

S/M/02.

MATRIX METHOD OF STRUCTURAL
ANALYSIS INCLUDING TEMPERATURE
EFFECTS

R. E. VALE

JULY 1959.



A. V. ROE CANADA LIMITED

MALTON - ONTARIO

TECHNICAL DEPARTMENT (Aircraft)

AIRCRAFT:

REPORT NO. 51M/02

FILE NO:

NO. OF SHEETS: 64

TITLE:

MATRIX METHOD OF STRUCTURAL
ANALYSIS INCLUDING TEMPERATURE
EFFECTS.

(STRESS DISTRIBUTION & INFLUENCE COEFF.)

PREPARED BY P.E. VALE DATE JULY/59

CHECKED BY _____ DATE _____

SUPERVISED BY _____ DATE _____

APPROVED BY _____ DATE _____

ISSUE NO.	REVISION NO.	REVISED BY	APPROVED BY	DATE	REMARKS



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

INTRODUCTION

An attempt is made to explain and illustrate the matrix method of structural analysis as developed by Dr. A. Grecozielski in Avro Report Gen/1090/336. In order to do this, the conventional redundant analysis with illustrative examples is outlined in Section 1.

This method of analysis is very similar to that in Sechler and Dunn: "Airplane Structural Analysis" p. 142.

The analysis in matrix notation is then outlined in Section 2 and purposely parallels that of Section 1 for aid in comparison. The effect of temperature is then treated in Section 3 using both notations again with an illustrative example.



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. 11

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

REFERENCES

1. DR. P. GREEDZIELSKI
"THEORY OF MULTI-SPAR AND MULTI-RIB WING
STRUCTURES" AVRO STRESS REPORT GEN/1020/336 - JUNE 1935
2. SECHLER AND DUNN
"AIRPLANE STRUCTURAL ANALYSIS AND DESIGN"
JOHN WILEY & SONS, INC. 1942
3. S. TIMOSHENKO
"THEORY OF ELASTICITY" - MCGRAW-HILL, 1934



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. 1-1

AIRCRAFT: _____

PREPARED BY

DATE

R. VALE

CHECKED BY

DATE

SECTION 1 :-

SIMPLY REDUNDANT STRUCTURES

BY NORMAL METHOD

IN THE USUAL TREATMENT OF REDUNDANT STRUCTURES, THE STRESS (OR LOAD) IN EACH MEMBER IS THOUGHT TO BE MADE UP OF TWO PARTS - THE STATICALLY DETERMINANT AND THE REDUNDANT.

$$S_i = \text{STRESS IN EACH MEMBER} \\ = \bar{S}_i + \bar{S}_i \quad (1)$$

where $\bar{S}_i$ = STATICALLY DETERMINANT STRESS
 $\bar{S}_i$ = STRESS DUE TO REDUNDANCIES

THE STRAIN ENERGY CAN BE WRITTEN:-

$$V = \sum_{i=1}^{i=n} M_i S_i^2$$

WHERE M_i IS A MODULUS WHICH DEPENDS ON THE ROLE PLAYED BY THE MEMBER.

ie:- $M_i = \frac{L_i A_i}{2E}$ FOR END LOAD CARRYING MEMBERS.



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

1-2

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

FOR SAKE OF SIMPLICITY WE WILL ASSUME THAT ALL MEMBERS DO CARRY END LOAD.

$$\text{THEN: } V = \sum_{i=1}^{i=n} \frac{L_i A_i}{2E} S_i^2 \quad (2)$$

$$= \sum_{i=1}^{i=n} \frac{L_i A_i}{2E} (\bar{S}_i + \bar{S}_i)^2 \quad (3)$$

LET k_{ij} = STRESS IN THE i TH MEMBER DUE TO THE j TH REDUNDANT LOAD EQUAL TO UNITY

AND F_j = ACTUAL VALUE OF THE j TH REDUNDANCY

$$V = \sum_{i=1}^{i=n} \frac{L_i A_i}{2E} (\bar{S}_i + k_{ij} F_j)^2 \quad (4)$$

BY CASTIGLIANO

$$\frac{\partial V}{\partial F_j} = \sum_{i=1}^{i=n} \frac{L_i A_i}{E} (\bar{S}_i + k_{ij} F_j) k_{ij} = 0 \quad (5)$$

THIS RESULTS IN j EQUATIONS FROM WHICH THE F_j REDUNDANCIES CAN BE SOLVED.



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

1-3

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

$$\sum_{i=1}^{i=n} \frac{L_i A_i}{E} \ddot{S}_i k_{ij} + \sum_{i=1}^{i=n} \frac{L_i A_i}{E} F_i k_{ij}^2 = 0 \quad (6)$$

$$F_i = - \frac{\sum \frac{L_i A_i}{E} \ddot{S}_i k_{ij}}{\sum \frac{L_i A_i}{E} k_{ij}^2} \quad (7)$$

THEN

$$S_i = \ddot{S}_i + k_{ij} F_j \quad (8)$$

$$= \ddot{S}_i - k_{ij} \frac{\sum \frac{L_i A_i}{E} \ddot{S}_i k_{ij}}{\sum \frac{L_i A_i}{E} k_{ij}^2} \quad (9)$$

THIS CAN BE DONE FOR EACH LOADING CASE OR FOR EACH UNIT LOAD.

LET US ASSUME THAT WE HAVE "a" LOADING POINTS AND THAT THE PROBLEM HAS BEEN SOLVED FOR A UNIT LOAD AT EACH OF THESE POINTS.

ie

$$S_{ia} = \ddot{S}_{ia} + k_{ij} F_{ja} \quad (10)$$

$$= \ddot{S}_{ia} - k_{ij} \left(\frac{\sum \frac{L_i A_i}{E} \ddot{S}_i k_{ij}}{\sum \frac{L_i A_i}{E} k_{ij}^2} \right)_a \quad (11)$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

1-4

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

THEN IF Q_a IS THE ACTUAL VALUE OF LOADS AT EACH LOAD POINT.

$$S_i = \sum_{a=1}^{a=m} S_{ia} Q_a \quad (12)$$

and $V = \sum_{i=1}^{i=n} \frac{L_i A_i}{2E} \left(\sum_{a=1}^{a=m} S_{ia} Q_a \right)^2 \quad (13)$

$$\frac{\partial V}{\partial Q_a} = \sum_b \sum_i \frac{L_i A_i}{E} S_{ia} S_{ib} Q_b \quad (14)$$

$$= \sum_b Z_{ab} Q_b \quad (15)$$

$= Z_{ab}$ = DISPLACEMENT AT a
DUE TO A LOAD AT b .

$$Z_{ab} = \sum_i \frac{L_i A_i}{E} S_{ia} S_{ib} \quad (16)$$

= RELATIONSHIP OF DISPLACEMENT
TO LOAD - IN OTHER WORDS
THE INFLUENCE COEFFICIENTS.



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 1-5

AIRCRAFT: _____

PREPARED BY

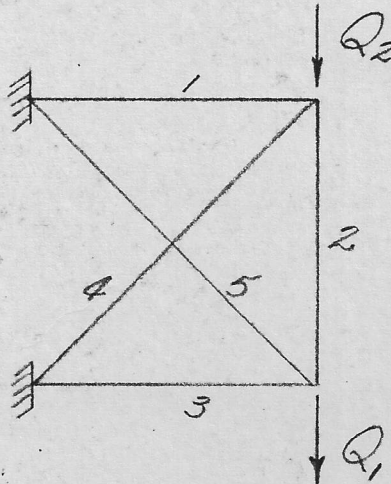
DATE

E. Vale

CHECKED BY

DATE

EXAMPLE PROBLEM N° 1



THERE ARE 5 MEMBERS $i.e.: i = 1, 2, 3, 4, 5$.

THERE ARE 2 LOADS $i.e.: a, b = 1, 2$.

THERE IS ONE REDUNDANCY $i.e.: j = 1$

TAKE MEMBER #4 AS THE REDUNDANT MEMBER.

AVRO AIRCRAFT LIMITED

TECHNICAL DEPT. (AIRFRAME)

REPORT NO. . _____

SHEET . . . 1-8

DATE

AIRCRAFT . . . _____

WEIGHT . . . _____

C. G. POSITION . _____

PREPARED BY - E. Vale

[illegible]



