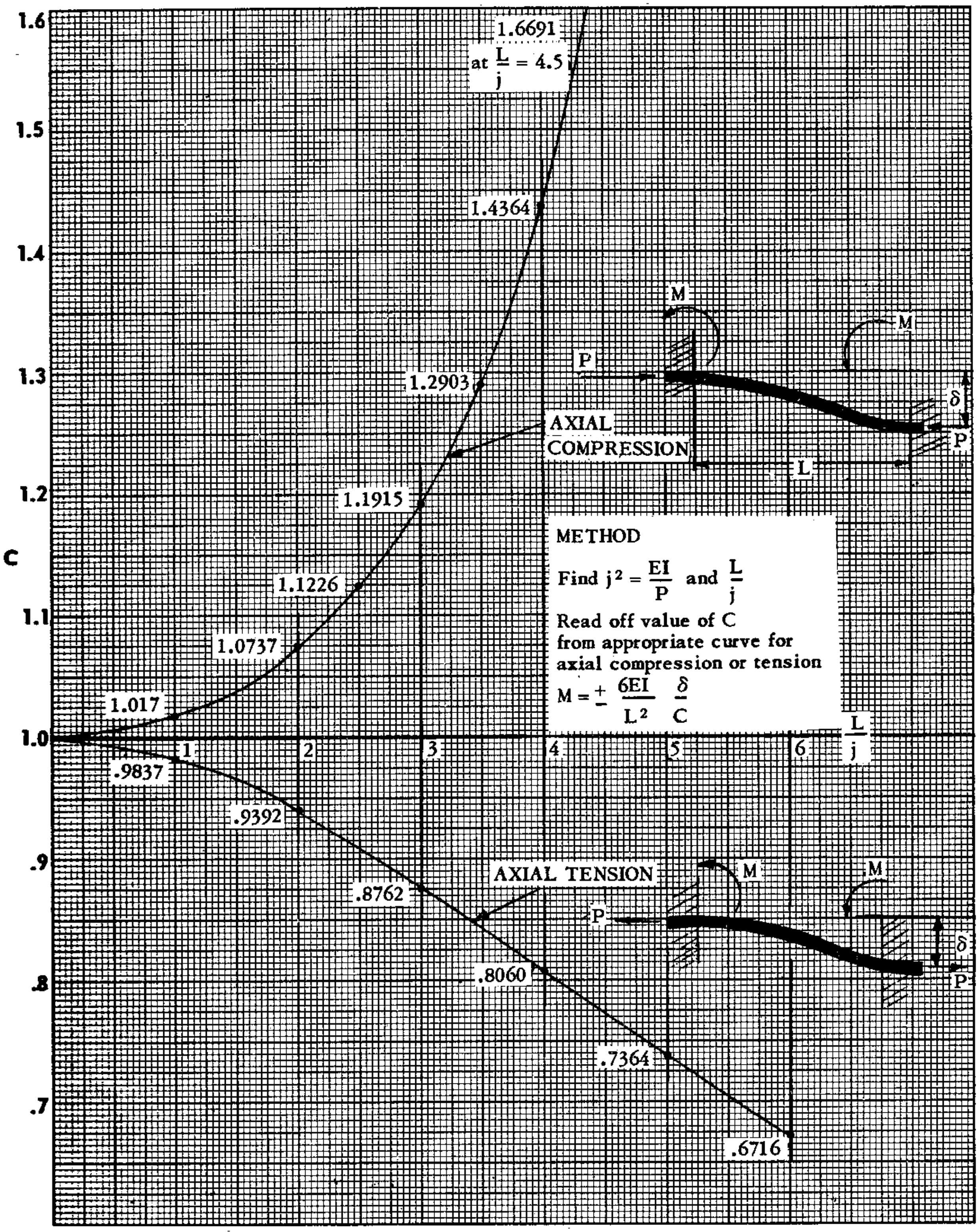


Fig. 6.5.2.3–5 End Moments for Beam Columns carrying a Uniformly Distributed Load over whole Span

6.5.2.3 Fixed End Beam-Columns: I Constant (cont'd)



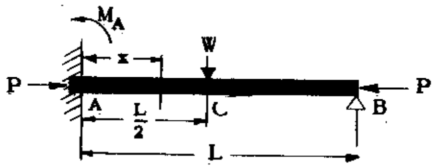
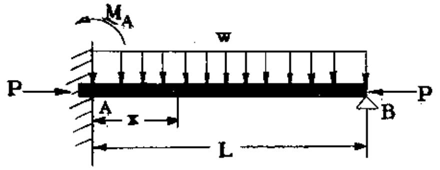
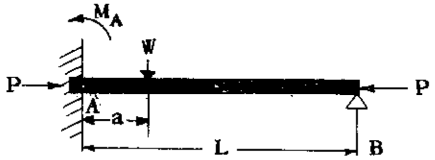
METHOD

Find $j^2 = \frac{EI}{P}$ and $\frac{L}{j}$

Read off value of C from appropriate curve for axial compression or tension

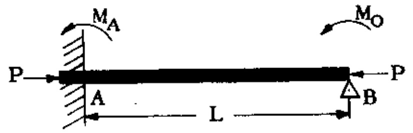
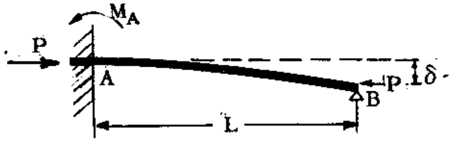
$M = \pm \frac{6EI}{L^2} \frac{\delta}{C}$

Fig. 6.5.2.3–6 End Moments for Beam-Column with Sinking Support

PROPPED CANTILEVER BEAM-COLUMN	BENDING MOMENT (M) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]
(1) AXIAL COMPRESSION + CONCENTRATED LATERAL LOAD AT CENTRE  $R_B = \frac{W}{2} - \frac{M_A}{L}$	C to B $M = M_A\left(\frac{\sin\frac{x}{j}}{\tan U} - \cos\frac{x}{j}\right) + Wj\left(\sin\frac{U}{2}\cos\frac{x}{j} - \frac{\sin\frac{U}{2}\sin\frac{x}{j}}{\tan U}\right)$ Max. $M = M_A = -\frac{WL}{2}\left[\frac{j\tan U(\sec\frac{U}{2} - 1)}{j\tan U - L}\right]$ Alternatively, adopt the treatment of Case 3 below For y, see Roark, p.133, Case 11.
(2) AXIAL COMPRESSION + UNIFORMLY DISTRIBUTED LATERAL LOAD  $R_B = \frac{1}{2}wL - \frac{M_A}{L}$	$M = M_A(\cot U\sin\frac{x}{j} - \cos\frac{x}{j}) + wj^2\left[\frac{\sin\frac{x}{j}}{\sin U}(1 - \cos U) + \cos\frac{x}{j} - 1\right]$ Max. $M = M_A$ $= -wLj\left[\frac{\tan U(\tan\frac{U}{2} - \frac{U}{2})}{\tan U - U}\right]$ Alternatively, adopt the treatment of Case 3 below For y, see Roark, p.133, Case 12
(3) AXIAL COMPRESSION + CONCENTRATED LATERAL LOAD ANYWHERE ON SPAN  	First consider as a beam-column fixed at both ends (Case 2A, Section 6.5.2.3) and find M_A and M_B. Then release B (by making $M_B \doteq 0$) and find the carry-over moment to A by using the carry-over factor for the appropriate L/j value, given in 6.5.7. This gives M_A.

6.5.2.4 Propped Cantilever Beam Columns: I Constant $j^2 = \frac{EI}{P}$ $U = \frac{L}{j}$

6.5.2.4 Propped Cantilever Beam-Columns: I Constant (cont'd) $j^2 = \frac{EI}{P}$ $U = \frac{L}{j}$

PROPPED CANTILEVER BEAM-COLUMN	BENDING MOMENT (M) [POSITIVE WHEN PUTTING TOP FLANGE IN COMPRESSION]	
(4) AXIAL COMPRESSION: NO LATERAL LOAD BUT END COUPLE  	$M_A = -M_O \frac{a}{2\beta}$ where $a = 6 \frac{(U \text{ cosec } U - 1)}{U^2}$ $\beta = 3 \frac{(\tan U - U)}{U^2 \tan U}$	
(5) AXIAL COMPRESSION: NO LATERAL LOAD BUT SUPPORT B SINKS  	$M_A = -\frac{3 EI \delta}{L^2 \beta}$ where $\beta = 3 \frac{(\tan U - U)}{U^2 \tan U}$ This result is analogous to that in GEN/1090/803, Special Case (b) without axial load. Proof in GEN/1090/811.7.5.	

6.5.3 Perry's Approximation

The bending moment M at any section X of a constant I beam-column pinned at both ends and carrying a uniformly distributed lateral load is proved in GEN/1090/811.7.1 to be:–

$$M = \frac{Q}{Q - P} M' \quad \text{(approx.)}$$

where M' = the bending moment at section X due to the lateral load alone that is, ignoring any load effect,

P = end load (less than Q)

and Q = Euler failing load for pinned ends = $\frac{\pi^2 EI}{L^2}$

This approximate formula gives results which are in close agreement with the secant formula of 6.5.2.2, Case 6, except when P approaches the value of Q. (See worked example in 6.5.4).

OTHER CASES

By writing the expression in the form

$$M = \frac{Q}{Q - CP} M'$$

an approximate expression for M for other loading cases and different end conditions results, C being a coefficient having values as below:

END CONDITION	LATERAL LOAD	C	REMARKS
Both Ends Pinned	Uniformly Distributed	1.0	B.M. at centre of Beam
Both Ends Pinned	Central Load	0.894	B.M. at centre of Beam
Both Ends Fixed	Uniformly Distributed	0.276	B.M. at centre of Beam
Both Ends Fixed	Uniformly Distributed	0.172	Fixed end moment
Both Ends Fixed	Central Load	0.212	Fixed end moments and B.M. at centre of beam

6.5.4 Worked Examples

6.5.4.1 Uniformly Distributed Load

Beam column shown in Fig. 6.5.4–1 for which $EI = 10^6$

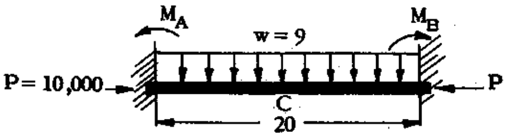


Fig. 6.5.4–1

END MOMENTS

$$M_A = M_B = -\frac{WL}{12}\,\beta'' \text{ from 6.5.2.3, Case 3A}$$

$$\beta'' = 3\left[\frac{1-\frac{U}{2}\cot\frac{U}{2}}{(\frac{U}{2})^2}\right] = \frac{3\times .358}{1.0} = 1.074$$

$$M_A = M_B = -300\times 1.074 = -\underline{322.2 \text{ lb. in.}}$$

This result was obtained in 3.3.3.

$$j^2 = \frac{EI}{P} = 100 \qquad j = 10$$

$$\frac{U}{2} = \frac{L}{2j} = \frac{20}{20} = 1.0 \text{ radian} = 57.3 \text{ deg.}$$

$$\cot\frac{U}{2} = .6420$$

$$(1-\cot\frac{U}{2}) = .358$$

$$W = 9\times 20 = 180$$

$$\frac{WL}{12} = \frac{180\times 20}{12} = 300$$

CHECK BY PERRY APPROXIMATION of 6.5.3

$$M = \frac{Q}{Q-CP}\quad M'$$

$$= 1.075\times 300 = \underline{322.5 \text{ lb. in.}}$$

$$Q = \frac{\pi^2 EI}{L^2} = \frac{9.85\times 10^6}{400} = 24625$$

$$C = .172 \qquad CP = .172\times 10000 = 1720$$

$$Q-CP = 22905 \qquad \frac{Q}{Q-CP} = \frac{24625}{22905} = 1.075$$

CENTRE MOMENT

$$M_c = \frac{WL}{24}\,a''$$

$$= 150\times 1.1298 = \underline{169.5 \text{ lb. in.}}$$

$$\frac{U}{2} = 1.0 \text{ rad.} = 57.3 \text{ deg.}$$

$$\operatorname{cosec}\frac{U}{2} = 1.1883$$

$$(\frac{U}{2}\operatorname{cosec}\frac{U}{2} - 1) = .1883$$

$$a'' = 6\left[\frac{\frac{U}{2}\operatorname{cosec}\frac{U}{2} - 1}{(\frac{U}{2})^2}\right]$$

$$= 6\times .1883 = 1.1298$$

CHECK BY PERRY APPROXIMATION

$$M_c = 1.126\times 150 = \underline{169 \text{ lb. in.}}$$

$$C = .276 \qquad CP = 2760$$

$$Q-CP = 21865 \qquad \frac{Q}{Q-CP} = 1.126$$

$$M' = \frac{WL}{24} = 150$$

6.5.4.2 Triangular Load

Simply supported beam column shown in Fig. 6.5.4–2. carrying triangular load.

