SOME EFFECTS OF INTERNAL HEAT SOURCES ON THE DESIGN OF FLIGHT STRUCTURES

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BY

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SUMMARY

A survey is given of the problems which arise due to heating of a structure. The magnitude and accuracy of the heat sources and their effects on the structure are discussed. This is followed by a more complete assessment of the engine bay problem, its heating, structural effects and design considerations. Conclusions relating to this latter problem are drawn.

In the appendices, the analytical treatments of the transient and steady state temperature distributions are given, as well as an analysis of the thermal stresses in a typical jet engine bay structure.

CONTENTS

LIST OF FIGURES

NOTATION

Symbol

a	-	Former Depth
c	-	Specific Heat
c_p	-	Specific Heat at Constant Pressure
h	-	Convective Heat Transfer Coefficient
k	-	Thermal Conductivity
l	-	Semi-former Spacing
n	-	A Constant ($n = 1/4$, Laminar flow; $n = 1/3$ Turbulent flow)
q	-	Rate of Internal Heat Generation
t	-	Time (hr.)
t	-	Thickness (ft.)
A	-	Bypass Area, (section area)
C	-	A Constant
D	-	Characteristic Length (ft.)
E	-	Young's Modulus ($lb/in.^2$)
E_T	-	Tangent Modulus ($lb/in.^2$)
F_ε	-	Emissivity Factor
F_F	-	Form Factor
m	-	Bypass Mass Flow (lb/sec.)
N_{GR}	-	Grashof Number
N_{PR}	-	Prandtl Number
Q	-	Heat Flow per Unit Area
T	-	Temperature (^oF)
α	-	Linear Coefficient of Expansion
λ	-	$\left(\frac{h\alpha + h\beta}{Kt}\right)^{1/2}$
ρ	-	Density ($lb./ft.^3$)
σ	-	Stefan - Boltzmann Constant, Stress

τ - Defined in Appendix I

K - ($\frac{k}{c\rho}$) Diffusivity (ft.^{2}/hr.)

P - Mean of Engine & Bypass Perimeters

M - Mach Number

<u>Subscripts</u>

c - Cooling Air

e - Environmental

F - Former

i - Insulation

s - Shroud

m - Mean Value

x, y, z - Rectangular Co-ordinates

α - Insulation Foil

β - Outer Shell

$\mathcal{E}$ - Engine

o - Initial

∞ - Infinite Former Spacing

1. Introduction

Until recently, the problem of internal heat sources was mainly confined to the engine and its accessories; the amount of heat transmitted to the structure being comparatively small. However, the demands imposed by high speed flight have resulted in increasing amounts of heat, emanating from both internal and external sources, having to be absorbed by the structure. The effects of these heat sources must therefore be considered in structural design.

The aim of the heat transfer analysis is to provide the temperature distribution from which the effects on the structure are determined. Any assumptions that are made, either to simplify the calculation, or because better data are not available, will therefore influence the structural analysis. For this reason, the problem of internal heat sources must be examined from the structures aspect as well as that of heat transfer.

As it is seldom possible to draw general conclusions when faced with all the possible combinations of structures and heat sources, a general review of the problem is first presented. This is followed by study of the particular case of the jet engine bay so that specific conclusions can be drawn.

2. GENERAL REVIEW

Since kinetic heating of structures became a problem, a great deal of analytical work and testing has been conducted. Fortunately, most of these data are also applicable to the case of internal heat sources, and a review of the common points and differences should provide a background to the latter problem.

2.1 Internal Sources of Heat

2.1.1 In general, temperature differences exist between various parts of a flight vehicle, and all elements of the vehicle act as heat sources or sinks. Structural effects, however, will be of importance only when the temperature, thermal gradients and heat transfer coefficients are high.

The important internal sources of heat are as follows:

(1) Engines a) Direct heat from engine skin and jet exhaust

b) Heat from engine auxiliaries and engine services

(2) Crew and passengers.

(3) Weapons, fuel and other stored chemicals.

In the case of vehicles which operate in the atmosphere, it is necessary also to consider air which leaks or is ducted to interior regions.

2.1.2 Discussion

Generally, the engine and its services give rise to most of the important problems of heating by internal sources. Later, some of these problems will be considered in detail. Physiological sources of heat generally are important only in that they constitute part of the load on the air conditioning system. Because of the high specific heat of fuel, it has a marked effect on the temperature of any structure with which it is in contact. Some equipment items store chemicals or gases, and the use of such equipment creates a heat source or sink. Weapons, storage batteries and generating equipment for oxygen or nitrogen are typical examples. Gas generation generally results in low temperatures. Usually these would be confined to the equipment proper, but a check should be made to ensure that the heat sinks are not affecting structural members, since thermal stresses, distortion and abnormal material properties can result from low as well as from high temperature conditions.