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

PREPARED BY _____

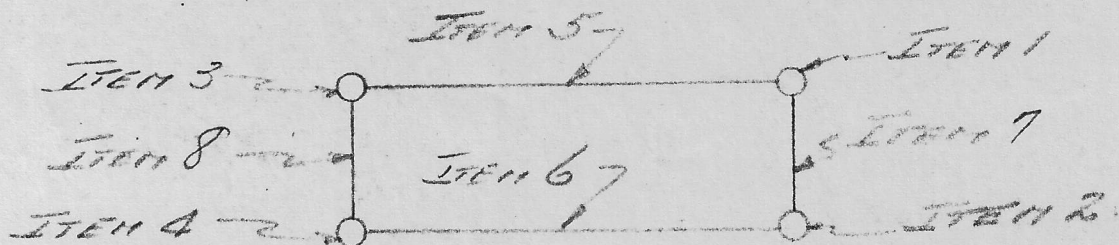
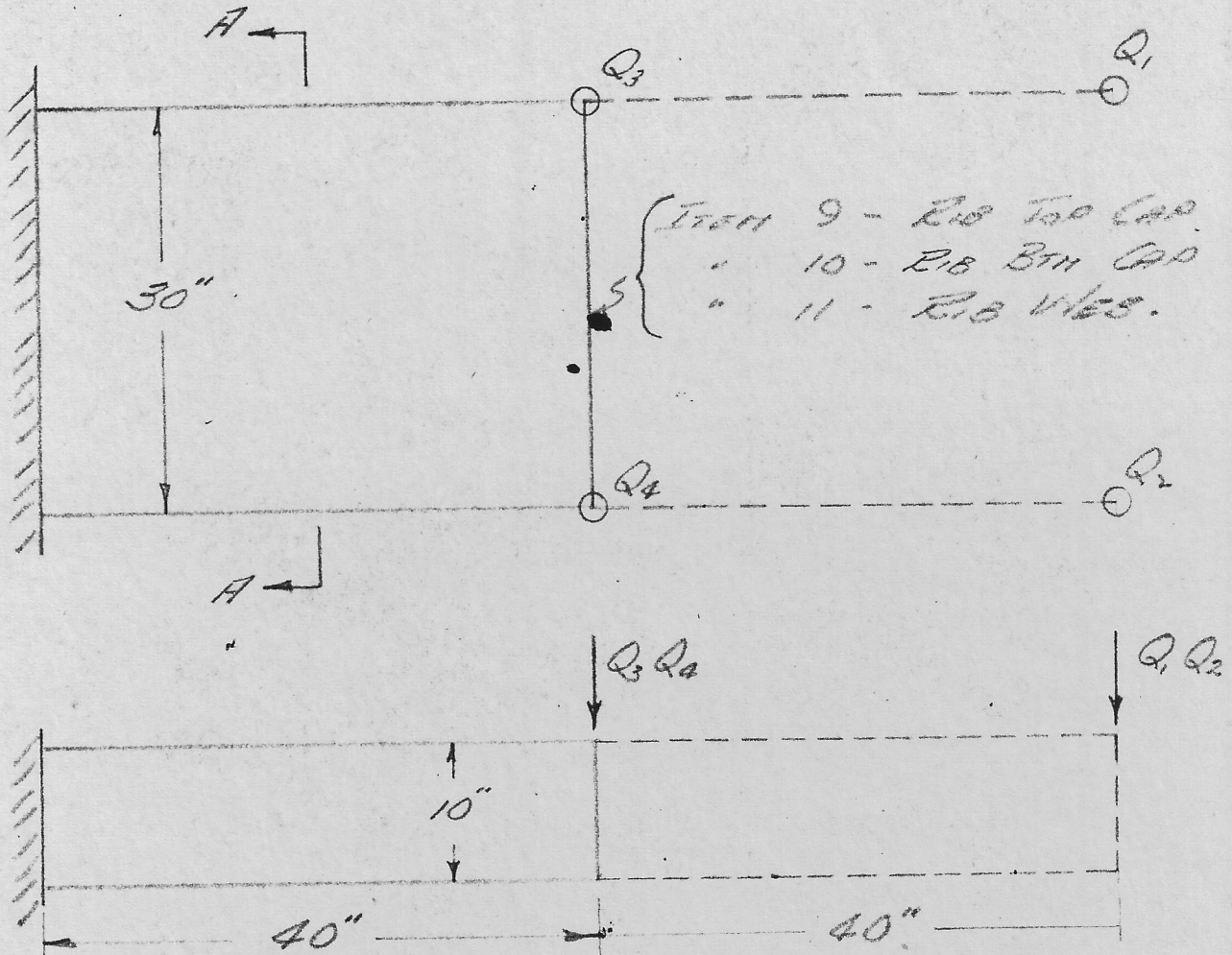
DATE _____

CHECKED BY _____

DATE _____

AIRCRAFT: .

EXAMPLE PROBLEM N° 2



SECTION A-A
(IDEALISED)



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

1-10

AIRCRAFT: _____

PREPARED BY

DATE

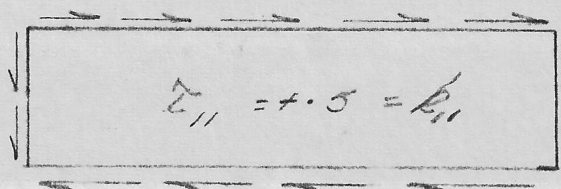
E. Vale

CHECKED BY

DATE

Take the two cover sheets and the rib as the redundant members. This leaves the two spars as the statically determinant structure. The values of $\bar{S}_{ia}$ on page 1-13 are then self-explanatory.

For the redundant load we will take the rib redistributing one lb. from one spar to the other. This sets up a shear flow in the structure of 0.1 lb/in. The redundant stress system is as shown below.



View Looking Inboard on Rib.



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

1-11

AIRCRAFT: _____

PREPARED BY _____

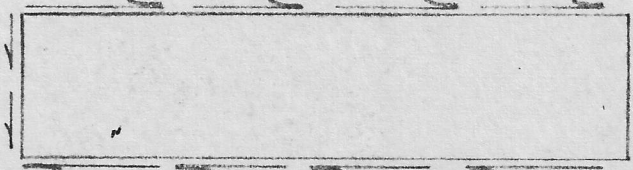
DATE _____

CHECKED BY _____

DATE _____

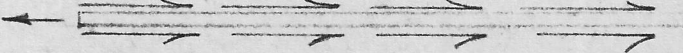
ITEM

7



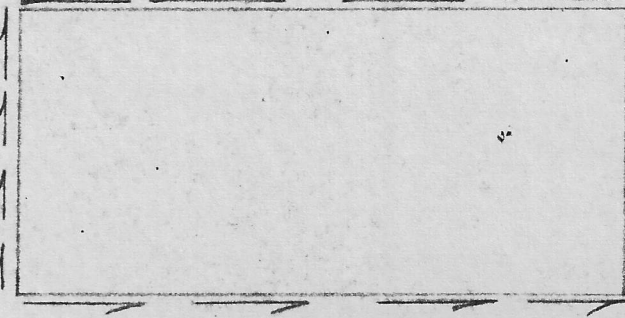
$$k_7 = -.5$$

1



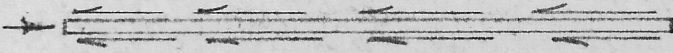
$$k_1 = \frac{4}{3}(1 - \frac{\pi}{2})$$

5



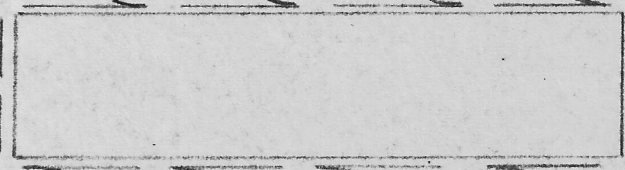
$$k_5 = .5$$

3



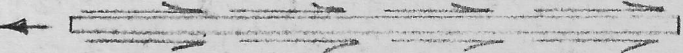
$$k_3 = -\frac{4}{3}(1 - \frac{\pi}{2})$$

8



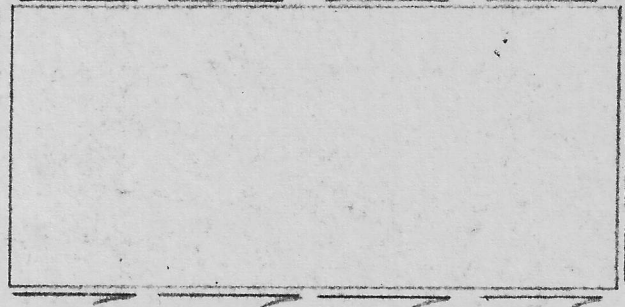
$$k_8 = -.5$$

4



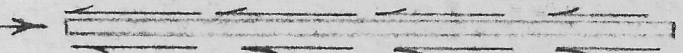
$$k_4 = \frac{4}{3}(1 - \frac{\pi}{2})$$

6



$$k_6 = .5$$

2



$$k_2 = -\frac{4}{3}(1 - \frac{\pi}{2})$$

$$q = 0.1 \text{ lb/in.}$$

$$\tau = 0.5 \text{ psi.}$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

1-12

AIRCRAFT: _____

PREPARED BY

DATE

R. Vale

CHECKED BY

DATE

In this problem:-

$$a, b = 1, 2, 3, 4.$$

$$j = 1$$

$$F_a = - \frac{\sum_i M_i \dot{S}_{aki}}{\sum_i M_i k_i^2}$$

$$\text{where } M_i = \frac{A_i}{E} \int_0^L \left(1 - \frac{x}{L}\right)^2 dx$$

$$= \frac{L A_i}{3}$$

$$\text{or } M_i = \frac{A_i}{E} \int_0^L \left(1 - \frac{x}{L}\right) \left(2 - \frac{x}{L}\right) dx$$

$$= \frac{5 L A_i}{6}$$

$$\text{or } M_i = \frac{A_i t_i E}{G}$$

EXAMPLE PROBLEM N°2

A. V. ROE CANADA LIMITED

MALTON, ONTARIO

TECHNICAL DEPT. (AIRFRAME)

REPORT NO.