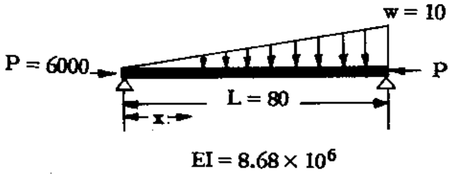


Fig. 6.5.4–2

By the formula of 6.5.2.2, Case 4,

$$M = wj^2\left[\frac{\sin\frac{x}{j}}{\sin U} - \frac{x}{L}\right]$$

$$j^2 = \frac{EI}{P} = \frac{8.68 \times 10^6}{6000} = 1.449 \times 10^3$$

$$j = 38 \qquad U = \frac{L}{j} = \frac{80}{38} = 2.105$$

$$= 120.7^{\circ}$$

$$\sin U = .8599$$

$$\text{Max. M occurs at } x = j \text{ arc cos } \left(\frac{\sin U}{U}\right)$$

$$\frac{\sin U}{U} = \frac{.8599}{2.105} = .411$$

The angle whose cosine is .411 is $65.7^{\circ} = 1.146$ rad.

$x = 38 \times 1.146 = \underline{43.6 \text{ in.}}$

Values of M at this and other stations are evaluated in the table below.

x	10	20	30	40	43.6	50	60	70	80
$\frac{x}{j}$ (rad.)	.263	.526	.789	1.052	1.146	1.315	1.578	1.841	2.104
$\frac{x}{j}$ (deg.)	15.1	30.2	45.3	60.4	65.7	75.5	90.6	105.7	120.8
$\sin\frac{x}{j}$	.2605	.503	.711	.869	.911	.968	1.0	.963	.859
$\frac{\sin x/j}{\sin U}$	.305	.585	.825	1.012	1.062	1.128	1.162	1.120	1.0
$\frac{x}{L}$	.125	.25	.375	.50	.544	.625	.75	.875	1.0
$\left[\frac{\sin x/j}{\sin U} - \frac{x}{L}\right]$	.18	.335	.45	.512	.518	.503	.412	.245	0
M	2600	4850	6510	7440	7510	7300	5970	3550	0

6.5.5 Polar Diagrams for Straight Compression Beam-Columns

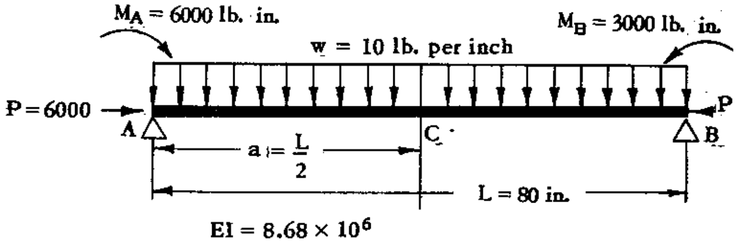
See GEN/1090/811.7.2 for explanation of theory.

TYPE		
1A	Constant I Uniformly Distributed Lateral Load	plus aggravating end moments
1B	Constant I Uniformly Distributed Lateral Load	plus one aggravating and one relieving end moment
1C	Constant I Uniformly Distributed Lateral Load	plus both relieving end moments
2	Constant I Sudden change in Uniformly Distributed Lateral Load	plus relieving end moments
3A	Constant I Single Concentrated Lateral Load	plus relieving end moments
*3B	Constant I Single Concentrated Lateral Load	plus relieving end moments
3C	Constant I Single Concentrated Lateral Load	plus aggravating end moments
4	Constant I Two Concentrated Lateral Loads	plus aggravating end moments
5A	I Varying Stepwise Uniformly Distributed Lateral Load	plus one aggravating and one relieving end moment
5B	No Lateral Load	plus end moments

*Type 3A differs from Type 3B only in so far as the relative value of W, the lateral load.

6.5.5 Type 1A

CONSTANT I: COMPRESSIVE END LOAD: UNIFORMLY DISTRIBUTED LATERAL LOAD + AGGRAVATING END MOMENTS



Procedure

(i) Find $a = \frac{L}{2} = 40$

$$\mu^2 = \frac{P}{EI} = \frac{6000}{8.68 \times 10^6} = 6.9 \times 10^{-4}$$

$$\mu = .0263$$

$$\mu a = .0263 \times 40 = 1.052 \text{ radians} = 60 \text{ degrees}$$

$$\frac{w}{\mu^2} = \frac{10}{6.9 \times 10^{-4}} = \underline{14500 \text{ lb. in.}}$$

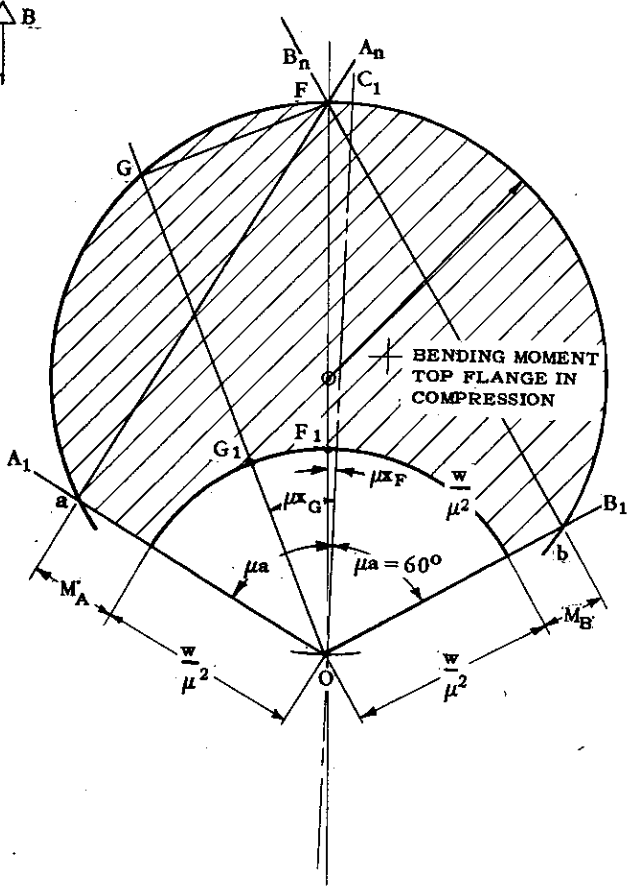
(ii) Draw OA_1 and OB_1 at 60 degrees to OC_1 and markoff points a and b. Draw normals aA_n and bB_n to meet at F. Construct a circle on OF as diameter.

(iii) Draw arc of radius $\frac{w}{\mu^2}$. Resultant B.M. is shown shaded. Max. M is given by FF_1 and equals <u>24,000</u> lb. in.

It occurs at $\mu x_F = 2.25$ degrees that is, at a distance $\frac{2.25}{60} \times 40 = 1.5$ in. to the left of C.

B.M. at any station G is GG_1 = <u>21700 lb. in.</u> which occurs at an angle μx_G from OC_1

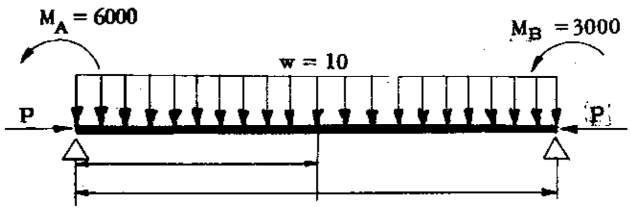
Shear at G is $FG \times \mu = 13350 \times .0263$
$= \underline{351 \text{ lb.}}$

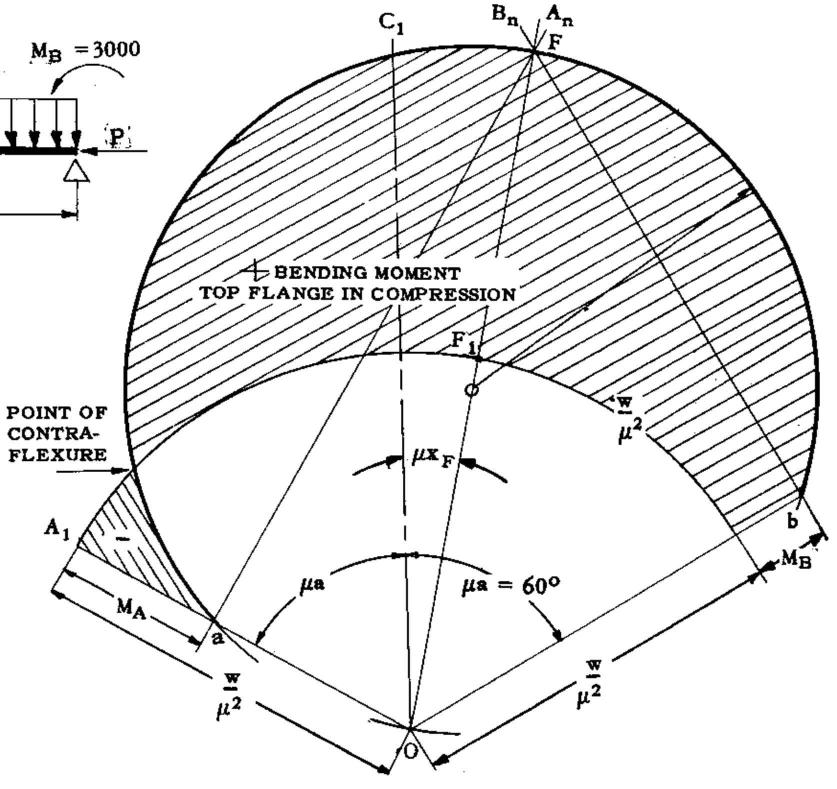


Type 1A

Type 1B

THE SAME AS TYPE 1A BUT M_A IS A RELIEVING MOMENT.