2.2 Heating

2.2.1 Modes of Heating

There are three distinct modes of heat transfer: conduction, convection and radiation. Convection is further differentiated into forced and free conditions. In general, a source will heat its environment by all three modes and, likewise, a structure will receive heat by all three. For most engineering work, it is sufficient to represent the rates of heat transfer by the various modes with the following formulae:

(1) Conduction at a point, in x direction

$$Q = -k\frac{\partial T}{\partial x}$$

(2) Convection - forced (Newton's Law)

$$Q = h\left(T - T_e\right)$$

(3) Convection - free

$$Q = C\left(N_{GR} N_{PR}\right)^n \frac{k}{D}\left(T - T_e\right)$$

(4) Radiation between two bodies

$$Q = \sigma . F_\varepsilon . F_F \left(T^4 - T_e^4\right)$$

and to these we can add the differential equation of heat conduction in solids

(5)

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right) + g$$

These equations and the conditions for which they apply have been studied in detail.[1]

Equation (2) is to be treated rather as a definition of h_c than as a physical law.

To readily assess the importance of the various heat transfer modes, it is sometimes advantageous to linearize the equations for radiation and free convection. Examples of how this can be done are given in Figures 1 and 2, using a simplified formula applicable to air[2] for free convection. Hence it is possible to determine an effective "overall" heat transfer coefficient which can be used over a limited range of temperature. Alternatively, conduction and forced convection can be expressed in terms of 4th or 5/4th power laws. One mode of heating will generally predominate for each heat source. Referring to the classifications in para. 2.1.2, the examples of the predominant modes are:

(1) Engine	Radiation from engine skin to structure
Engine services:	
Oil System)	Forced convection from heat exchanger efflux
Hydraulic system)	
Electrical system	Free convection to compartment interior
Air conditioning	Free convection to compartment structure

(2) Crew and passengers — Free convection to cabin air

2.2.2 Thermal States

Any case of structural heating will be either transient or steady state. Whether or not steady state is reached will depend upon the relative size of the heat load and heat sink, and upon the duration of heating. For instance, structural temperatures which depend mainly upon the flight condition, may not reach equilibrium, because of a short time of flight or because of changing ambient conditions. However, in some cases of internal heating, the heat input is relatively independent of flight condition. For example, engine temperatures can remain fairly constant over a wide range of flight conditions, as can the temperature of air conditioned equipment and compartments. Hence, the structure adjacent to such heat sources may be at a steady temperature for long periods.

It is much easier to calculate steady state temperature distributions in a structure than to calculate the transient state distributions. Because of this there has been a tendency in the past to design for the steady state, whether or not it is reached. This practice, although convenient for preliminary design, is undesirable for two reasons. First, since the reduction in material strength is great at elevated temperatures, too conservative a design could result, with attendant weight penalty. Second, as the most sever thermal stress occurs when the temperature gradients across a structure are highest, design to the maximum steady state temperature may not be safe.

Thus it is often necessary to analyze the transient temperature distributions and the thermal effects induced in the structure. It must also be remembered that in dealing with internal heat sources, achievement of a steady state may not result in uniform temperatures. The equilibrium case can involve severe temperature gradients and correspondingly severe stress and distortion levels.

2.2.3 Discussion

Complete exact solutions of the heat transfer equations are very seldom possible in practical cases.

Heating by conduction can be rigorously treated for simple structures. If, however, radiation and convection occur, as in most cases encountered in practice, a full treatment is possible only be involved numerical methods. Alternatively, some simplifying assumptions can be made about either the geometry of the problem or the method of analysis.

In the analysis of steady state problems, simplification generally takes the form of choosing a one dimensional heat path that will include the critical points of the structure. Sometimes conformal transformations can be used to make two dimensional flow problems more tractable. If these simplifications do not lead to equations which can be solved conveniently, numerical or relaxation techniques can be employed.

For the analytical treatment of transient state problems, it is usually necessary to simplify the structure and to select values of thermal properties which are independent of the temperature. Also, it is convenient to use a step input function. This restriction is not essential to the analysis, but often can be justified since modern vehicles change their flight conditions so rapidly.

If it is desired and worthwhile to solve complex or non-linear problems more exactly, then a new method recently introduced by Biot[4] can be used.