SHEET 1-13

DATE

PREPARED BY E. Vale

AIRCRAFT

WEIGHT

C. G. POSITION

ITEM <i>i</i>	1	2	3	4 M_i $= L_i A_i$ or $A_i t_i \frac{E}{g}$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
	L_i	A_i	t_i		k_i	$M_i k_i$	$\bar{S}_{i1}$	$\bar{S}_{i2}$	$\bar{S}_{i3}$	$\bar{S}_{i4}$	$M_i \bar{S}_{i1}$	$M_i k_i \bar{S}_{i2}$	$M_i k_i \bar{S}_{i3}$	$M_i k_i \bar{S}_{i4}$					
1	40	6		240	$\frac{4}{3}(1-\frac{2}{3})$	142.222	$\frac{2}{3}+$ $\frac{2}{3}(1-\frac{2}{3})$	0	$\frac{2}{3}(1-\frac{2}{3})$	0	177.7778			71.1111					
2	40	6		240	$\frac{4}{3}(1-\frac{2}{3})$	142.222	$\frac{2}{3}-$ $\frac{2}{3}(1-\frac{2}{3})$	0	$\frac{2}{3}(1-\frac{2}{3})$	0	177.7778			71.1111					
3	40	6		240	$\frac{4}{3}(1-\frac{2}{3})$	142.222	0	$\frac{2}{3}+$ $\frac{2}{3}(1-\frac{2}{3})$	0	$\frac{2}{3}(1-\frac{2}{3})$		177.7778		71.1111					
4	40	6		240	$\frac{4}{3}(1-\frac{2}{3})$	142.222	0	$\frac{2}{3}-$ $\frac{2}{3}(1-\frac{2}{3})$	0	$\frac{2}{3}(1-\frac{2}{3})$		177.7778		71.1111					
5		1200	.2	630	.5	157.5000	0	0	0	0									
6		1200	.2	630	.5	157.5000	0	0	0	0									
7		400	.2	210	.5	52.5000	- .5	0	- .5	0	52.5000		52.5000						
8		400	.2	210	.5	52.5000	0	.5	0	.5		52.5000		52.5000					
9	30	8		240	0	0	0	0	0	0									
10	30	8		240	0	0	0	0	0	0									
11		300	.2	157.5	.5	39.3750	0	0	0	0									
Σ						1028. 2638					408. 0556	408. 0556	194. 7222	194. 7222					
	$E = 10.5 \times 10^6$						$F_1 = -39684 = -F_2$												
	$G = 4.0 \times 10^6$						$F_3 = -18937 = -F_4$												

EXAMPLE PROBLEM N° 2

A. V. ROE CANADA LIMITED

MALTON, ONTARIO
TECHNICAL DEPT. (AIRFRAME)

REPORT NO. . _____

SHEET . . 1-14

DATE . . . _____

PREPARED BY - T. J. [unclear]

AIRCRAFT - - - - -

WEIGHT	-	-	_____	_____
--------	---	---	-------	-------

C. G. POSITION - _____

[illegible]

FORM 1543

AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

1-15

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

The influence coefficients
can be obtained from

$$E Z_{ab} = \sum_i M_i S_{ia} S_{ib}$$

where

$$M_i = L_i A_i C_i$$

$$\text{or} = A_i t_i \frac{E}{9}$$

where C_i is a constant from the
integration of the cap stresses
over the length of the cap.

$$\int_0^l \left(1 - \frac{x}{2l}\right)^2 dx = \frac{7}{12} l$$

$$\int_0^l \left(1 - \frac{x}{2l}\right) \left(1 - \frac{x}{2l}\right) dx = \frac{5}{12} l$$

$$\int_0^l \left(1 - \frac{x}{l}\right)^2 dx = \frac{1}{3} l$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

1-16

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

$$M_1 S_{11}^2 = \left(1.3333 \times \frac{7}{12} - 2 \times 1.3333 \times .52912 \times \frac{5}{12} + .52912^2 \times \frac{1}{3} \right) L.A.$$

$$= (1.03704 - .58791 + .09332) 240$$

$$= .54245 \times 240$$

$$= 130.19$$

$$= M_2 S_{21}^2 = M_3 S_{32}^2 = M_4 S_{42}^2$$

$$M_3 S_{31}^2 = .52912^2 \times \frac{1}{3} \times 240$$

$$= 22.40$$

$$= M_4 S_{41}^2 = M_1 S_{12}^2 = M_2 S_{22}^2$$

$$M_1 S_{13}^2 = .41418^2 \times \frac{1}{3} \times 240$$

$$= 13.72$$

$$= M_2 S_{23}^2 = M_3 S_{34}^2 = M_4 S_{44}^2$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

1-17

AIRCRAFT: _____

PREPARED BY

DATE

E. Vale

CHECKED BY

DATE

$$M_3 S_{33}^2 = .25249^2 \times \frac{1}{3} \times 240$$

$$= 5.10$$

$$= M_4 S_{34}^2 = M_1 S_{14}^2 = M_2 S_{24}^2$$

$$M_1 S_{11} S_{12} = \left(1.3333 \times .52912 \times \frac{5}{12} - .52912^2 \times \frac{1}{3} \right) 240$$

$$= (-.29395 - .09332) 240$$

$$= .20063 \times 240$$

$$= 48.15$$

$$= M_2 S_{21} S_{22} = M_3 S_{31} S_{32} = M_4 S_{41} S_{42}$$

$$M_1 S_{11} S_{13} = \left(1.33333 \times .41418 \times \frac{5}{12} - .52912 \times .41418 \times \frac{1}{3} \right) 240$$

$$= (-.23010 - .07305) 240$$

$$= .15705 \times 240$$

$$= 37.69$$

$$= M_2 S_{21} S_{23} = M_3 S_{32} S_{34} = M_4 S_{42} S_{44}$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

1-18

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

$$M_3 S_{13} S_{33} = .52912 \times .25249 \times \frac{1}{3} \times 240$$

$$= 10.69$$

$$= M_4 S_{41} S_{43} = M_1 S_{12} S_{14} = M_2 S_{22} S_{24}$$

$$M_1 S_{11} S_{14} = (.33333 \times .25249 \times \frac{5}{12} - .52912 \times .25249 \times \frac{1}{3}) \times 240$$

$$= (.14027 - .04453) \times 240$$

$$= .09574 \times 240$$

$$= 22.98$$

$$= M_2 S_{21} S_{24} = M_3 S_{32} S_{33} = M_4 S_{42} S_{43}$$

$$M_3 S_{31} S_{34} = .52912 \times .41418 \times \frac{1}{3} \times 240$$

$$= 17.53$$

$$= M_4 S_{41} S_{44} = M_1 S_{12} S_{13} = M_2 S_{22} S_{23}$$

$$M_1 S_{13} S_{14} = .41418 \times .25249 \times \frac{1}{3} \times 240$$

$$= 8.37$$

$$= M_2 S_{23} S_{24} = M_3 S_{33} S_{34} = M_4 S_{43} S_{44}$$

EXAMPLE PROBLEM N° 2

INFLUENCE COEFFICIENTS

A. V. ROE CANADA LIMITED

MALTON, ONTARIO

TECHNICAL DEPT. (AIRFRAME)

AIRCRAFT

WEIGHT

C. G. POSITION

REPORT NO.

SHEET

DATE

PREPARED BY

1-19

E. Vale

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Item	$M_i S_{i1}^2$	$M_i S_{i2}^2$	$M_i S_{i3}^2$	$M_i S_{i4}^2$	$M_i S_{i1} S_{i2}$	$M_i S_{i1} S_{i3}$	$M_i S_{i1} S_{i4}$	$M_i S_{i2} S_{i3}$	$M_i S_{i2} S_{i4}$	$M_i S_{i3} S_{i4}$									
1	130.19	22.40	13.72	5.10	48.15	37.69	22.98	17.53	10.69	8.37									
2	130.19	22.40	13.72	5.10	48.15	37.69	22.98	17.53	10.69	8.37									
3	22.40	130.19	5.10	13.72	48.15	10.69	17.53	22.98	37.69	8.37									
4	22.40	130.19	5.10	13.72	48.15	10.69	17.53	22.98	37.69	8.37									
5	24.80	24.80	5.65	5.65	24.80	11.84	11.84	11.84	11.84	5.65									
6	24.80	24.80	5.65	5.65	24.80	11.84	11.84	11.84	11.84	5.65									
7	19.10	8.27	34.50	1.88	12.57	25.67	6.00	16.89	3.95	8.06									
8	8.27	19.10	1.88	34.50	12.57	3.95	16.89	6.00	25.67	8.06									
9	-	-	-	-	-	-	-	-	-	-									
10	-	-	-	-	-	-	-	-	-	-									
11	6.20	6.20	1.41	1.41	6.20	2.96	2.96	2.96	2.96	1.41									
Σ	388.35	388.35	86.73	86.73	161.94	153.02	77.27	77.27	153.02	36.89									
$\Sigma/10.5$	36.986	36.986	8.260	8.260	15.423	14.573	7.359	7.359	14.573	3.513									
	Z_{11}	Z_{22}	Z_{33}	Z_{44}	Z_{12}	Z_{13}	Z_{14}	Z_{23}	Z_{24}	Z_{34}									



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

2-1

AIRCRAFT: _____

PREPARED BY _____

DATE _____

R. Sale

CHECKED BY _____

DATE _____

SECTION 2- SIMPLY REDUNDANT

STRUCTURES BY MATRIX METHODS.

The use of matrix algebra in redundant structures generally leads to a more organized procedure. In simply redundant structures this system may look more cumbersome (and usually is) but is presented here to give an insight into the significance of each of the matrices used.

In this section, the analysis of the simply redundant structure of section one is rewritten in matrix



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 2-2

AIRCRAFT: _____

PREPARED BY

DATE

CHECKED BY

DATE

notation. The numbering of the equations of this section will correspond to that of section one.