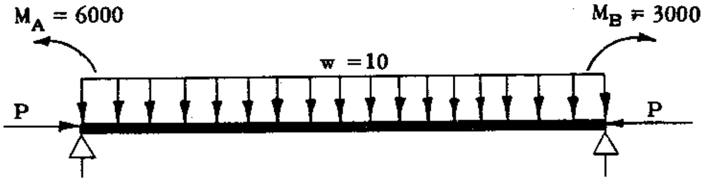
Procedure

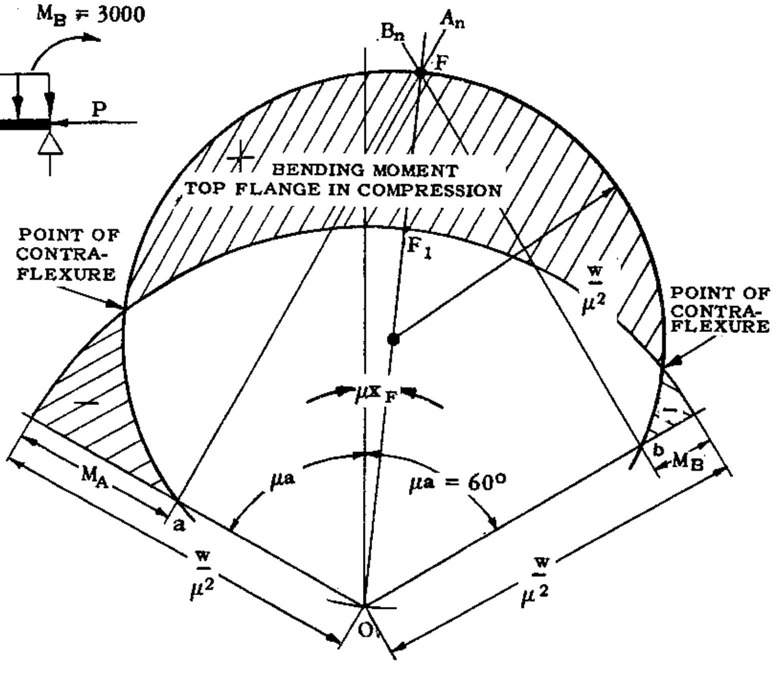
Similar to Type 1A except that point a is positioned differently Max. M = FF_1 = <u>11700 lb. in.</u> at μx_F = 11.5 degrees to the right of C

Type 1B

Type 1C

THE SAME AS TYPE 1A BUT M_A AND M_B ARE BOTH RELIEVING MOMENTS.





Procedure

Similar to Type 1A

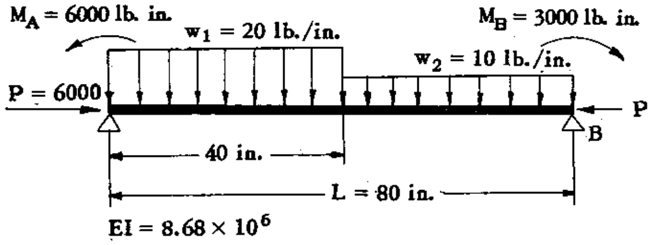
Max. M = FF_1 = <u>6100 lb. in.</u>

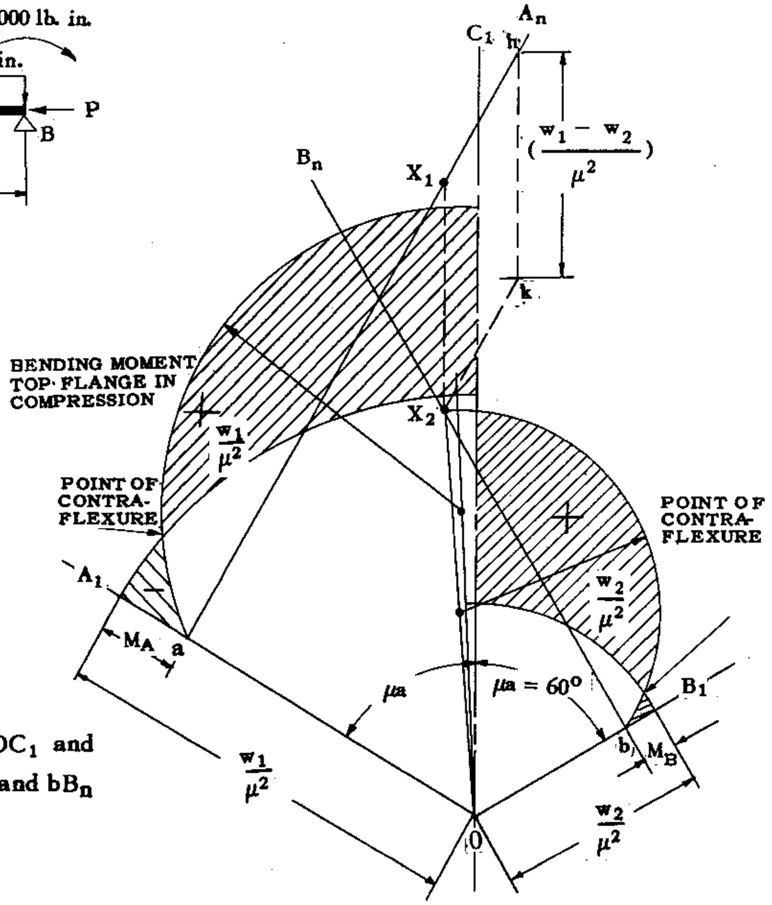
At μx_F = 6° to the right of C.

Type 1C

6.5.5 Type 2

CONSTANT 1. COMPRESSIVE END LOAD. SUDDEN CHANGE IN UNIFORMLY DISTRIBUTED LATERAL LOAD. + RELIEVING END MOMENTS.





Type 2

Procedure

(i) $\mu a = 60$ degree as for Case 1 A

$$\frac{w_1}{\mu^2} = 29000 \qquad \frac{w_2}{\mu^2} = 14{,}500$$

(ii) Draw OA_1 and OB_1 at 60 degrees to OC_1 and mark off points a and b Draw normals aA_n and bB_n

(iii) Draw arcs of radius $\frac{w_1}{\mu^2}$ and $\frac{w_2}{\mu^2}$

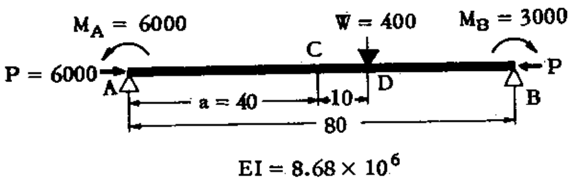
From any point h on aA_n draw hk of length $(\frac{w_1 - w_2}{\mu^2})$ parallel to OC_1 and from k draw kX_2 parallel to aA_n to meet normal bB_n in X_2. Draw the part-circle on OX_2 as diameter. From X_2 draw X_2X_1 parallel to OC_1 to meet normal aA_n in X_1. Construct part-circle on OX_1 as diameter.

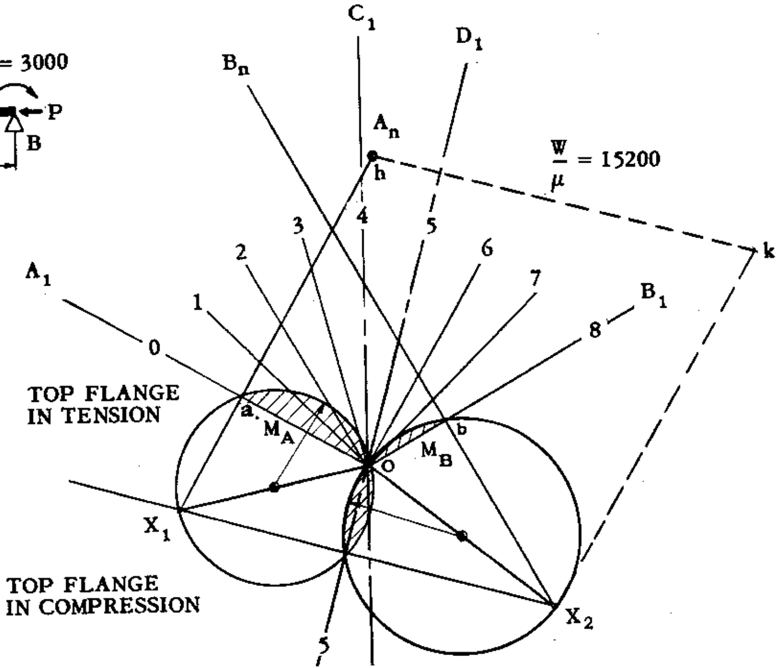
Resultant B.M. is shown shaded.

The same method can be employed for any combination of end moments.

6.5.5 Type 3A

CONSTANT I: COMPRESSIVE END LOAD. CONCENTRATED LATERAL LOAD + RELIEVING END MOMENTS.





Type 3A

Procedure

(i) $\mu a = 60$ degree as for Case I A

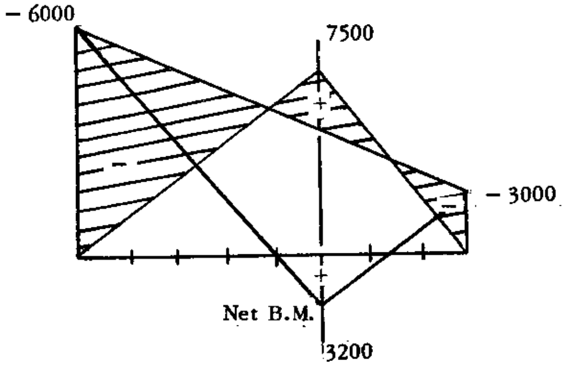
$$\frac{W}{\mu} = \frac{400}{.0263} = 15200$$

(ii) Draw OA_1 and OB_1 at 60 degree to OC_1 and draw OD_1 at 15 degree to OC_1 to represent the position of D (where W acts). Mark off points a and b on OA_1 and OB_1 and construct the normals aA_n and bB_n.

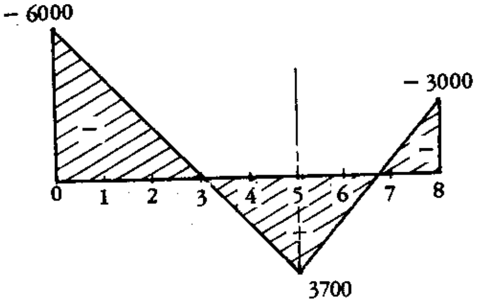
(iii) From any point h on aA_n draw (to the right of aA_n) a line hk of length $\frac{W}{\mu}$ at right angles to OD_1; from k draw kX_2 parallel to A_na to meet B_nb produced in X_2. Construct a part circle on OX_2 as diameter.

From X_2 draw X_2X_1 parallel to kh to meet A_na produced in X_1. Draw the part-circle on OX_1 as diameter. Resultant B.M. is shaded.

$$\text{Free BM} = \frac{Wab}{L} = \frac{400 \times 50 \times 30}{80} = 7500$$



B.M. Diagram without End Load on a Straight Base

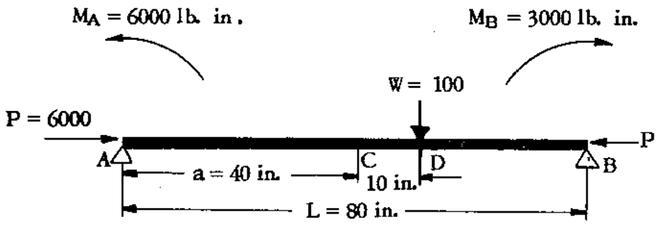


B.M. Diagram with End Load on a Straight Base

6.5.5 Type 3B

CONSTANT I, COMPRESSIVE END LOAD, CONCENTRATED LATERAL LOAD + RELIEVING END MOMENTS

This case differs from 3A only in so far as W (and hence $\frac{W}{\mu}$) is smaller. This results in a different diagram, as will be understood by considering the analogous cases when there is no axial load.