2.3 Effects on the Structure

2.3.1 Introduction

Adding temperature to the loading and time parameters has the effect of producing interactions between those parameters. The change between the room temperature and the elevated temperature interactions can be represented approximately as shown in Figures 3a and 3b.

When examining these diagrams, two things should be realized. First, that some additional interactions exist (e.g. effect of rate of loading on strength properties). Secondly, that strictly speaking the diagram titled "elevated temperatures" also applies to room temperature effects, and it is only because these effects are small for standard structures that they can be neglected.

To produce an exact structural analysis with thermal effects presents difficulties, which are caused mainly by the temperature response being a function of time. This is particularly true for structural elements which are affected by kinetic heating, with or without additional exposure to internal heat sources.

Therefore, it is usual to first assess the importance of various effects by taking a severe case and assuming a total time for its occurrence. In this manner, we are led to simplifications of the overall problem which will enable an appropriate analysis to be made.

It is fortunate for the internal heat source problem that transients usually represent a small proportion of the total time, so that the assumption of a severe case and its corresponding time is easier to make.

2.3.2 Strength Properties

Perhaps the most drastic effect of temperature on the structure is the corresponding drop in strength properties. Given the working temperature range of a structure, it can be seen which materials are definitely unsuitable. However, the final choice should not be based on the strength values only, but should take into account the function of the structure and the magnitude of the thermal effects which can arise.

For this summary only, three typical materials are examined. The approximate strength properties of these materials are shown on Figures 4 to 7. The specific strengths at elevated temperatures (Figures 6 and 7) give a measure of the efficiency of the alloys for ultimate and yield conditions. It can be seen that the aluminum alloys become inefficient above 300-350°F, and titanium at around 800-1,000°F. Thus of the three materials considered only stainless steel can be used efficiently above that value.

There are a few points worthy of note regarding aluminum alloys. First, at elevated temperatures they are affected considerably by the strain and heating rate. Second, their properties are dependent upon the length of exposure at temperature; this effect being cumulative. This is a characteristic of precipitation hardened materials. On the other hand, the recovery values represent the room temperature strengths of a material after it has been exposed to a high temperature for a given time. The aluminum alloys once more show a large decrease in properties as well as a marked dependence upon time of exposure. The other alloys do not seem appreciably affected, and their recovery values are equal to their original room temperature values.

2.3.3 Thermal Stresses

Differences of temperature within a structure produce unequal strains, which if restrained, will result in thermal stresses. However, as it is usual to calculate the temperature distribution by using a simplified heat transfer model, an arbitrary subdivision of the term thermal stress has arisen. The value, which is calculated by using the thermodynamic model isolated from the rest of the structure, is called the free thermal stress. Then introducing the interaction of the surrounding structure on that component ("restraint stresses"), the sum produces finally, the thermal stress proper as previously defined.

Thermal stress = Free thermal stress + Restraint stresses.

Another difference should be noted between engineering theory and secondary effects thermal stresses. Engineering theory thermal stresses are defined as the stresses calculated assuming a linear relationship between stress and strain, and that physical properties (like α and E) are constant with temperature. The secondary effects thermal stresses would take into account plasticity, effects of buckling and creep, as well as the change of material properties with temperature.

Investigations of the general problem have been conducted by Shuh [5], Goldberg [6], and others. The engineering theory free thermal stresses for a spar type of structure being found for conditions involving symmetry or both heating and structure.

Unfortunately, the secondary effect thermal stresses cannot be generalized as easily, but their influence is sufficient to produce large magnitude deviations from the engineer's theory values.

As a special case of thermal stress, we must introduce the thermal shock problem. This occurs when the surface of a material suffers a sudden change of temperature. The large localized strains thereby induced may cause failure of a brittle material. This may occur at the surface of a material of poor conductivity or at the centre of a material with good conductivity.

2.3.4 Stiffness

Heating of the structure can cause a marked reduction in the stiffness, which may introduce some aeroelastic and/or stability problems. If this reduction appears critical, the stiffness analysis may well have to be conducted at intervals throughout the aircraft's mission in order to give the stiffness as a function of time for consideration in the aeroelastic analysis.

The most direct change of stiffness is caused by the reduction in the shear and Young's modulii. Figures 8 and 9 show typical variations of Young's modulus and non-dimensional tangent modulus with temperature. The reduction in the shear modulus is similar but the effect is even more pronounced.