The notation used here is that of GEN/1090/336 and meaning and interchangeability of indices is explained in that report on page 35 et seq.

$$S_i = \bar{S}_i + \bar{S}_i \quad (1)$$

$$V = \frac{1}{2} S_i C_{ik} S_k \quad (2)$$

where

$$S_i = | S_1, S_2, S_3, \dots |$$

$$S_k = \begin{vmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ \vdots \end{vmatrix}$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 2-3

AIRCRAFT: _____

PREPARED BY

DATE

E. Vale.

CHECKED BY

DATE

$$C_{ik} = \frac{1}{E} \times \begin{vmatrix} l_1 A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_2 A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & l_3 A_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_4 A_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

here again assuming all members are end load carrying.

$$V = \frac{1}{2} (\bar{S}_i + \bar{S}_i) C_{ik} (\bar{S}_k + \bar{S}_k) \quad (3)$$

Let K_{ip} be the matrix relating the stress in the i^{th} member to the p^{th} redundancy.

$$ie \quad \bar{S}_i = K_{ip} F_p$$

$$\begin{vmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ \vdots \end{vmatrix} = \begin{vmatrix} K_{11} & K_{12} & K_{13} & K_{14} & \vdots \\ K_{21} & K_{22} & K_{23} & K_{24} & \vdots \\ K_{31} & K_{32} & K_{33} & K_{34} & \vdots \\ K_{41} & K_{42} & K_{43} & K_{44} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} \times \begin{vmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \end{vmatrix}$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

2-4

AIRCRAFT: _____

PREPARED BY _____

DATE _____

E. Vale

CHECKED BY _____

DATE _____

$$V = \frac{1}{2} (\dot{S}_i + K_{ip} F_p) C_{ik} (\dot{S}_k + K_{kq} F_q)$$

$$= \frac{1}{2} \dot{S}_i C_{ik} \dot{S}_k + \frac{1}{2} \dot{S}_i C_{ik} K_{kq} F_q + \frac{1}{2} F_p K_{pi} C_{ik} \dot{S}_k + \frac{1}{2} F_p K_{pi} C_{ik} K_{kq} F_q$$

$$= \frac{1}{2} \dot{S}_i C_{ik} \dot{S}_k + \dot{S}_i C_{ik} K_{kq} F_q + \frac{1}{2} F_p K_{pi} C_{ik} K_{kq} F_q \quad (4)$$

In order to clarify the meaning of the indices, equation 4 of section one is expanded so that the two equations can be compared.

$$V = \sum_i \frac{L_i A_i}{E} (\dot{S}_i + k_{ip} F_p)^2$$

$$= \sum_i \frac{L_i A_i}{E} \dot{S}_i \dot{S}_k$$

$$+ \sum_i \frac{L_i A_i}{E} \dot{S}_i k_{ip} F_p$$

$$+ \sum_i \frac{L_i A_i}{2E} k_{ip} k_{kp} F_p^2$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. 2-5

AIRCRAFT: _____

PREPARED BY

DATE

CHECKED BY

DATE

where

$$\begin{aligned} \sum \dot{S}_i \dot{S}_j = & \dot{S}_1^2 + \dot{S}_1 \dot{S}_2 + \dot{S}_1 \dot{S}_3 + \dots \\ & + \dot{S}_2^2 + \dot{S}_2 \dot{S}_1 + \dot{S}_2 \dot{S}_3 + \dots \\ & + \dot{S}_3^2 + \dot{S}_3 \dot{S}_1 + \dot{S}_3 \dot{S}_2 + \dots \\ & + \dots \end{aligned}$$

$$\begin{aligned} \sum \dot{S}_i k_{ip} = & \dot{S}_1 k_{1p} + \dot{S}_1 k_{2p} + \dot{S}_1 k_{3p} + \dots \\ & + \dot{S}_2 k_{1p} + \dot{S}_2 k_{2p} + \dot{S}_2 k_{3p} + \dots \\ & + \dot{S}_3 k_{1p} + \dot{S}_3 k_{2p} + \dot{S}_3 k_{3p} + \dots \\ & + \dots \end{aligned}$$

$$\begin{aligned} \sum k_{ip} k_{jp} = & k_{1p} k_{1p} + k_{1p} k_{2p} + k_{1p} k_{3p} + \dots \\ & + k_{2p} k_{1p} + k_{2p} k_{2p} + k_{2p} k_{3p} + \dots \\ & + k_{3p} k_{1p} + k_{3p} k_{2p} + k_{3p} k_{3p} + \dots \\ & + \dots \end{aligned}$$

$$\text{Let } H_{iq} = C_{ik} K_{kq}$$

$$D_{pq} = K_{pi} H_{iq} = K_{pi} C_{ik} K_{kq}$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

2-6

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

$$V = \frac{1}{2} \dot{S}_i C_{ik} \dot{S}_k + \dot{S}_i H_{iq} F_q + \frac{1}{2} F_p D_{pq} F_q \quad (4a)$$

$$\frac{\partial V}{\partial F_q} = \dot{S}_i H_{iq} + F_p D_{pq} = 0 \quad (5)$$

Let $\bar{D}_{pq}^{-1}$ be the inverse D_{pq}

then

$$F_p = -\bar{D}_{pq}^{-1} H_{qk} \dot{S}_k \quad (7)$$

and

$$S_i = \dot{S}_i + K_{ip} F_p \quad (8)$$

$$= \dot{S}_i - K_{ip} \bar{D}_{pq}^{-1} H_{qk} \dot{S}_k \quad (9)$$

$$S_{ia} = \dot{S}_{ia} - K_{ip} F_{pa} \quad (10)$$

$$= \dot{S}_{ia} - K_{ip} \bar{D}_{pq}^{-1} H_{qk} \dot{S}_{ka} \quad (11)$$

= Stress in each member due to unit load at a.

We now change the notation. Up to now S_{ia} has been the stress in each member due to a unit load at a. Let T_{ia} be used for this and

$$\dot{S}_{ia} = T_{ia} Q_a$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

2-7

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

where S_{ia} is the stress due to the actual load Q_a at a .

$$S_{ia} = T_{ia} - K_{ip} D_{pq} H_{qi} T_{ka}$$

$$\text{and } S_i = S_{ia} Q_a \quad (12)$$

= Stress in each member due to all loads Q_a

$$V = \frac{1}{2} Q_a S_{ai} C_{ik} S_{kb} Q_b \quad (13)$$

$$\frac{\partial V}{\partial Q_a} = S_{ai} C_{ik} S_{kb} Q_b \quad (14)$$

$$= Z_{ab} Q_b \quad (15)$$

= Z_{ab} = displacement at a due to a load Q_b at b .

$$\text{where } Z_{ab} = S_{ai} C_{ik} S_{kb} \quad (16)$$

= displacement to load matrix.

An alternative form of Z_{ab} can be obtained.

$$V = \frac{1}{2} \dot{S}_i C_{ik} \dot{S}_k + S_i H_{iq} F_q + \frac{1}{2} F_p D_{pq} F_q$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

2-8

AIRCRAFT: _____

PREPARED BY _____

DATE _____

R. Vale.

CHECKED BY _____

DATE _____

From Equation 5

$$(\dot{S}_i H_{iq} + F_p D_{pq}) \dot{F}_q = 0$$

$$V = \frac{1}{2} \dot{S}_i C_{ik} \dot{S}_k + \frac{1}{2} F_p H_{pk} \dot{S}_k$$

$$= \frac{1}{2} (\dot{S}_i + F_p K_{pi}) C_{ik} \dot{S}_k$$

$$= \frac{1}{2} \dot{S}_i C_{ik} \dot{S}_k$$

$$= \frac{1}{2} Q_a S_{ai} C_{ik} T_{ib} \dot{Q}_b$$

$$= \frac{1}{2} Q_a T_{ai} C_{ik} S_{kb} \dot{Q}_b$$

from which

$$Z_{ab} = T_{ai} C_{ik} S_{kb} \quad (16a)$$

This saves the effort of rearranging S_{ia} to S_{ai} and provides a check. Since Z_{ab} must be symmetric.