$EI = 8.68 \times 10^6$

Procedure

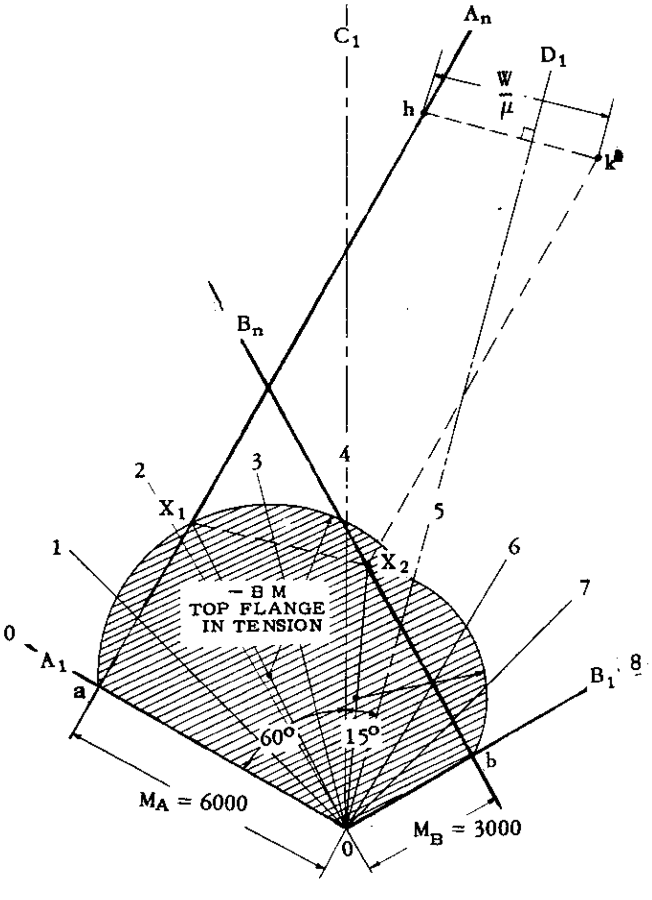
(i) $\mu a = 60$ degree as for Case 1 A
$\frac{W}{\mu} = \frac{100}{.0263} = 3800$

(ii) Draw OA_1 and OB_1 at 60 degree to OC_1 and draw OD_1 at 15 degree to OC_1 to represent the position of D (where W acts). Mark off points a and b on OA_1 and OB_1 and construct the normals aA_n and bB_n.

(iii) From any point h on aA_n draw (to the right of aA_n) a line hk of length $\frac{W}{\mu}$ at right angles to OD_1; from k draw kX_2 parallel to $A_n a$ to meet bB_n in X_2. Construct a part-circle on OX_2 as diameter.

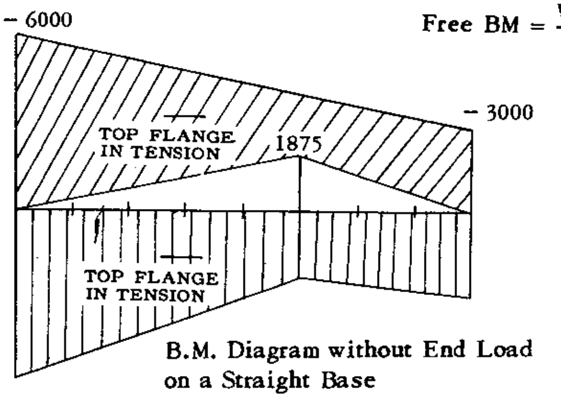
From X_2 draw X_2X_1 parallel to kh to meet aA_n in X_1. Draw the part-circle on OX_1 as diameter.

Resultant B.M. is shaded.

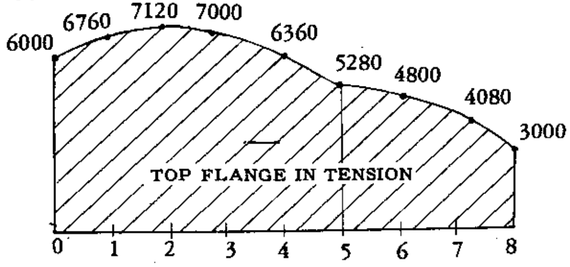


Type 3 B

$$\text{Free BM} = \frac{Wab}{L} = \frac{100 \times 50 \times 30}{80} = 1875$$



B.M. Diagram without End Load on a Straight Base

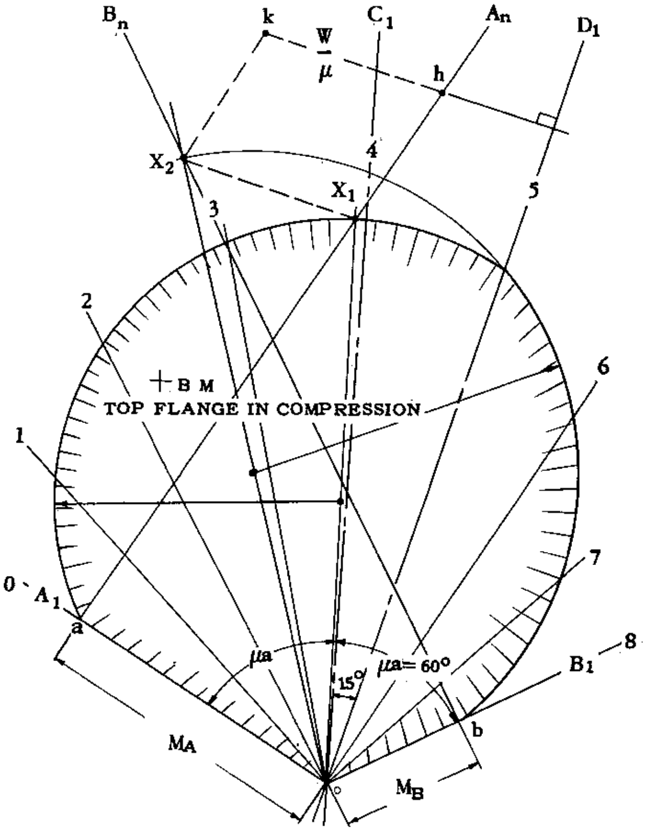


B.M. Diagram with End Load on a Straight Base

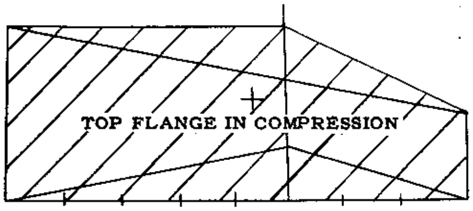
6.5.5 Type 3C

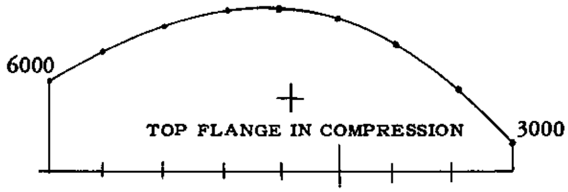
AS FOR TYPES 3A AND 3B BUT WITH AGGRAVATING END MOMENTS

The procedure is the same except that hk is struck off to the *left* of aA_n.



Type 3C



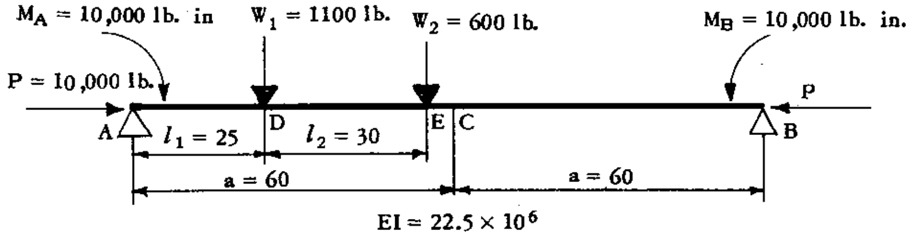


B.M. Diagram without End Load on a Straight Base

B.M. Diagram with End Load on a Straight Base

6.5.5 Type 4

CONSTANT I: COMPRESSIVE END LOAD. TWO CONCENTRATED LATERAL LOADS + AGGRAVATING END MOMENTS.



Procedure

(i) $\mu = \sqrt{\dfrac{P}{EI}} = \sqrt{\dfrac{10^4}{22.5 \times 10^6}} = .02105$

$\mu l_1 = .02105 \times 25 = .527 \text{ radians } = 30°11^1$

$\mu l_2 = .02105 \times 30 = .633 \text{ radians } = 36°12^1$

$\mu a = .02105 \times 60 = 1.261 \text{ radians } = 72°24^1$

$\dfrac{W_1}{\mu} = \dfrac{1100}{.02105} = 52250$

$\dfrac{W_2}{\mu} = \dfrac{600}{.02105} = 28500$

(ii) Draw OC_1 vertically and mark off OA_1 and OB_1 on each side of it at $72°24^1$. Then draw in OD_1 at $30°11^1$ to OA_1 and OE_1 at $36°12^1$ to OD_1.

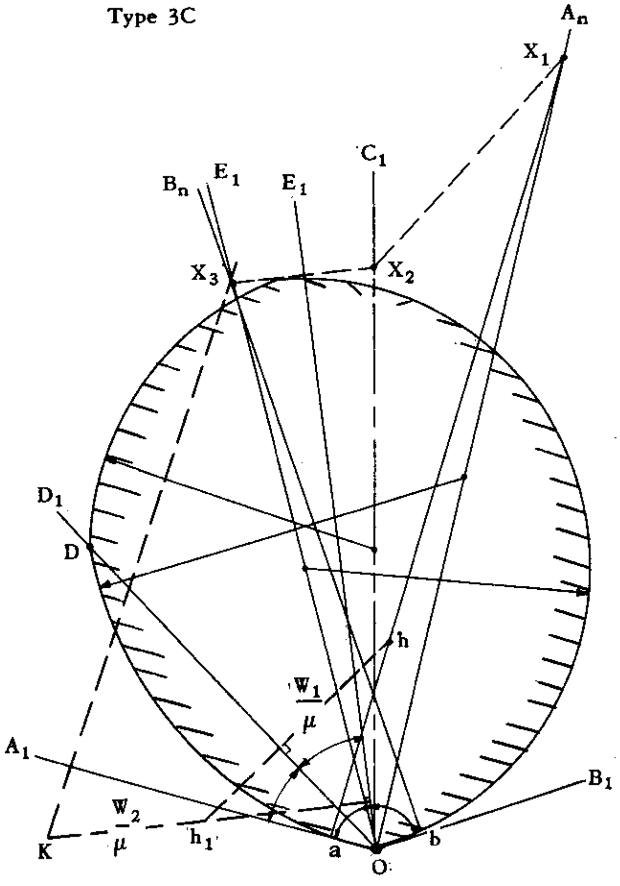
Mark off Oa to represent M_A and Ob to represent M_B and construct the normals aA_n and bB_n.

From any point h on aA_n draw hh_1 normal to OD_1 and equal to $\frac{W_1}{\mu}$. From h_1 draw h_1K equal to $\frac{W_2}{\mu}$ and normal to OE_1. From K draw KX_3 parallel to aA_n and cutting bB_n in X_3. From X_3 draw X_3X_2 parallel to Kh_1 to cut OC_1 in X_2. From X_2 draw X_2X_1 parallel to h_1h to cut aA_n in X_1.

Construct part-circles on OX_1, OX_2 and OX_3 as diameters.

Resultant B.M. is shaded.

This diagram is similar in general characteristics to Type 3C

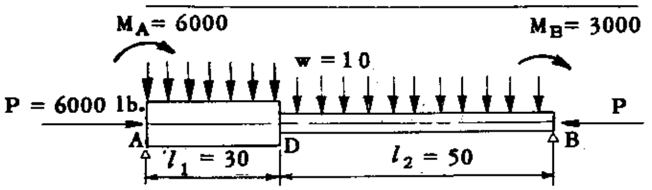


Type 4.