Stiffness can be further reduced by thermal stresses. The classical example of this is the effect on a solid diamond-shaped thin wing, where the leading and trailing edges heat up quickly and become loaded in compression. Then due to twist, the vertical components of the compressive force act in the direction of the applied torque, producing a reduction in torsional stiffness.[7] There is a bending effect also, which is due to the antielastic curvature producing an offset of the leading and trailing edges, whose compressive forces bend the wing further.[8]

Buckling will also cause a reduction of stiffness, due to loss of effective material. This can be caused by thermal stresses and/or a decrease of local stiffness modulus due to temperature. Creep similarly can reduce the stiffness by causing buckling to occur. The net result of buckling is to further lower the value of the elastic modulus. As this reduction may not be uniform, a change of vibration mode shape can occur.

2.3.5 Fatigue

This term is applied to the failure of a structure due to the repetition of a stress and/or a temperature cycle which it has previously withstood. The possible combinations of loading and temperature introduce another difficulty to this already intriciate problem. It is known that fatigue is affected not only by the loading and temperature cycles, but also by the surface conditions, stress concentrations and the environmental conditions.

We can however differentiate between static or creep fatigue and cyclic fatigue. In the former, the total duration of the loading is the important parameter, whereas in the second the influence of the number of cycles predominates. Quite often, at elevated temperature, both these effects are present and combined. Due to lack of knowledge it is practically impossible to interrelate all the effects influencing fatigue, but even if this were possible the problem of forecasting the history of the temperature and load cycles would remain.

Figure 10 shows the approximate effect of temperature and number of cycles on the endurance limit of three materials. In most cases the increase in temperature causes a reduction in endurance limit. The stainless steel, however, behaves similarly up to the onset of plasticity, and then shows an increase with a peak at about 1,200°F, followed by another change of slope and a decline of the endurance limit.

2.3.6 Deformations

The remaining major effects on the structure are produced by deformations due to temperature. These can be classified as either dependent or independent of time, and their influence is felt on the deformation of the structure as well as on the stability of its components.

Deformations, which are independent of time, result from the thermal expansion of the structure. These may cause an appreciable reduction of clearances between components. The analysis however is comparatively simple; the main difficulty arising in estimating the interaction of other components.

Time independent stability problems include buckling and crippling of components, such as columns and plates. When the temperature is uniform the same analytical procedures as for room temperature can be used, with a due allowance being made for the temperature effects on the stress strain curve. However, when the structural component has a non-uniform temperature distribution (thermal buckling) the analysis is not on such firm footing. The variation of temperature produces bowing and therefore an eccentricity in the component, thermal stresses are introduced and there is a variation of material properties across the section. In the usual analysis, the thermal stresses are added to the applied stresses and the eccentricity is taken into account together with a stress-strain curve weighted in favour of the higher temperature. Although thermal stresses lower the buckling strength of stiffened panels, in general the post buckling ultimate strength is relatively unaffected. Further studies and tests are required to develop or confirm the various methods of analysis which have been proposed.

Creep is the time dependent deformation imposed by the application of stress. A characteristic creep curve shows an initial extension that occurs immediately after application of the load, followed by a decreasing creep rate (1st stage), then a constant creep rate (2nd stage), until finally an accelerated creep rate followed by failure (3rd stage). Some typical data are shown in Figures 11 to 13. A marked dependence on temperature exists for all materials, and is particularly pronounced for aluminum alloy. The difficulty in analysis lies in the prediction of the time-loading-temperature history and its correlation to creep data, as well as the influence of residual stresses which exist after removal of the load. However, the importance of creep depends very much on the function of the structure and its temperature levels.

The principal time dependent stability problem is creep buckling. The one dimensional problem is the analysis of a column with slight but unavoidable eccentricity. This produces a bending moment under which the initial curvature of the column increases due to creep. The increased curvature produces a larger bending moment and hence more rapid creep, and so on until failure occurs. This has been treated extensively, and a satisfactory solution seems possible by using the iso-stress-strain curves. These represent a plot of stress versus strain, with time as a parameter.

The standard room temperature formulae are then used with the iso-stress-strain curves replacing the actual stress-strain curves. It is interesting to note that tests of columns under various combinations of load and time seem to indicate that failure occurs once a critical deflection has been reached. The study of the two and three-dimensional problems are faced with the difficulty that little is known about its creep laws. A few problems have been solved and empirical formula derived from tests.

3. THE ENGINE BAY PROBLEM

The engine is generally the most powerful internal heat source in an aircraft. Therefore an investigation of the heat transfer, effects on the structure, and design considerations of a simplified jet engine bay should disclose most of the typical problems produced by other heat sources,and suggest means for solving them.