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

2-9

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

Example Problem N°1

Consider the truss of the first problem of section one.

$$\dot{S}_1 = 0 + 0$$

$$\dot{S}_2 = 0 - 2Q_2$$

$$\dot{S}_3 = -2Q_1 - 2Q_2$$

$$\dot{S}_4 = 0 + 0$$

$$\dot{S}_5 = 28284Q_1 + 28284Q_2$$

$$\dot{S}_i = 1/a Q_a$$

$$\begin{vmatrix} \dot{S}_i \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & -2 \\ -2 & -2 \\ 0 & 0 \\ 28284 & 28284 \end{vmatrix} \times \begin{vmatrix} Q_1 \\ Q_2 \end{vmatrix}$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

2-10

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

$$\bar{S}_1 = -0.14142 F_1$$

$$\bar{S}_2 = -0.14142 F_1$$

$$\bar{S}_3 = -0.14142 F_1$$

$$\bar{S}_4 = 0.2 F_1$$

$$\bar{S}_5 = 0.2 F_1$$

$$\bar{S}_i = K_{ip} F_p \quad (\text{in this case } p=1)$$

$$\begin{vmatrix} \bar{S}_i \end{vmatrix} = \begin{vmatrix} -0.14142 \\ -0.14142 \\ -0.14142 \\ 0.20000 \\ 0.20000 \end{vmatrix} \times \begin{vmatrix} F_1 \end{vmatrix}$$

$$V = \frac{1}{2} S_i C_{ik} S_k$$

$$= \frac{1}{2} \begin{vmatrix} S_1 & S_2 & S_3 & S_4 & S_5 \end{vmatrix} \times 10^7 \times$$

$$\begin{vmatrix} 50 & 0 & 0 & 0 & 0 \\ 0 & 50 & 0 & 0 & 0 \\ 0 & 0 & 50 & 0 & 0 \\ 0 & 0 & 0 & 10.711 & 0 \\ 0 & 0 & 0 & 0 & 10.711 \end{vmatrix} \times \begin{vmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{vmatrix}$$



AVRO AIRCRAFT LIMITED
MALTON, ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

2-11

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

$$H_{ig} = C_{ik} K_{ig}$$

$$= \begin{vmatrix} -7.0711 \\ -7.0711 \\ -7.0711 \\ 14.1420 \\ 14.1420 \end{vmatrix}$$

$$D_{pg} = K_{pi} H_{ig}$$

$$= 8.6568 \times 10^{-7}$$

$$\dot{D}_{pg} = .115516 \times 10^{-7}$$

$$\dot{D}_{pg} H_{gi} = \begin{vmatrix} -.816826 \\ -.816826 \\ -.816826 \\ 1.633629 \\ 1.633629 \end{vmatrix}$$

$$F_{pa} = (\dot{D}_{pg} H_{gi}) T_{ia}$$

$$= \begin{vmatrix} .625421 \\ .788786 \end{vmatrix}$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

2-12

AIRCRAFT: _____

PREPARED BY

DATE

R. Vale

CHECKED BY

DATE

$$Nia = Kip Fox$$

$$= \begin{vmatrix} -.08845 & -.11155 \\ -.08845 & -.11155 \\ -.08845 & -.11155 \\ .12508 & .15776 \\ .12508 & .15776 \end{vmatrix}$$

$$Sia = Tia - Nia$$

$$= \begin{vmatrix} .08845 & .11155 \\ .08845 & -.08845 \\ -.11155 & -.08845 \\ -.12508 & -.15776 \\ .15776 & .12508 \end{vmatrix}$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

2-13

AIRCRAFT: _____

PREPARED BY _____

DATE _____

E. V. L.

CHECKED BY _____

DATE _____

$$O_{ai} = T_{ab} S_{ib}$$

$$= \begin{vmatrix} 0 & 0 & -10 & 0 & 20 \\ 0 & -10 & -10 & 0 & 20 \end{vmatrix} \times 10^{-7}$$

$$Z_{ab} = O_{ai} S_{ib}$$

$$= \begin{vmatrix} 4.2707 & 3.3861 \\ 3.3862 & 4.2706 \end{vmatrix} \times 10^{-7}$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

2-14

AIRCRAFT: _____

PREPARED BY _____

DATE _____

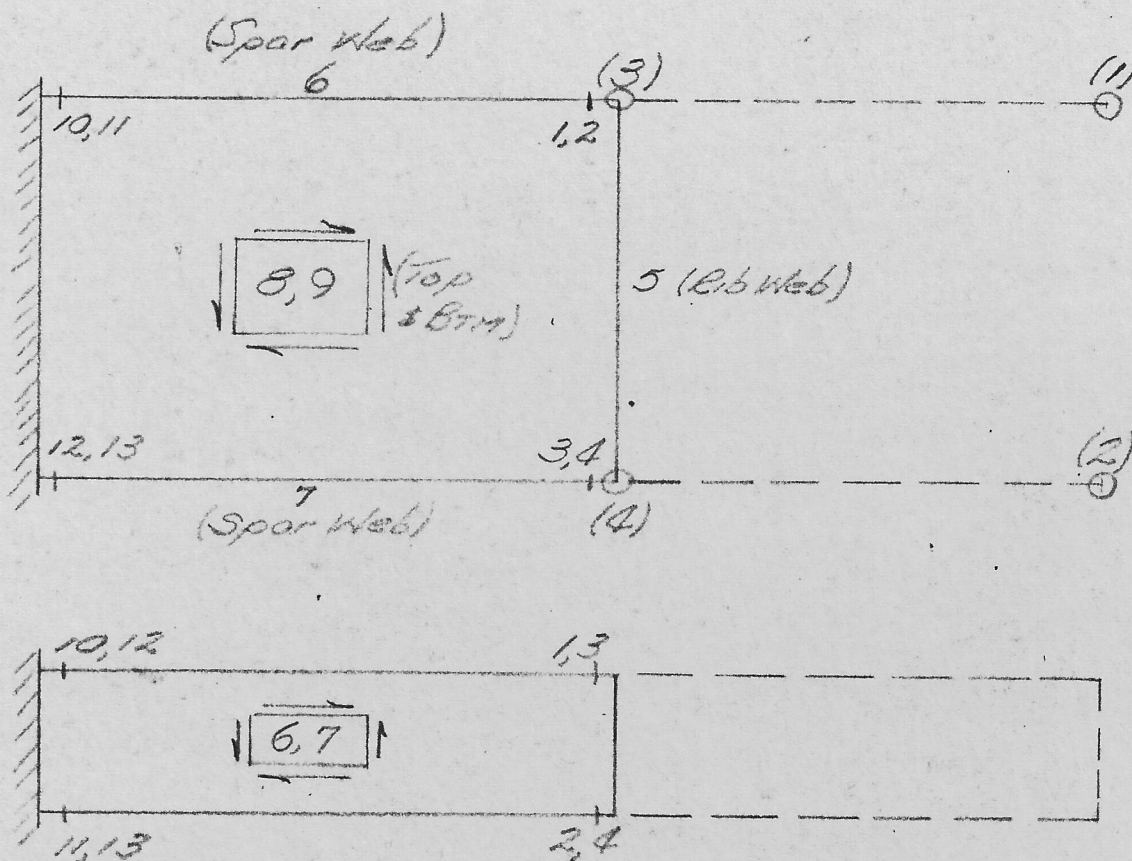
E. Vale

CHECKED BY _____

DATE _____

Example Problem No 2

Consider the box of the second example of section one.



Load Points (x)

Stress Points x



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

AIRCRAFT:

REPORT NO.

SHEET NO.

PREPARED BY

DATE

CHECKED BY

DATE

To establish the C_{ij} matrix:-
Axial Element:-

$$\sigma = \sigma_1 \left(1 - \frac{x}{l}\right) + \sigma_2 \frac{x}{l}$$

$$2V = \frac{A}{E} \int_0^l \left[\sigma_1 \left(1 - \frac{x}{l}\right) + \sigma_2 \frac{x}{l} \right]^2 dx$$

$$= \frac{A}{E} \left[\frac{l}{3} \sigma_1^2 + \frac{2l}{6} \sigma_1 \sigma_2 + \frac{l}{3} \sigma_2^2 \right]$$

$$C_{11} = C_{22} = C_{33} = C_{44} = C_{10,10} = C_{11,11}, C_{12,12} = C_{13,13}$$

$$= \frac{Al}{3E} = \frac{6 \times 40}{3 \times 10.5 \times 10^6} = 0.00762 \times 10^{-3}$$

$$C_{1,10} = C_{2,11} = C_{3,12} = C_{4,13}$$

$$= \frac{Al}{6E} = 0.00381 \times 10^{-3}$$

Web Elements

$$2V = \frac{At}{9} z^2$$

$$C_{55} = \frac{30 \times 10 \times 2}{4 \times 10^6} = 0.01500 \times 10^{-3}$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

2-16

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

$$C_{6,6} = C_{7,7} = \frac{40 \times 10 \times 2}{4.0 \times 10^6} = .02000 \times 10^{-3}$$

$$C_{8,8} = C_{9,9} = \frac{40 \times 30 \times 2}{4.0 \times 10^6} = .06000 \times 10^{-3}$$

To establish the K_{ip} matrix:-

Let the redundancy be a transfer of one in-lb of moment between spars at the root end.

$$\text{ie } T_{10} = T_{11} = T_{12} = T_{13} = \frac{1}{10 \times 6} = .01667$$

$$= K_{10,1} = -K_{11,1} = K_{12,1} = K_{13,1}$$

$$f = \frac{1}{10 \times 40 \times 2} = .00125$$

$$\delta = \frac{.00125}{.2} = .00625$$

$$= K_{5,1} = -K_{6,1} = K_{7,1} = -K_{8,1} = K_{9,1}$$

By referring to section one, the derivation of the T_{ia} matrix is straightforward.