ANGLE A_1OC_1 = ANGLE B_1OC_1 = $72^0\ 24^1$
ANGLE A_1OD_1 = $30^0\ 11^1$
ANGLE D_1OE_1 = $36^0\ 12^1$

6.5.5 Type 5A

I VARYING STEPWISE. COMPRESSIVE END LOAD. UNIFORMLY DISTRIBUTED LATERAL LOAD + ONE AGGRAVATING AND ONE RELIEVING END MOMENT



$(EI)_1 = 12 \times 10^6$

$(EI)_2 = 8.68 \times 10^6$

Procedure

(i) $\mu_1{}^2 = \frac{P}{(EI)_1} = \frac{6000}{12 \times 10^6} = 5.0 \times 10^{-4}$

$\mu_2{}^2 = \frac{P}{(EI)_2} = \frac{6000}{8.68 \times 10^6} = 6.9 \times 10^{-4}$

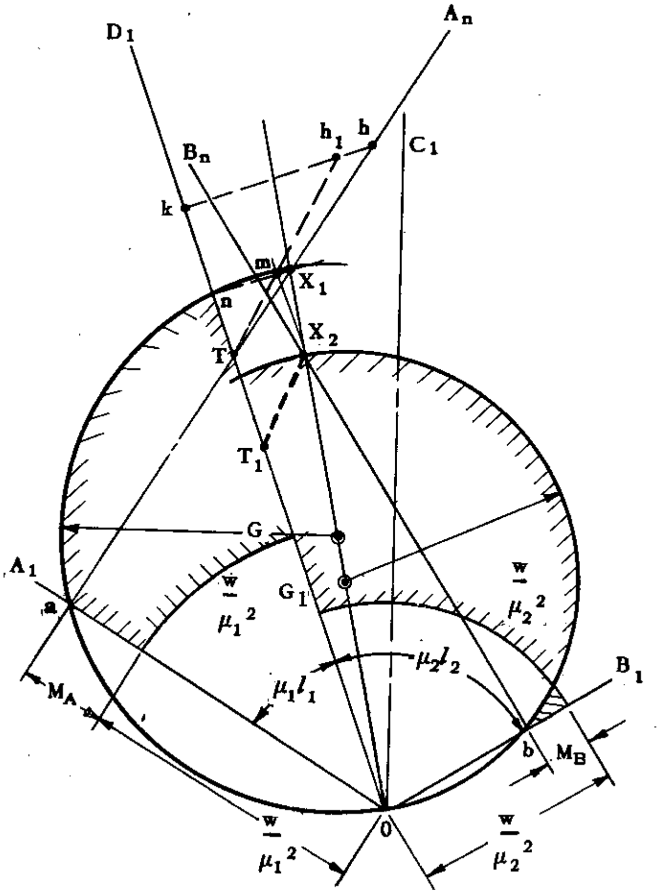
$\left.\begin{array}{l}\mu_1 = .0224 \\ \mu_2 = .0263\end{array}\right] \quad \frac{\mu_1}{\mu_2} = .852$

$\mu_1 l_1 = .0224 \times 30 = .672 \text{ radians} = 38^0 30^1$

$\mu_2 l_2 = .0263 \times 50 = 1.315 \text{ radians} = 75^0 30^1$

$\overline{114^0}$

$\frac{w}{\mu_1{}^2} = 20000 \qquad \frac{w}{\mu_2{}^2} = 14500$



Type 5A

(ii) Draw OC_1 vertically and OA_1 and OB_1 at $\frac{114}{2} = 57$ degrees on each side of it. Then put in OD_1 at an angle $\mu_1 l_1 = 38^0 30^1$ to OA_1. Mark off $\frac{w}{\mu_1{}^2}$ and M_A on OA_1 to give point a and $\frac{w}{\mu_2{}^2}$ and M_B on OB_1 to give point b. Draw normals aA_n and bB_n.

(iii) From any point h on aA_n draw hk normal to OD_1 to meet OD_1 in k. Choose a point h_1 on hk such that $\frac{hk}{h_1k} = \frac{\mu_2}{\mu_1}$ or $h_1k = \frac{\mu_1}{\mu_2}\, hk = .852\, hk$.

Join h_1 to T, the point of intersection of aA_n and OD_1 Mark off a length TT_1 on OD_1 equal to GG_1 that is, equal to $(\frac{w}{\mu_1{}^2} - \frac{w}{\mu_2{}^2})$, and from T_1 draw T_1X_2 parallel to Th_1 to cut bB_n in X_2. Draw a part circle on OX_2 diameter. From X_2 draw X_2m parallel to OD_1 to cut Th_1 in m.

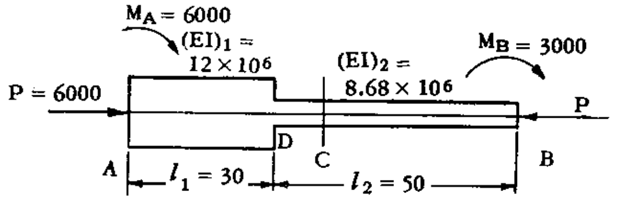
Construct the normal to OD_1 viz nmX_1 to cut aA_n in X_1.

Draw the part circle on OX_1 as diameter.

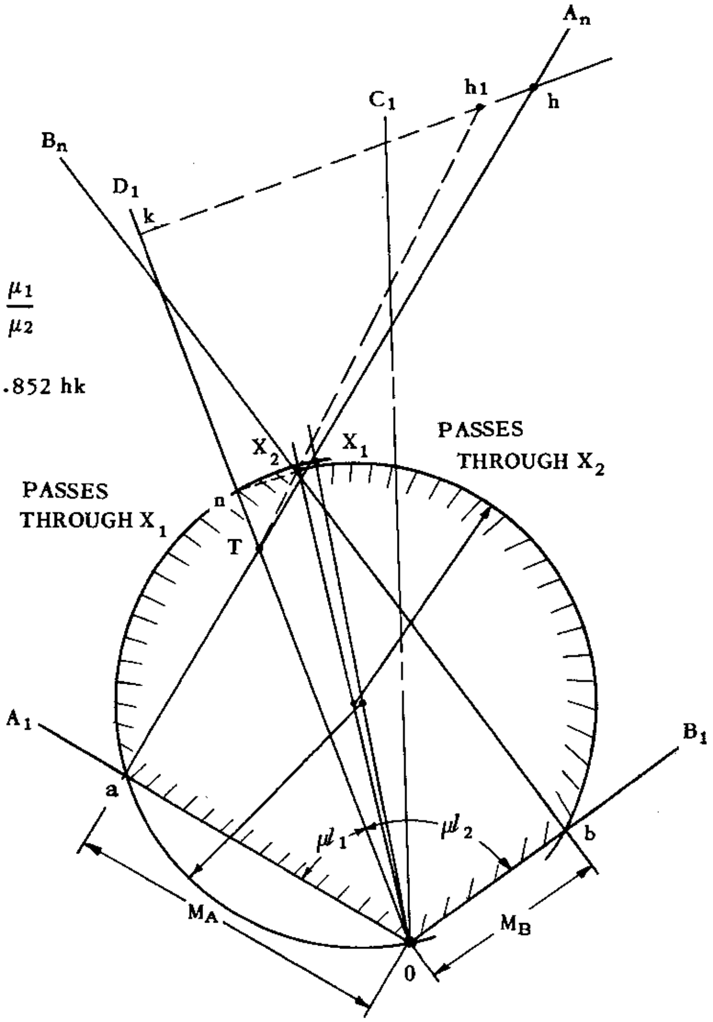
Resultant B.M. is shaded.

6.5.5 Type 5B

I VARYING STEPWISE. NO LATERAL LOAD BUT END MOMENTS.



Same construction as Type 5A, but since there is no lateral load X_2 coincides with m and T with T_1.



$$\frac{h_1k}{hk} = \frac{\mu_1}{\mu_2}$$

$$h_1k = .852\ hk$$

Type 5B

6.5.6 Tapered Beam-Columns: General Method

A general method for determining the distribution of bending moment, shear and deflection of a beam-column of varying I acted on by end moments and carrying a fairly complex lateral load is developed in "Aircraft Engineering", February 1953, p.56 by Gittleman.

Briefly the method is as follows:- Considering any segment KL of a deflected beam-column AB having ordinates x_K and x_L, such that $l = (x_L - x_K)$ Fig. 6.5.6–1(a), it is shown that if

M_K and M_L = bending moment ⎤ at K and L
θ_K and θ_L = slope ⎦ respectively

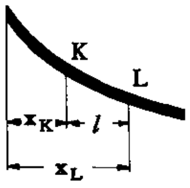


Fig. 6.5.6–1(a)

$$\theta_L = -M_K \frac{\mu}{P} \sin \mu l + \theta_K \cos \mu l + \alpha$$

$$M_L = M_K \cos \mu l + \theta_K \frac{P}{\mu} \sin \mu l + \beta$$

where

$$\alpha = \frac{\Sigma W - Q}{P} (1 - \cos \mu l) - \frac{w}{P} \left(\frac{\sin \mu l}{\mu} + x_K \cos \mu l - x_L \right)$$

$$\beta = -(\Sigma W - Q) \frac{\sin \mu l}{\mu} - \frac{w}{\mu^2} + \frac{w}{\mu} \left(\frac{\cos \mu l}{\mu} - x_K \sin \mu l \right)$$

$$\mu^2 = \frac{P}{EI} \quad \text{and} \quad Q = R_A{}' + \frac{M_B - M_A}{L} \quad \text{where } R_A{}' \text{ is the free reaction at A.}$$

α and β depend solely upon the loading and beam geometry.

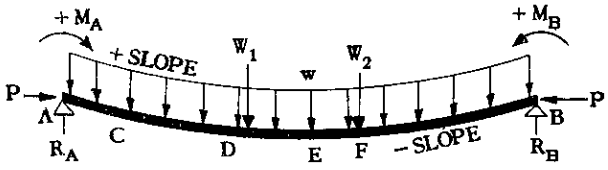


Fig. 6.5.6–1(b)

By first assuming that K coincides with A and that the slope there, viz $\theta_K = \theta_A$, is zero (which it may very well not be of course), it is possible to calculate α and β and find θ_L and M_L at C.

Having found these values for segment AC, consider the next segment CD and use the L values just found as K values for this second segment and find θ_L and M_L at D.

In this way θ_L at B = θ_B and M_L at B = M_B are ultimately found (by tabulation).

A further relationship is:– $M_B = a\,\theta_A + b$ where a and b are constants.

The calculation is repeated for another value of θ_A, two trial solutions being necessary.

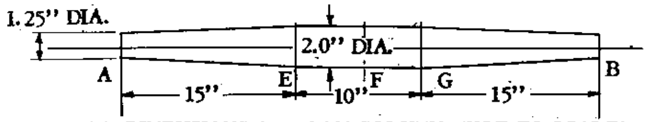
6.5.6 Tapered Beam-Columns: General Method (cont'd)

NUMERICAL EXAMPLE

The problem considered is the same as in the reference quoted above.

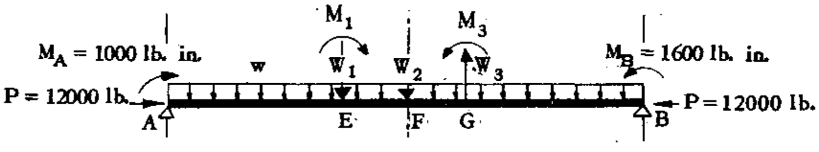
Although this particular beam-column is symmetrical about its mid-span, it is not necessary for this to be so, for the method is a general one in this respect.