Of the various structures which are possible in the engine bay region, a circular surrounding structure comprises of an inner shell or shroud and an outer shell reinforced by formers has been chosen for discussion. It is assumed that the primary purposes of the structure are either to house the engine and provide it with a bypass air duct or to guide the flow around the jet exhaust.

3.1 Heating

A typical engine bay structure for many flight vehicles is shown in Figure 14.

The thermal problem can be considered in two parts: first, the determination of the insulation foil temperature and second, the determination of structural temperature distributions; treating the foil as the source of heat.

3.1.1 Heat Input

We assume that the foil surface temperature is determined mainly by the balance between heat radiated from the engine and heat convected to the bypass air. Hence as a first approximation the foil temperature can be determined independently of the structure. The foil, at constant temperature, can then be treated as the source of heat to the structure.

Thus to find the temperature of the foil, we need to determine the temperature and heat transfer coefficient of the bypass air and the temperature of the engine. Air taken into the bypass will be at stagnation temperature, and will be heated in its passage along the bypass, both by the engine and by the foil.

It is a simple matter to assume a foil temperature distribution along the bypass, and evaluate the increase in air temperature for successive length elements, and thus arrive at the air temperature distribution.

Heat transfer coefficients can be evaluated by the method outlined in Appendix I. Thus, for a one-foot length of bypass duct, the increment in air temperature is

$$\Delta T_c = \frac{h P \left\{ (T_\varepsilon - T_c) + (T_\alpha - T_c) \right\}}{3600 \, C_p A m^{1/5}}$$

With this approximate distribution of air temperature the foil temperatures can be calculated, and if necessary, a second iteration can be made.

Now a simple heat balance can be written for the foil temperature, using an assumed value for mean shroud temperature. This assumption can be relaxed if necessary by a second iteration.

$$h_{v\varepsilon}(T_\varepsilon - T_\alpha) = h_{vc}(T_\alpha - T_c) + h_{v\alpha}(T_\alpha - T_{SM})$$

The foil temperature is then our input for the further structural heating analysis.

If there is no insulation, a similar analysis can be carried out for the shroud, neglecting as a first approximation, the effect of the formers, in finding the temperature of the bypass air. Then our heat balance can be written

$$h_{v\varepsilon}(T_\varepsilon - T_\infty) = h_{vc}(T_\infty - T_c) + h_{v\beta}(T_\infty - T_\beta)$$

where T_∞ is the temperature which the shroud would reach in the absence of formers. Later, the effect on shroud temperature due to heat flow through the formers to the outer shell can be checked and an iteration made if necessary.

Typical variations of foil temperature with Mach number, bypass flow and engine temperature are given in Figures 15 and 16.

3.1.2 Steady State Heating

The steady state case can be solved in considerable detail if required. We have assumed that the insulation foil can be considered at constant temperature over any length 2ℓ along the structure. We further assume the following:

(a) The capability of the boundary layer for removal of heat is so great that the outer shell is maintained at a constant temperature.

(b) No heat is lost from the former web.

(c) Heat transfer from the shroud to the outer shell whatever mechanism can be approximated by an overall heat transfer coefficient.

The details of the analysis and some extensions to this simple case are given in Appendix II.

The solution gives the following temperature distribution throughout the structure:

$$T_{Former} = T_\beta + \frac{(T_\infty - T_\beta)(a+x)\lambda \sinh \lambda \ell}{a\lambda \sinh \lambda \ell + \gamma \cosh \lambda \ell}$$

$$T_{Shroud} = T_\alpha - \frac{(T_\infty - T_\beta)\gamma \cosh \lambda(\ell - x)}{a\lambda \sinh \lambda \ell + \gamma \cosh \lambda \ell}$$

A typical structure which we shall examine is that shown in Figure 17. For a basic case corresponding to a M = 3 flight condition the values of the parameters are

$$h_\alpha = 1.0 \ \frac{BTU}{hr. ft^2 \, ^\circ F} \qquad h_\beta = 0.5 \ \frac{BTU}{hr. ft^2 \, ^\circ F}$$

$$T_\alpha = 800^\circ F \qquad T_\beta = 560^\circ F$$

The resulting temperature distribution is shown in Figure 18. The significant structural parameters can be grouped as:

$$\ell^2/kt \qquad \ell/a \qquad \gamma = k_F t_F / 2kt$$

in which the quantities without subscript pertain to the shroud the variations from the basic case of the mid-shroud temperature and the shroud-former junction temperature with each of these parameters are given in Figures 19 to 23.