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. 2-17

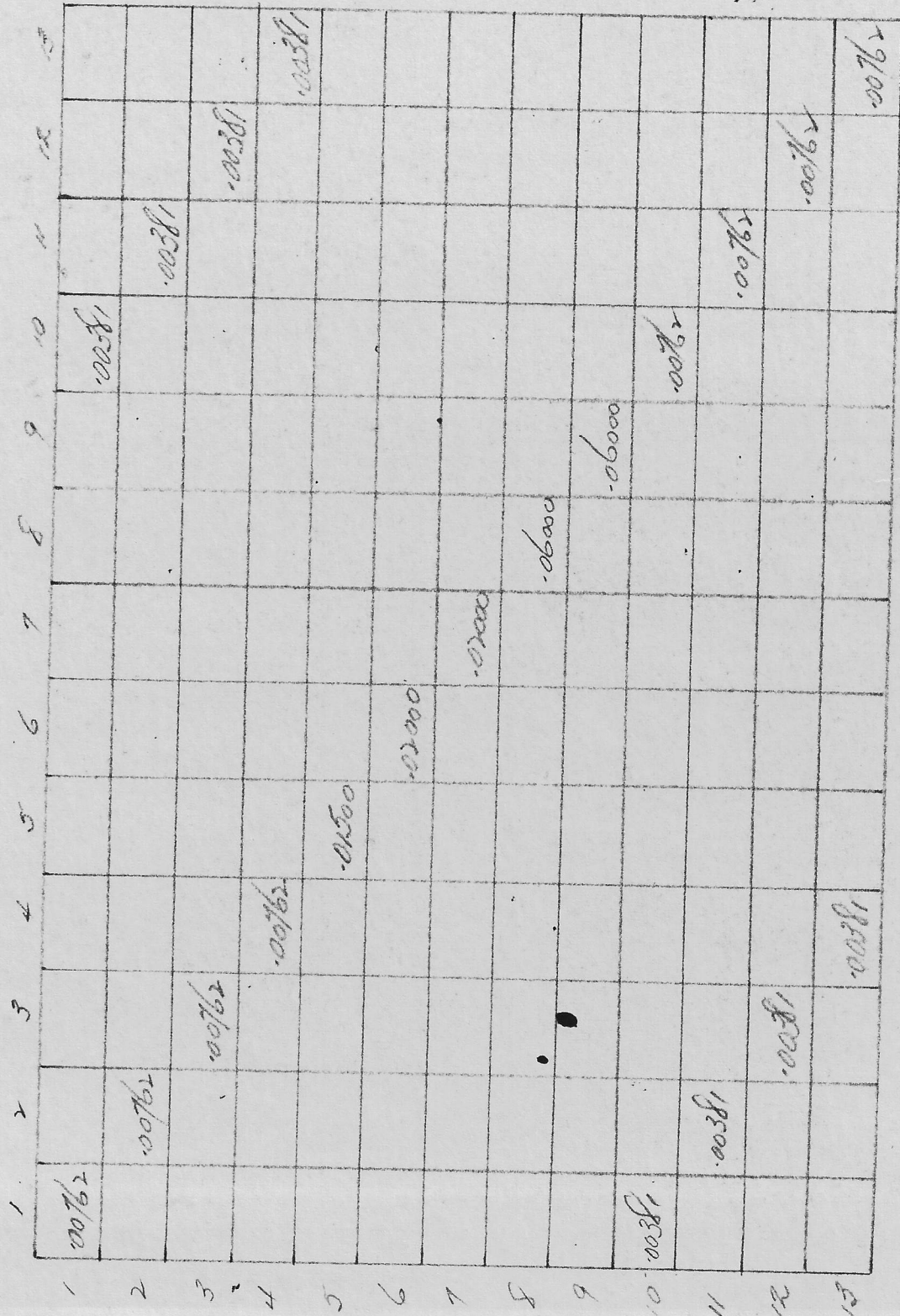
AIRCRAFT: _____

PREPARED BY

DATE

CHECKED BY

DATE





AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 2-18

AIRCRAFT: _____

PREPARED BY

DATE

CHECKED BY

DATE

1/2

1 2 3 4

1	.66667			
2	.66667			
3		.66667		
4		.66667		
5	.50000		.50000	
6		.50000		.50000
7				
8				
9				
10	1.33333		.66667	
11	1.33333		.66667	
12		1.33333		.66667
13		1.33333		.66667

Kip

1	
2	
3	
4	
5	.00625
6	.00625
7	.00625
8	.00625
9	.00625
10	.01667
11	.01667
12	.01667
13	.01667



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 2-19

AIRCRAFT: _____

PREPARED BY

DATE

C. Vale

CHECKED BY

DATE

The program will be as follows:-

$C_{ik}, K_{ip} \text{ \& } T_{ia}$ are given.

$$H_{ip} = C_{ik} K_{kp} \quad (13 \times 1)$$

$$D_{pq} = K_{pi} H_{iq} \quad (1 \times 1)$$

$$D_{pq} \rightarrow \bar{D}_{pq} \quad (1 \times 1)$$

$$M_{pi} = \bar{D}_{pq} H_{qi} \quad (1 \times 13)$$

$$F_{pa} = -M_{pi} T_{ia} \quad (1 \times 4)$$

$$N_{ia} = K_{ip} F_{pa} \quad (13 \times 4)$$

$$\rightarrow S_{ia} = T_{ia} + N_{ia} \quad (13 \times 4)$$

$$O_{ai} = T_{ak} C_{ki} \quad (4 \times 13)$$

$$\rightarrow Z_{ab} = O_{ai} S_{ib} \quad (4 \times 4)$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. 2-20

AIRCRAFT: _____

PREPARED BY

DATE

CHECKED BY

DATE

Hip

1	.0000635127
2	.0000635127
3	.0000635127
4	.0000635127
5	.0000937500
6	.0001250000
7	.0001250000
8	.0003750000
9	.0003750000
10	.0001270254
11	.0001270254
12	.0001270254
13	.0001270254

Dpg

.00001530599

Dpg

65333.8936



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 2-21

AIRCRAFT: _____

PREPARED BY

DATE

CHECKED BY

DATE

Mip

1	4.14953198
2	4.14953198
3	4.14953198
4	4.14953198
5	6.125052525
6	8.166736700
7	8.166736700
8	24.500210100
9	24.500210100
10	8.299063968
11	8.299063968
12	8.299063968
13	8.299063968

Fap

1	31.74688725
2	31.74688725
3	15.14884230
4	15.14884230



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. 2-22

AIRCRAFT: _____

PREPARED BY

DATE

CHECKED BY

DATE

Nia

	1	2	3	4
1	—	—	—	—
2	—	—	—	—
3	—	—	—	—
4	—	—	—	—
5	$\overline{.19842}$	$\overline{.19842}$	$\overline{.09468}$	$\overline{.09468}$
6	$\overline{.19842}$	$\overline{.19842}$	$\overline{.09468}$	$\overline{.09468}$
7	$\overline{.19842}$	$\overline{.19842}$	$\overline{.09468}$	$\overline{.09468}$
8	$\overline{.19842}$	$\overline{.19842}$	$\overline{.09468}$	$\overline{.09468}$
9	$\overline{.19842}$	$\overline{.19842}$	$\overline{.09468}$	$\overline{.09468}$
10	$\overline{.52922}$	$\overline{.52922}$	$\overline{.25253}$	$\overline{.25253}$
11	$\overline{.52922}$	$\overline{.52922}$	$\overline{.25253}$	$\overline{.25253}$
12	$\overline{.52922}$	$\overline{.52922}$	$\overline{.25253}$	$\overline{.25253}$
13	$\overline{.52922}$	$\overline{.52922}$	$\overline{.25253}$	$\overline{.25253}$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. 2-23

AIRCRAFT: _____

PREPARED BY

DATE

CHECKED BY

DATE

Sia

	1	2	3	4
1	.66667			
2	.66667			
3		.66667		
4		.66667		
5	.19842	.19842	.09468	.09468
6	.30158	.19842	.40532	.09468
7	.19842	.30158	.09468	.40532
8	.19842	.19842	.09468	.09468
9	.19842	.19842	.09468	.09468
10	.80411	.52922	.41414	.25253
11	.80411	.52922	.41414	.25253
12	.52922	.80411	.25253	.41414
13	.52922	.80411	.25253	.41414



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No.

SHEET No.

2-24

AIRCRAFT:

PREPARED BY

DATE

CHECKED BY

DATE

Oia

	1	2	3	4
1	.01016		.00254	
2	.01016		.00254	
3		.01016		.00254
4		.01016		.00254
5				
6	.01000		.01000	
7		.01000		.01000
8				
9				
10	.01270		.00508	
11	.01270		.00508	
12		.01270		.00508
13		.01270		.00508



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 2-25

AIRCRAFT: _____

PREPARED BY

DATE

E. J. L.

CHECKED BY

DATE

Lab

	1	2	3	4
1	.036987	.015426	.014572	.007361
2	.015426	.036987	.007361	.014572
3	.014572	.007361	.008261	.003512
4	.007361	.014572	.003512	.008261



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 3-1

AIRCRAFT: _____

PREPARED BY

DATE

CHECKED BY

DATE

Section 3:-

Stress Distribution and
Influence Coefficients
Including Temperature Effects.

The influence of temperature on structural analysis can be handled with little extra complication as long as non-linearities such as creep and yielding are not considered. Excluding these non-linearities the problem is reduced to the consideration of the additional strain due to thermal expansion and the change of elastic moduli. In order to obtain an insight into these effects we will consider them one at a time. In all this, we are assuming that the temperature distribution is given and we will go on from there.



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 3-2

AIRCRAFT: _____

PREPARED BY

DATE

R. Vale
CHECKED BY

DATE

The effect of change of moduli is a relatively unimportant one from the standpoint of complexity. Knowing the temperature of each structural element means that we also know its modulus and by introducing this new modulus in the C_{ik} matrix this effect is taken care of.