Fig. 6.5.6–2(a) shows the dimensions of the beam (not to scale), (b) shows the complex loading and (c) the method of step-wise division, referred to below

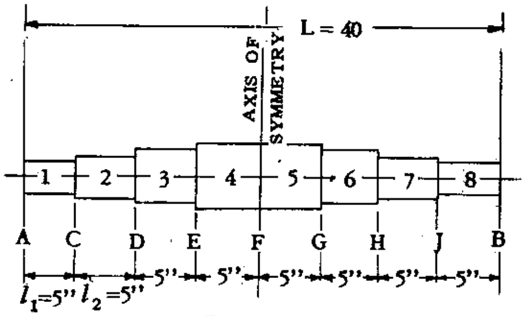


(a) DIMENSIONS OF BEAM-COLUMN (NOT TO SCALE)

M_1 = 600 lb. in. ↻
M_3 = 400 lb. in. ↺
W_1 = 800 lb. ↓
W_2 = 500 lb. ↓
W_3 = 800 lb. ↑
w = 20 lb./in. ↓



(b) LOADING ON BEAM-COLUMN



(c) STEP-WISE DIVISION

Fig. 6.5.6–2

PROCEDURE

(1) Divide the span into convenient segments. In this case each is 5 in. long, such that l = 5 constant, but it is neither necessary nor desirable that l should be constant in some instances. In fact, the subdivision into segments must be guided by the rule that *no concentrated load or moment must be applied within the length of a segment.* Note, for example, that the concentrated loads and moments occur at E, F and G at one end of a segment.

The diameter of cross-section of each segment is considered as uniform over its length and equal to that at the mid-section, that is, mid-way between A and C, C and D, etc.

(2) Consider the beam-column *as a beam pinned at the ends* and calculate the free reaction R_A' at A. By moments about B,

$$R_A' \times 40 + 600 - 400$$
$$= \frac{20}{2} \times 40^2 + 800 \times 25 - 800 \times 15 + 500 \times 20$$

$$R_A' = 845 \text{ lb.}$$

$$Q = R_A' + \frac{M_B - M_A}{L}$$
$$= 845 + \frac{1600 - 1000}{40} = 845 + 15 = \underline{860 \text{ lb.}}$$

(3) Find the sum of the concentrated loads ΣW at any station and hence (ΣW − Q).

For each segment calculate the values in Table 6.5.6–1 and then *a* (Table 6.5.6–2) and β (Table 6.5.6–3).

6.5.6 Tapered Beam-Columns: General Method (cont'd)

TABLE 6.5.6–1

SEGMENT		1	2	3	4	5	6	7	8
MEAN DIAMETER	D	1.375	1.625	1.875	2.0	2.0	1.875	1.625	1.375
	D^4	3.56	6.96	12.25	16.0	16.0	12.25	6.96	3.56
$I = \frac{\pi}{64} D^4$		.1748	.3415	.602	.786	.786	.602	.3415	.1748
$EI \times 10^{-6}$		5.244	10.245	18.06	23.58	23.58	18.06	10.245	5.244
$\mu^2 \times 10^4$		22.9	11.71	6.65	5.08	5.08	6.65	11.71	22.9
μ		.0477	.0341	.0257	.0226	.0226	.0257	.0341	.0477
μl (RADIANS)		.2385	.1705	.1285	.1130	.1130	.1285	.1705	.2385
(DEGREES)		13.7	9.77	7.36	6.48	6.48	7.36	9.77	13.7
$\sin \mu l$		.2368	.1696	.1282	.1126	.1126	.1282	.1696	.2368
$\cos \mu l$		.9717	.9855	.9917	.9936	.9936	.9917	.9855	.9717
Σw		–	–	–	800	1300	500	500	500
$(\Sigma w - Q)$		−860	−860	−860	- 60	440	−360	-360	-360

TABLE 6.5.6–2 FOR FINDING a

$\frac{(\Sigma w - Q)}{P}$	−.07167	−.07167	−.07167	−.005	.03667	−.030	−.030	−.030
$(1 - \cos \mu l)$	.0283	.0145	.0083	.0064	.0064	.0083	.0145	.0283
$10^3 \times \frac{(\Sigma w - Q)}{P}(1 - \cos \mu l)$	−2.028	−1.039	−.595	−.032	−.235	−.249	−.435	−.849
$\frac{\sin \mu l}{\mu}$	4.9539	4.9736	4.9883	4.9823	4.9823	4.9883	4.9736	4.9539
x_k	0	5	10	15	20	25	30	35
$x_k \cos \mu l$	0	4.9275	9.917	14.904	19.872	24.7925	29.565	34.0095
x_L	5	10	15	20	25	30	35	40
$(\frac{\sin \mu l}{\mu} + x_K \cos \mu l - x_L)$	−.0461	−.0989	−.0947	−.1137	−.1457	−.2192	−.4614	−1.0366
$\frac{w}{P}$	CONSTANT AT		1.666×10^{-3}					
$10^4 \frac{w}{P}(\frac{\sin \mu l}{\mu} + x_K \cos \mu l - x_L)$	−.77	−1.65	−1.58	−1.90	−2.43	−3.65	−7.69	−17.28
$10^4 a$	−19.51	−8.74	−4.37	1.58	4.78	1.16	3.34	8.79

6.5.6 Tapered Beam-Columns: General Method (cont'd)

TABLE 6.5.6–3 FOR FINDING β

	1	2	3	4	5	6	7	8
$(\Sigma W - Q)$	− 860	− 860	− 860	− 60	440	− 360	− 360	− 360
$\dfrac{\sin \mu l}{\mu}$	4.9539	4.9736	4.9883	4.9823	4.9823	4.9883	4.9736	4.9539
$(\Sigma W - Q)\dfrac{\sin \mu l}{\mu}$	−4260	−4277	−4290	− 299	2192	−1796	−1790	−1783
$\dfrac{W}{\mu^2}$	8790	17199	30280	39159	39159	30280	17199	8790
$\dfrac{\cos \mu l}{\mu}$	20.3711	28.9003	38.5875	43.9646	43.9646	38.5875	28.9003	20.3711
$x_K \sin \mu l$	0	.8480	1.2820	1.6890	2.2520	3.2050	5.0880	8.2705
$\dfrac{\cos \mu l}{\mu} - x_K \sin \mu l$	20.3711	28.0523	37.3055	42.2756	41.7126	35.3825	23.8123	12.1006
$\dfrac{W}{\mu}(\cos \dfrac{\mu l}{\mu} - x_K \sin \mu l)$	8542	16453	29031	37414	36916	27535	13966	5074
β	4012	3531	3041	−1446	−4435	− 949	−1443	−1933

(4) Assume that K coincides with A and take $M_K = 1000$ } (the known fixing moment at A) and $\theta_K = 0$, as an initial assumption.

Then for segment AC

$$\theta_L = -M_K \frac{\mu}{P} \sin \mu l + a$$

$$= -1000 \times .939 \times 10^{-6} - 19.51 \times 10^{-4}$$

$$= -9.39 \times 10^{-4} - 19.51 \times 10^{-4} = -\underline{28.90 \times 10^{-4}}$$

$$M_L = 1000 \cos \mu l + \beta$$

$$= 971.7 + 4012 = \underline{4984}$$

Assume now that K coincides with C and L with D

$$M_K = M_L \text{ above} = 4984$$

$$\theta_K = \theta_L \text{ above} = -28.9 \times 10^{-4}$$

Hence determine θ_L and M_L for segment CD and for all segments, by tabulation, as shown in Tables 6.5.6–4 and –5.

It is found that M_L at B = $M_B = -2867$ }

6.5.6 Tapered Beam-Columns: General Method (cont'd)

TABLE 6.5.6–4

	1	2	3	4	5	6	7	8
$\frac{\mu}{P}\sin\mu l \times 10^{-6}$	.939	.482	.275	.212	.212	.275	.482	.939
M_K	1000	4984	8271	11476	9432	3879	2152	− 116
$-M_K\frac{\mu}{P}\sin\mu l \times 10^{-4}$	−9.39	−24.02	−22.75	−24.33	−20.00	−10.67	−10.37	−1.09
$\theta_K \times 10^{-4}$	0	−28.9	−61.24	−87.85	−110.04	−124.56	−133.05	−138.15
$\cos\mu l$	.9717	.9855	.9917	.9936	.9936	.9917	.9855	.9717
$\theta_K \cos\mu l \times 10^{-4}$	0	−28.48	−60.73	−87.29	−109.34	−123.54	−131.12	−134.24
$10^4\,\alpha$	−19.51	−8.74	−4.37	1.58	4.78	1.16	3.34	8.79
$10^4\,\theta_L$	− 28.9	−61.24	−87.85	−110.04	−124.56	−133.05	−138.15	−124.36

TABLE 6.5.6–5

	1	2	3	4	5	6	7	8
M_K	1000	4984	8271	11476	9432	3879	2152	−116
$M_K \cos\mu l$	972	4912	8202	11403	9372	3847	2121	−113
$\frac{P}{\mu}\sin\mu l$	59446	59683	59860	59788	59788	59860	59683	59446
$\theta_K \frac{P}{\mu}\sin\mu l$	0	−172	−367	−525	−658	−746	−794	−821
β	4012	3531	3041	−1446	−4435	−949	−1443	−1933
M_L	4984	8271	10876	9432	4279	2152	−116	$\boxed{-2867}$

Note that when transferring M_L from column 3 to M_K in column 4, it is necessary to add on the applied moment of 600 lb.in. at E, thus making 10876 + 600 = 11476 as indicated by the dotted line in Table 6.5.6–5.

Similarly when transferring M_L = 4279 from column 5 to column6 it is reduced by 400 to 3879, because of the applied moment at G.

6.5.6 Tapered Beam-Columns: General Method (cont'd)

The deflected shape of the beam, *relative to A*, is shown diagrammatically in Fig. 6.5.6–3. (It is thus because all the values of θ_L are negative).

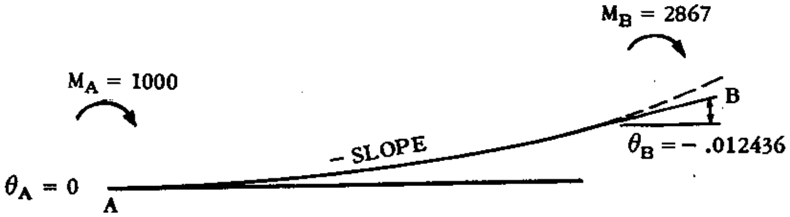


Fig. 6.5.6–3

From the relationship

$$\begin{aligned} M_B &= a\theta_A + b \\ -2867 &= 0 \quad + b \\ b &= -\ 2867 \end{aligned}$$

SECOND APPROXIMATION

Assume a second value $\theta_A = 0.10$ and take $M_A = 1000$ again. Carry out a similar calculation (not shown here or in the original reference) to give $M_B = 34968$

$$\begin{aligned} M_B &= a\theta_A + b \\ 34968 &= a \times 0.10 - 2867 \\ 0.10a &= 37835 \\ a &= 378{,}350 \end{aligned}$$

It is known that the actual value of $M_B = 1600$

$$\begin{aligned} \text{Hence } 1600 &= 378{,}350\ \theta_A - 2867 \\ \text{and } \theta_A &= 0.011807 \end{aligned}$$

FINAL SOLUTION

Start with $\theta_A = 0.011807$ and $M_A = 1000$) and work through as in Table 6.5.6–6 to find θ_L and M_L for all segments

Values at B are $\theta_B = -\ 0.010075$
$M_B = 1598$)

SHEAR AND DEFLECTION

Refer to the original article for the method to be used.