Thermal expansion due to temperature changes is the more complicated effect to handle. However, to think about it in a simple way it can be thought of as a lack of fit of a particular structural element. This lack of fit can be treated by introducing a fictitious external load that produces a fictitious stress in the heated element such that it undergoes



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

3-3

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

AIRCRAFT: _____

a additional strain equal to that produced by the temperature change. In the case of the simple axial element, the strain due to temperature change αT can be accounted for by introducing a fictitious stress $E\alpha T$ and the problem treated in the conventional manner of non-heated structures. It always must be remembered however that this fictitious stress must be subtracted from the answer to obtain the final true stress. In the case of a two-dimensional element the fictitious stress is $\frac{E\alpha T}{1-\nu}$ and for the three-dimensional element it is $\frac{E\alpha T}{1-2\nu}$. (See Timoshenko's Theory of Elasticity page 203 et seq.) To illustrate this, consider a simple statically determinate



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

3-4

AIRCRAFT: _____

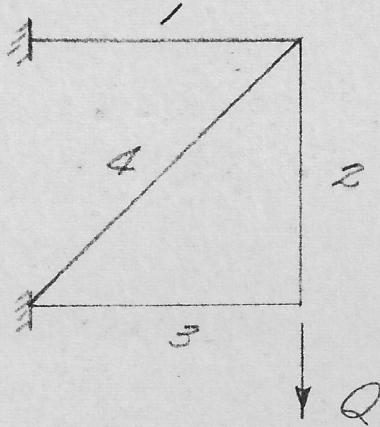
PREPARED BY _____

DATE _____

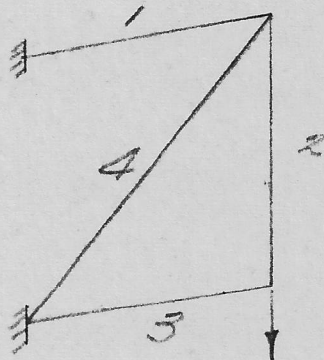
CHECKED BY _____

DATE _____

structure. Take as an example the following truss.



Let member 4 be heated and the truss will take up the following shape.



It is evident that the loads in each element are unchanged but the truss undergoes a deflection.



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

3-5

AIRCRAFT: _____

PREPARED BY

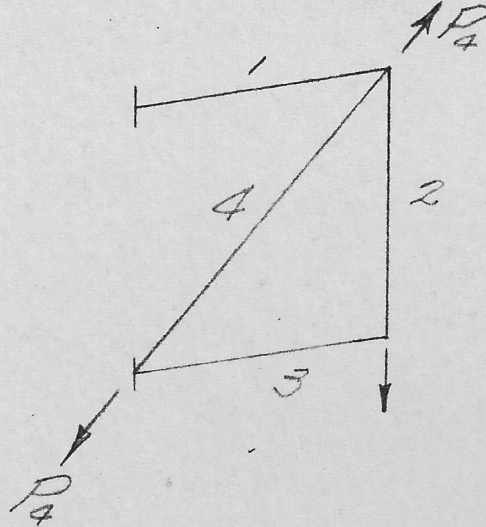
DATE

E. Vale

CHECKED BY

DATE

The same effect can be produced by an external load as shown.



where $P_4 = A_4 E \alpha t_4$ $t_4 = \text{temp. change of element 4}$
 $\sigma_4 = E \alpha t_4$

This is obviously true for any element

ie. $\sigma_i = E \alpha t_i$

and for unit temperature rise

$\sigma_i = E \alpha$

$V = \sum_i \frac{1}{2} M_i (\dot{S}_i + \sigma_i t_i)^2$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 3-6

AIRCRAFT: _____

PREPARED BY

DATE

E. Vale

CHECKED BY

DATE

$$\text{Let } \dot{S}_i = T_i Q$$

then

$$V = \sum_i \frac{L_i A_i}{2E} (T_i Q + \dot{S}_i t_i)^2$$

$$\frac{\partial V}{\partial Q} = \sum_i \frac{L_i A_i}{E} (T_i Q + \dot{S}_i t_i) T_i$$

$$= \sum_i \frac{L_i A_i T_i^2}{E} Q + \sum_i \frac{L_i A_i T_i \dot{S}_i t_i}{E}$$

$$= S_Q + S_t$$

$$= S$$

where

S_Q = Deflection due to load

S_t = Deflection due to temp.

It has been shown then that the structure can be treated as if the stress in each element is $T_i Q + \dot{S}_i t_i$. The true stress in each



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

3-7

AIRCRAFT: _____

PREPARED BY

DATE

K. Vale

CHECKED BY

DATE

element is $(T_i Q + \bar{\sigma}_i t_i) - \bar{\sigma}_i t_i = T_i Q$

The same reasoning will apply to redundant structures.

$$S_i = \bar{S}_i + \bar{\sigma}_i t_i + k_{ij} F_j$$

$$V = \sum_i \frac{L_i A_i}{E} (\bar{S}_i + \bar{\sigma}_i t_i + k_{ij} F_j)^2$$

$$\frac{\partial V}{\partial F_j} = \sum_i \frac{L_i A_i}{E} (\bar{S}_i + \bar{\sigma}_i t_i + k_{ij} F_j) k_{ij} = 0$$

from which

$$F_j = - \frac{\sum_i \frac{L_i A_i}{E} (\bar{S}_i + \bar{\sigma}_i t_i) k_{ij}}{\sum_i \frac{L_i A_i}{E} k_{ij}^2}$$

$$F_{j, \text{ast}} = - \frac{\sum_i \frac{L_i A_i}{E} T_{ia} k_{ij} Q_a}{\sum_i \frac{L_i A_i}{E} k_{ij}^2} - \frac{\sum_i \frac{L_i A_i}{E} k_{ij} \bar{\sigma}_i t_i}{\sum_i \frac{L_i A_i}{E} k_{ij}^2}$$

= Redundant loads for various loads Q_a and various temperature distributions.



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

5-8

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

The above will now be evolved in matrix notation.

Let a, b = Loading points
 c, d = Temperature points
 i, k = Stress points

$$S_i = \dot{S}_i + \tau_i + K_{ip} F_p$$

$$2V = S_i C_{ik} S_k$$

$$= (\dot{S}_i + \tau_i + K_{ip} F_p) C_{ik} (\dot{S}_k + \tau_k + K_{kq} F_q)$$

$$= \dot{S}_i C_{ik} \dot{S}_k + 2 \dot{S}_i C_{ik} \tau_k + \tau_i C_{ik} \tau_k$$

$$+ 2 \dot{S}_i C_{ik} K_{kq} F_q + 2 \tau_i C_{ik} K_{kq} F_q$$

$$+ F_p K_{pi} C_{ik} K_{kq} F_q$$

As before let,

$$H_{iq} = C_{ik} K_{kq}$$

$$D_{pq} = K_{pi} H_{iq} = K_{pi} C_{ik} K_{kq}$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 3-9

AIRCRAFT: _____

PREPARED BY

DATE

CHECKED BY

DATE

$$2V = \dot{S}_i C_{ik} \dot{S}_k + 2 \dot{S}_i C_{ik} T_k + \dot{T}_i C_{ik} \dot{T}_k \\ + 2 \dot{S}_i H_{iq} F_q + 2 \dot{T}_i H_{iq} F_q + F_p D_{pq} F_q$$

$$\frac{\partial V}{\partial F_q} = \dot{S}_i H_{iq} + \dot{T}_i H_{iq} + F_p D_{pq} = 0$$

$$F_p = - (\dot{S}_i H_{iq} + \dot{T}_i H_{iq}) D_{pq}^{-1}$$

then

$$S_i = \dot{S}_i + \dot{T}_i - K_{ip} D_{pq}^{-1} H_{qk} (\dot{S}_k + \dot{T}_k)$$

To get the true stress,
we now have to subtract the
fictitious initial stress.

$$S_i^* = S_i - \dot{T}_i$$

$$= \dot{S}_i - K_{ip} D_{pq}^{-1} H_{qk} (\dot{S}_k + \dot{T}_k)$$

or, for unit load & unit temp. rise.

$$S_{i,ac}^* = T_{ia} - K_{ip} D_{pq}^{-1} H_{qk} (T_{ka} + \theta_{kc})$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

3-10

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

where $T_i = \theta_{ic} Q_c$

= fictitious stress at
each point i for
temperature Q_c at c .

Let $N_{iac} = K_{ip} F_{pac}$

$F_{pac} = - M_{pi} T_{iac}$

$M_{pi} = D_{pq} H_{qi}$

$T_{iac} = T_{ia} + \theta_{ic}$

Then $S_{iac}^* = T_{ia} + N_{ic}$

and $S_{iac} = S_{iac}^* + \theta_{ic}$

Going back to the energy
expression and remembering
that

$(\dot{S}_i H_{iq} + T_i H_{iq} + F_p D_{pq}) F_q = 0$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 3-11

AIRCRAFT: _____

PREPARED BY

DATE

CHECKED BY

DATE

$$\begin{aligned} 2V &= \dot{S}_i C_{ik} \dot{S}_k + 2 \dot{S}_i C_{ik} T_k + T_i C_{ik} T_k \\ &\quad + \dot{S}_i H_{iq} F_q + T_i H_{iq} F_q \\ &= (\dot{S}_i + T_i + H_{iq} F_q) C_{ik} (\dot{S}_i + T_i) \\ &= (\dot{S}_i^* + T_i) C_{ik} (\dot{S}_i + T_i) \\ &= (\dot{S}_{iac}^* + \theta_{ic}) Q_c C_{ik} (T_{kb} Q_b + \theta_{id} Q_d) \end{aligned}$$

$$\begin{aligned} \frac{\partial V}{\partial Q_b} &= (\dot{S}_{iac}^* + \theta_{ic}) Q_c C_{ik} T_{kb} \\ &= \text{Deflection at } b \text{ due to} \\ &\quad \text{temperatures } Q_c \\ &= Z_{bac} Q_{ac} \end{aligned}$$

where $Z_{bac} = (\dot{S}_{iac}^* + \theta_{ic}) C_{ik} T_{kb}$

Let $O_{ib} = C_{ik} T_{kb}$

$Z_{bac} = (\dot{S}_{iac}^* + \theta_{ic}) O_{ib}$

= Influence coefficients.



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

3-12

AIRCRAFT: _____

PREPARED BY _____

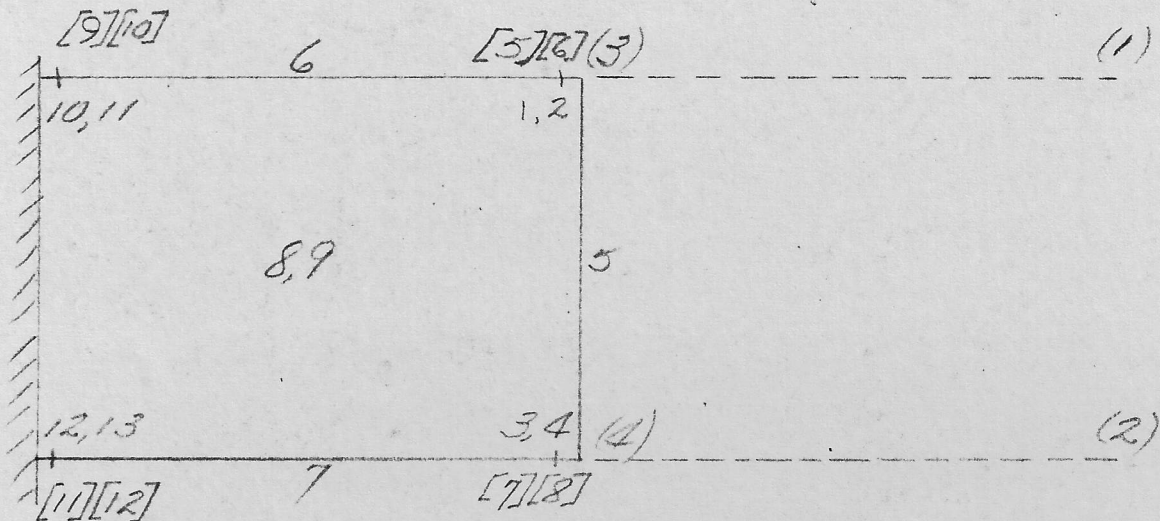
DATE _____

CHECKED BY _____

DATE _____

Example Problem:

Consider the box of the previous section.



LOAD POINTS (x)

Stress Points x

Temp. Points [x]

The temperature points have been numbered carrying on from the load points. The reason for this can be seen in the $T_{i,jac}$ & $S_{i,jac}^*$ matrices.

The construction of the θ_{ic} matrix is straightforward.



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. _____

3-13

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

$$\theta_{1,5} = \theta_{2,6} = \theta_{3,7} = \theta_{4,8} = \theta_{10,9} = \theta_{11,10} = \theta_{12,11} = \theta_{13,12}$$

$$= \frac{E \alpha}{1-\nu}$$

$$= \frac{10.5 \times 10^6 \times 12.7 \times 10^{-6}}{.6667}$$

$$= 200073 \times 10^3$$

The program is:-

K_{ip} , θ_{ic} , M_{pi} , T_{ia} are given

$$T_{iac} = T_{ia} + \theta_{ic}$$

$$F_{p,ac} = -M_{pi} T_{iac}$$

$$N_{i,ac} = K_{ip} F_{p,ac}$$

$$\rightarrow \overset{*}{S}_{i,ac} = T_{ia} + N_{i,ac}$$

$$S_{i,ac} = \overset{*}{S}_{i,ac} + \theta_{ic}$$

$$\rightarrow Z_{a,bd} = O_{ai} S_{i,bd}$$



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 3-14

AIRCRAFT: _____

PREPARED BY

DATE

E. Vale

CHECKED BY

DATE

01C

C →

<i>1</i>	<i>5</i>	<i>6</i>	<i>7</i>	<i>8</i>	<i>9</i>	<i>10</i>	<i>11</i>	<i>12</i>
<i>1</i>	<i>.200073</i>							
<i>2</i>		<i>.200073</i>						
<i>3</i>			<i>.200073</i>					
<i>4</i>				<i>.200073</i>				
<i>5</i>								
<i>6</i>								
<i>7</i>								
<i>8</i>								
<i>9</i>								
<i>10</i>					<i>.200073</i>			
<i>11</i>						<i>.200073</i>		
<i>12</i>							<i>.200073</i>	
<i>13</i>								<i>.200073</i>



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 3-15

AIRCRAFT: _____

PREPARED BY

DATE

E. Vale

CHECKED BY

DATE

ac *TAC*

1	2	3	4	5	6	7	8	9	10	11	12
1	66667										
2	66667				200073						
3	66667				200073						
4	66667					200073					
5											
6	5000										
7	5000										
8											
9											
10	13333		66667					200073			
11	13333		66667					200073			
12		13333	66667						200073		
13		13333	66667							200073	



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. 3-16

AIRCRAFT: _____

PREPARED BY

DATE

E. Vale

CHECKED BY

DATE

Fasp

p →

ac

1	31.74688725
2	31.74688725
3	15.14884230
4	15.14884230
5	-83020931
6	-83020931
7	-83020931
8	-83020931
9	1.66041862
10	1.66041862
11	1.66041862
12	1.66041862



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. _____

3-17

AIRCRAFT: _____

PREPARED BY _____

DATE _____

CHECKED BY _____

DATE _____

ac →

Niac

	1	2	3	4	5	6	7	8	9	10	11	12
1												
2												
3												
4												
5	.19842	.19842	.09468	.09468	.00519	.00519	.00519	.00519	.01038	.01038	.01038	.01038
6	.19842	.19842	.09468	.09468	.00519	.00519	.00519	.00519	.01038	.01038	.01038	.01038
7	.19842	.19842	.09468	.09468	.00519	.00519	.00519	.00519	.01038	.01038	.01038	.01038
8	.19842	.19842	.09468	.09468	.00519	.00519	.00519	.00519	.01038	.01038	.01038	.01038
9	.19842	.19842	.09468	.09468	.00519	.00519	.00519	.00519	.01038	.01038	.01038	.01038
10	.52922	.52922	.25253	.25253	.01384	.01384	.01384	.01384	.02768	.02768	.02768	.02768
11	.52922	.52922	.25253	.25253	.01384	.01384	.01384	.01384	.02768	.02768	.02768	.02768
12	.52922	.52922	.25253	.25253	.01384	.01384	.01384	.01384	.02768	.02768	.02768	.02768
13	.52922	.52922	.25253	.25253	.01384	.01384	.01384	.01384	.02768	.02768	.02768	.02768



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT No. _____

SHEET No. 3-18

AIRCRAFT: _____

PREPARED BY

DATE

CHECKED BY

DATE

* Siac (True Stress Matrix)

ac →

	1	2	3	4	5	6	7	8	9	10	11	12
1	.66667											
2	.66667											
3		.66667										
4			.66667									
5	.19842	.19842	.09468	.09468	.00519	.00519	.00519	.00519	.01038	.01038	.01038	.01038
6	.30108	.19842	.40532	.09468	.00519	.00519	.00519	.00519	.01038	.01038	.01038	.01038
7	.19842	.30108	.09468	.40532	.00519	.00519	.00519	.00519	.01038	.01038	.01038	.01038
8	.19842	.19842	.09468	.09468	.00519	.00519	.00519	.00519	.01038	.01038	.01038	.01038
9	.19842	.19842	.09468	.09468	.00519	.00519	.00519	.00519	.01038	.01038	.01038	.01038
10	.80411	.52932	.41414	.25253	.01384	.01384	.01384	.01384	.02768	.02768	.02768	.02768
11	.80411	.52932	.41414	.25253	.01384	.01384	.01384	.01384	.02768	.02768	.02768	.02768
12	.55932	.80411	.25253	.41414	.01384	.01384	.01384	.01384	.02768	.02768	.02768	.02768
13	.55932	.80411	.25253	.41414	.01384	.01384	.01384	.01384	.02768	.02768	.02768	.02768

TECHNICAL DEPARTMENT

REPORT No.

SHEET NO.

3-19

AIRCRAFT:

PREPARED BY

DATE

CHECKED BY

DATE

[illegible]



AVRO AIRCRAFT LIMITED
MALTON - ONTARIO

TECHNICAL DEPARTMENT

REPORT NO. _____

SHEET NO. 3-20

AIRCRAFT: _____

PREPARED BY

DATE

CHECKED BY

DATE

Za, bd

	1	2	3	4	5	6	7	8	9	10	11	12
1	.036987	.015426	.014572	.007361	.001629	.001629	.000403	.000403	.001734	.001734	.000807	.000807
2	.015426	.036987	.007361	.014572	.000403	.000403	.001629	.001629	.000807	.000807	.001734	.001734
3	.014572	.007361	.008261	.003512	.000316	.000316	.000192	.000192	.000631	.000631	.000385	.000385
4	.007361	.014572	.003512	.008261	.000192	.000192	.000316	.000316	.000385	.000385	.000631	.000631