6.5.6 Tapered Beam-Columns: General Method (cont'd)

TABLE 6.5.6–6

	1	2	3	4	5	6	7	8
$\frac{\mu}{P} \sin \mu l \times 10^{-6}$	.939	.482	.275	.212	.212	.275	.482	.939
M_K	1000	5686	9647	13498	12068	7096	5904	4083
$-M_K \frac{\mu}{P} \sin \mu l \times 10^{-4}$	−9.39	−27.41	−26.53	−28.62	−25.59	−19.51	−28.46	−38.34
$\theta_K \times 10^{-4}$	118.07	85.83	48.44	17.14	−10.01	−30.76	−48.86	−73.27
$\cos \mu l$	.9717	.9855	.9917	.9936	.9936	.9917	.9855	.9717
$\theta_K \cos \mu l \times 10^{-4}$	14.73	84.59	48.04	17.03	−9.95	−30.51	−48.15	−71.20
$10^4 \alpha$	−19.51	−8.74	−4.37	1.58	4.78	1.16	3.34	8.79
$10^4 \theta_L$	85.83	48.44	17.14	−10.01	−30.76	−48.86	−73.27	−100.75

TABLE 6.5.6–7

M_K	1000	5686	9647	13498	12068	7096	5904	4083
$M_K \cos \mu l$	972	5604	9567	13412	11991	7037	5818	3967
$\frac{P}{\mu} \sin \mu l$	59446	59683	59860	59788	59788	59860	59683	59446
$\theta_K \frac{P}{\mu} \sin \mu l$	702	512	290	102	−60	−184	−292	−436
β	4012	3531	3041	−1446	−4435	−949	−1443	−1933
M_L	5686	9647	12898	12068	7496	5904	4083	$\boxed{1598}$

6.5.7 Beam-Columns

STIFFNESS AND CARRY-OVER CHARTS (to include effect of End Load)

Stiffness and carry-over factors, required for carrying out moment distribution calculations of beams with axial load and lateral load, are given herein in chart form.

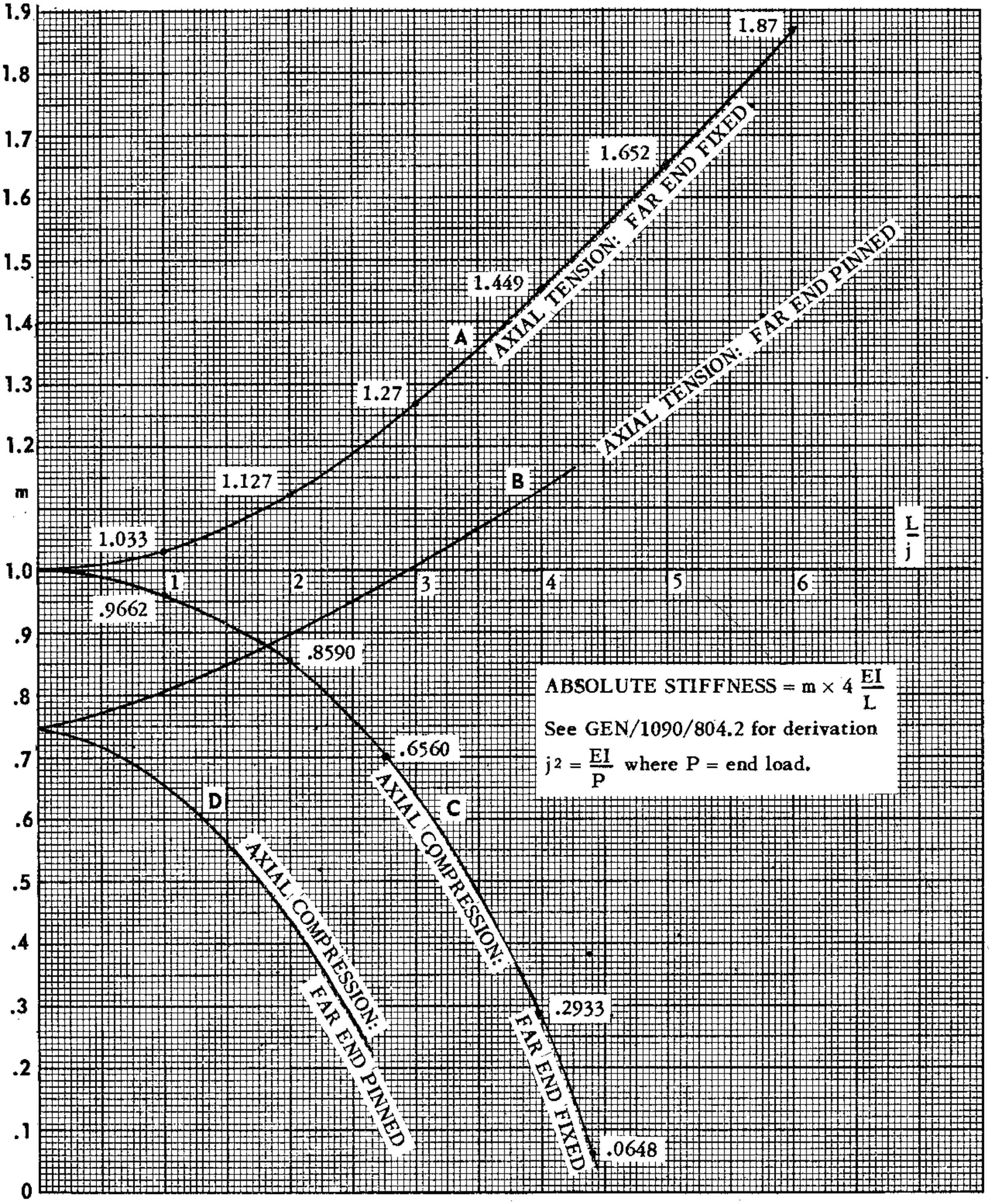


Fig. 6.5.7–1 Stiffness Factor m for Constant I Beam-Columns (From N.A.C.A., T.N. 534)

6.5.7 Beam-Columns (cont'd)

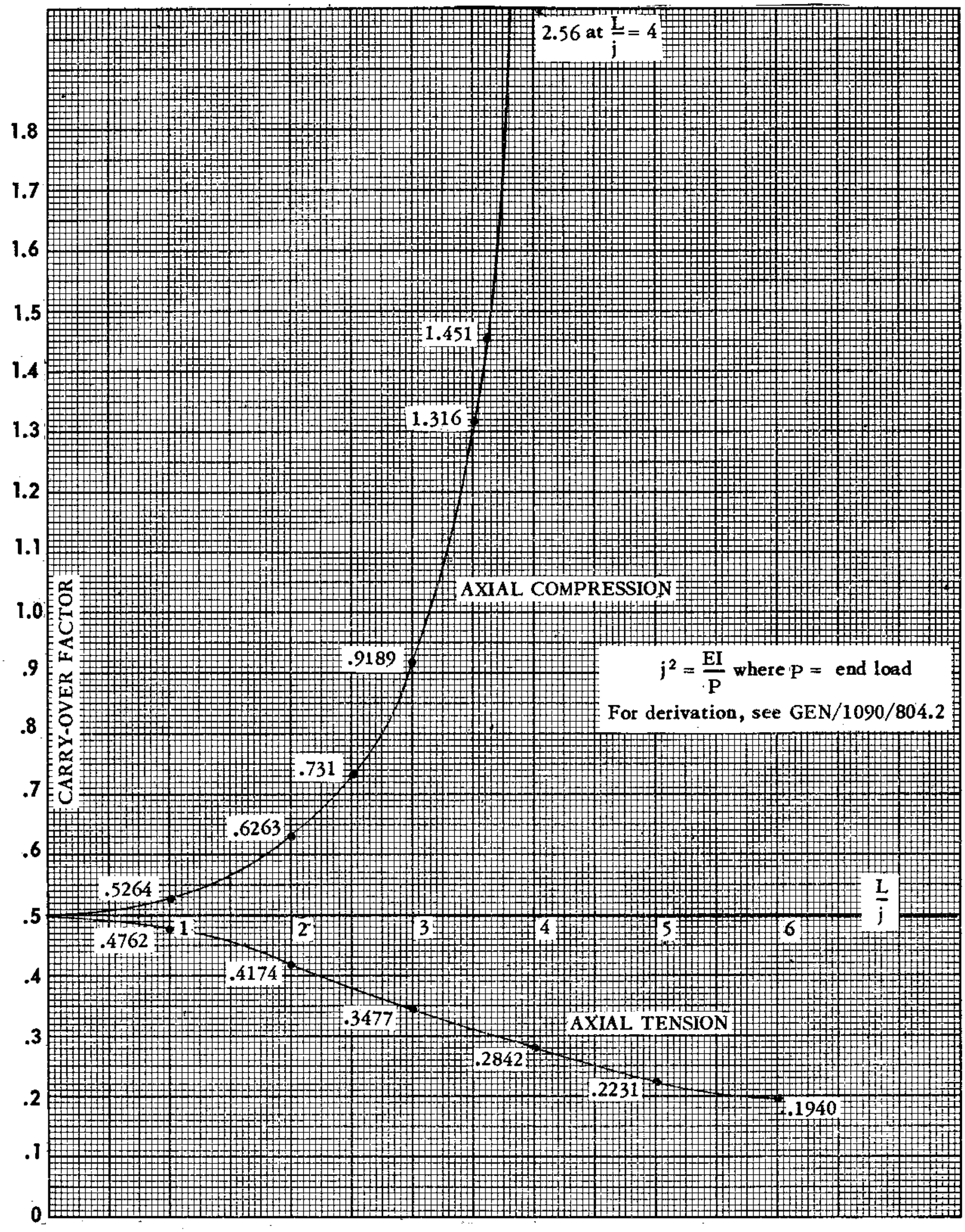


Fig. 6.5.7–2 Carry-Over Factors for Constant I Beam-Columns (From N.A.C.A., T.N. 534)

6.6 COLUMNS WITH SMALL INITIAL BOW

TABLE OF CONTENTS

6.6.1 Columns with Small Initial Bow

$j^2 = \frac{EI}{P}$ $U = \frac{L}{j}$

<table>
<tr><th>COLUMNS WITH SMALL INITIAL BOW</th><th>+ B.M. PUTS TOP FLANGE IN COMPRESSION</th></tr>
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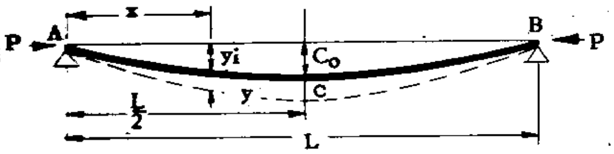
(1)

SIMPLY SUPPORTED AT ENDS
COMPRESSIVE END LOAD
NO LATERAL LOAD



(Deflected shape shown dotted)

INITIAL BOW (SINUSOIDAL) $y_i = C_o \text{ SIN } \frac{\pi x}{L}$

Where C_o = initial bow at centre C

</td><td>

At any station distant x from A,

$$M = \frac{Q}{Q-P} P.C_o \sin \frac{\pi x}{L} \quad \text{(Perry approximation)}$$

where Q = Euler failing load for pinned ends = $\frac{\pi^2 EI}{L^2}$

Max. $M = \frac{Q}{Q-P} P.C_o$ at C

Deflection relative to line A B is

$$y + y_i = \frac{Q}{Q-P} C_o \sin \frac{\pi x}{L}$$

$$= \frac{Q}{Q-P} C_o \text{ at C where it is a max.}$$

For derivation see GEN/1090/811.8

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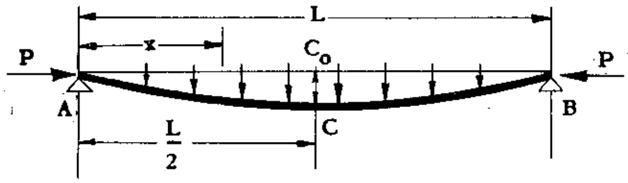
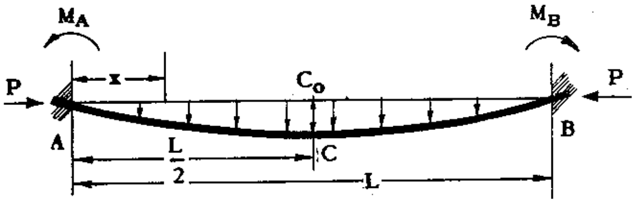
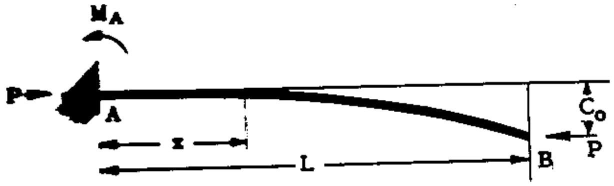
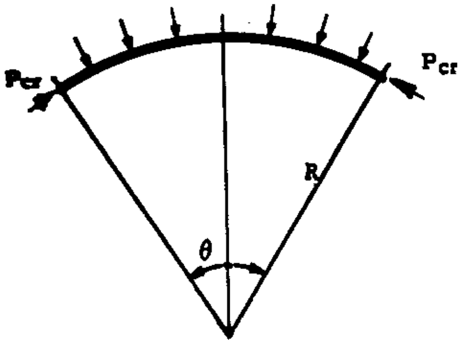
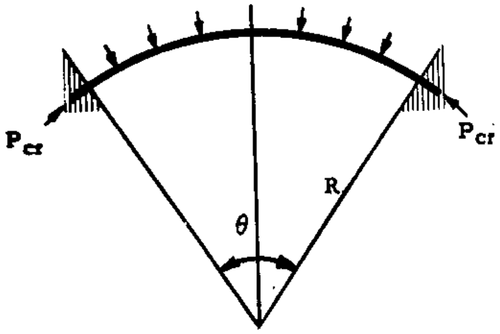
6.6.1 Columns with Small Initial Bow (cont'd)

$j^2 = \frac{EI}{P} \quad U = \frac{L}{j}$

COLUMNS WITH SMALL INITIAL BOW	+ B.M. PUTS TOP FLANGE IN COMPRESSION.
(2) SIMPLY SUPPORTED AT ENDS COMPRESSIVE END LOAD + UNIFORMLY DISTRIBUTED LATERAL LOAD (INCREASING INITIAL BOW)  INITIAL BOW (PARABOLIC) $y_i = Kx\,(L-x)$ Where $k = \frac{4C_o}{L^2}$ and C_o = initial bow at centre C	At any station distant x from A, $M = A\cos\frac{x}{j} + B\sin\frac{x}{j} - j^2(w+2Pk)$ where $A = j^2(w+2Pk)$, $B = j^2(w+2Pk)\tan\frac{U}{2}$ Alternatively, use a polar diagram, as in 6.6.3 Max. $M = j^2(w+2Pk)\left(\sec\frac{U}{2} - 1\right)$ at C These expressions are basically similar to those in 6.5.2.2, Case 3A with the addition of a term involving k, due to the initial bow. When the lateral load *decreases* the initial bow, use $(w-2Pk)$ instead of $(w+2Pk)$ throughout.
(3) FIXED AT ENDS. COMPRESSIVE END LOAD + UNIFORMLY DISTRIBUTED LATERAL LOAD (INCREASING INITIAL BOW)  INITIAL BOW (PARABOLIC) $y_i = Kx\,(L-x)$ Where $k = \frac{4C_o}{L^2}$ and C_o = initial bow at centre C	$M_A = M_B = -j^2(w+2Pk)\left[1 - \frac{U}{2}\cot\frac{U}{2}\right]$ At any station distant x from C, $M = A\cos\frac{x}{j} + B\sin\frac{x}{j} - j^2(w+2Pk)$ where $A = j^2(w+2Pk) - M_A$ $B = j^2\left[(w+2Pk) - M_A\right]\tan\frac{U}{2}$ Alternatively, use a polar diagram, as in 6.6.3. Max. $+M = j^2(w+2Pk)\left[\frac{U}{2}\operatorname{cosec}\frac{U}{2} - 1\right]$ at C. When the lateral load *decreases* the initial bow, use $(w-2Pk)$ instead of $(w+2Pk)$ throughout.

6.6.1 Columns with Small Initial Bow (cont'd)

$j^2 = \frac{EI}{P}$ $U = \frac{L}{j}$

COLUMNS WITH SMALL INITIAL BOW	+ B.M. PUTS TOP FLANGE IN COMPRESSION
(4) CANTILEVER  INITIAL BOW $y_i = C_o \sin \frac{\pi x}{L}$	$M_A = - \frac{Q}{Q-P} P.Co$ where Q = Euler failing load for pinned ends $= \frac{\pi^2 EI}{L^2}$
(5) ARCH PINNED AT ENDS EXTERNAL RADIAL PRESSURE 	$P_{cr} = \left[4\left(\frac{\pi}{\theta}\right)^2 - 1\right] \frac{EI}{R^3}$ Proof in Den Hartog Advanced strength of Materials, p. 282.
(6) ARCH FIXED AT ENDS EXTERNAL RADIAL PRESSURE 	$P_{cr} = \left[9\left(\frac{\pi}{\theta}\right)^2 - 1\right] \frac{EI}{R^3}$

6.6.2 Numerical Example

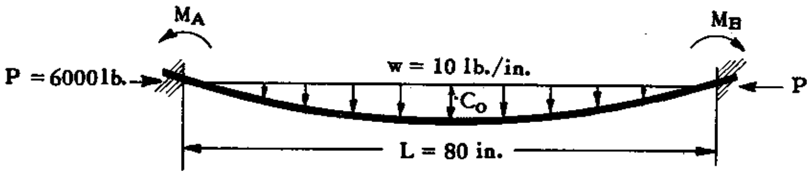


Fig. 6.6.2–1

Slightly bowed column of Fig. 6.6.2–1 having max. bow at centre $c_o = 2$ in. To find M_A, M_B and M_C. $EI = 8.68 \times 10^6$.

From tabular summary 6.6.1, Case (3):

$$M_A = -j^2\ (w + 2Pk)\ \left[1 - \frac{U}{2}\cot\frac{U}{2}\right]$$

$$j^2 = \frac{EI}{P} = \frac{8.68 \times 10^6}{6000} = 1448$$

$$\begin{aligned} M_A &= -\,1448\ (10 + 15)\ \left[1 - .602\right] \\ &= -\,3.62 \times 10^4 \times .398 \\ &= -\,14400 \end{aligned}$$

$$M_C = j^2\ (w + 2Pk) \left[\frac{U}{2}\ \text{cosec}\ \frac{U}{2} - 1\right]$$

$$\begin{aligned} &= 3.62 \times 10^4\ \left[1.212 - 1\right] \\ &= 3.62 \times 10^4 \times .212 \\ &= \underline{7670} \end{aligned}$$

$$k = \frac{4c_o}{L^2} = \frac{4 \times 2}{6400} \doteq 12.5 \times 10^{-4}$$

$$2Pk = 2 \times 6000 \times 12.5 \times 10^{-4} = 1.2 \times 12.5 = 15$$

$$\frac{U}{2} = \frac{L}{2j} = \frac{80}{76.2} = 1.051 \text{ radians} = 60.2 \text{ degrees}$$

$$\cot\frac{U}{2} = .5727$$

$$\begin{aligned} \frac{U}{2}\cot\frac{U}{2} &= 1.051 \times .5727 \\ &= .602 \end{aligned}$$

$$\text{cosec}\frac{U}{2} = 1.1524$$

$$\begin{aligned} \frac{U}{2}\text{cosec}\ \frac{U}{2} &= 1.051 \times 1.1524 \\ &= 1.212 \end{aligned}$$

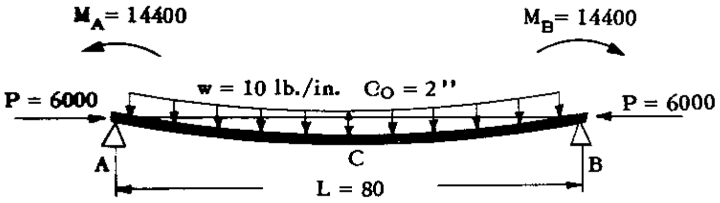
This value checks with that obtained from the Polar Diagram in 6.6.3.

6.6.3 Polar Diagrams

The method of drawing the polar diagram for a column with small initial bow is shown below.

The derivation is in "Aircraft Engineering", Oct. 1952 p. 288 "Solutions of Problems on the Slightly Curved Beam" by Ratzersdorfer.

CONSTANT I: COMPRESSIVE END LOAD. UNIFORMLY DISTRIBUTED LATERAL LOAD (AGGRAVATING INITIAL BOW) + RELIEVING END MOMENTS



GIVEN DATA: Initial bow $c_O = 2.0$ in.

$EI = 8.68 \times 10^6$
$\mu^2 = 6.9 \times 10^{-4}$
$\frac{w}{\mu^2} = 14500$
$\mu a = 60$ degrees

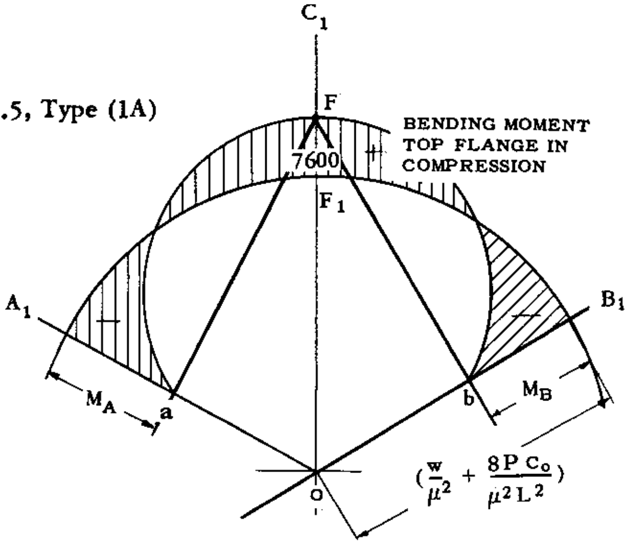
From Section 6.5.5, Type (1A)

$$8Pc_O = 8 \times 6000 \times 2 = 9.6 \times 10^4$$

$$\mu^2 L^2 = 6.9 \times 10^{-4} \times 6400 = 4.42$$

$$\frac{8Pc_O}{\mu^2 L^2} = \frac{9.6 \times 10^4}{4.42} = 2.175 \times 10^4$$

$$\left(\frac{w}{\mu^2} - \frac{8Pc_O}{\mu^2 L^2}\right) = 14500 + 21750 = \underline{36250}$$



Procedure

As for 6.5.5, Type IC except that an arc of radius $\left(\frac{w}{\mu^2} + \frac{8Pc_O}{\mu^2 L^2}\right)$, and not $\frac{w}{\mu^2}$, is struck from O

If w acts so as to relieve the initial bow, strike an arc from O of radius $\left(\frac{w}{\mu^2} - \frac{8Pc_O}{\mu^2 L^2}\right)$

If M_A or M_B is an aggravating moment the position of a and b will be changed accordingly.

B.M. at C is FF_1 = 7650 lb. in.