Typical structures for flight at M = 2 and M = 4 would have temperature distributions as shown in Figures 24 and 25 respectively.

3.1.3 Transient State Heating

The transient state heating of the typical Mach 3 structure (Figure 17) has been calculated by a finite difference method under the assumptions made in para. 3.1.2. Figure 26 gives the heating history found by this method. Considering the nature of the input data, a simple approach such as this will generally suffice. If a more detailed study is warranted the theory of Biot[4] can be used to develop an accurate analytical solution (ref. Appendix III). Figure 27 shows a series of temperature distributions determined by this analytical method.

3.2 Effects on Structure

From the definition of its purpose, the engine bay structure can be assumed to be only moderately loaded by effects other than those produced simultaneously by the temperature and pressure loadings.

In addition the design is such that in general, the pressure stresses are low, particularly when the temperature loading is high.

3.2.1 Strength Properties

The remarks of the previous discussion on strength properties still apply and Figures 4 to 7 are representative of materials considered for use in the jet engine bay within a speed range of $M = 2.0$ to $M = 4.0$.

3.2.2 Thermal Stresses

The thermal stresses represent the principal thermal source of loading on the structure, and as such an appreciation of the various effects on their distribution and magnitude is instructive.

The transient heating case gives rise to the largest thermal stresses, but fortunately this occurs for short periods only. This case was analysed on the assumption that:

(a) The former and shells can be treated as a beam restrained in bending because the former is free to expand radially but cannot bend.

(b) An effective width of the shells acts with the former. This is estimated from the curve of die away of shell hoop stress due to a load uniformly distributed along a circular section, see Figure 28 and Appendix IV.

(c) The section does not buckle or creep and αE is constant with temperature and stress level.

With these assumptions the equation for stress is:

$$\sigma = \alpha E \left(-T + T_M\right)$$

However, once the peak stress in a shell has been calculated, the stress distribution follows the shape of the die-away curve.

Using this approximate method, the stresses were calculated for the results of two transient temperature distribution calculations on a typical M = 3.0 structure. See Figures 29 and 30.

For the steady state case, the same method of analysis was used to calculate:

(a) The stress distribution in three structures typical for M = 2, 3 and 4. See Figures 31 to 33.

(b) The change in stresses due to variations in heating and geometrical parameters on a typical M = 3.0 structure. See Figures 34 to 38.

Then an exact analysis is carried out on the typical M = 3.0 structure (ref. Appendix IV). The results are shown on Figure 39.

Reviewing the stresses obtained for the various conditions and methods we can conclude that:

The effective width method gives an excellent evaluation of the hoop stresses. A correction for the radius can be introduced which brings the results even closer. However, this method does not evaluate the bending stresses which are of the same order as the hoop stresses.

In all cases investigated, the hoop stresses are low, and as expected the transient cases give the maximum values. To represent the true conditions however, the stresses which arise from the internal pressure on the shroud (see Figure 40) should be added to those values. This addition is algebraic only as long as either stress or the sum of the stresses remains in the elastic range. In the plastic range this addition must be done on the basis of strains.

Plasticity will also cause some residual stresses to be present when the structure returns to initial conditions. The next loading cycle will start from a pre-stressed condition, thus giving a different stress distribution. A similar effect is also known to exist for creep. Taking these effects into account will give the secondary thermal stresses which can differ considerably from the engineer's value. It has been assumed in the stress analysis that there was no end restraints to the longitudinal expansion of the shells. Should end conditions however be such that this is prevented, a longitudinal stress of the order of 6-7000 psi (standard case) is induced and will tend to produce longitudinal buckling of the shells.

3.2.3 Stiffness

This effect is important only inasmuch as the lower value of Young's modulus decreases the panel's buckling strength. Thermal stresses produce no change of stiffness, due to the symmetry of the heating and of the structure. In any case, the effect of an overall decrease in stiffness of the engine bay does not produce noticeable aeroelastic effects. However, there can be a risk of the decreased stiffness allowing the structure to deflect longitudinally so as to interfere with the symmetry of clearance around the engine.

3.2.4 Fatigue

With the comparatively low stress levels encountered in the steady state condition, there are few fatigue problems. The stress concentrations, which arise at rivet holes for instance, are relieved to a large extent by the temperature allowing local yielding. The problem can arise however if this region is exposed to the high intensity noise levels of the engine efflux, or if the loading is such that thermal buckling, yielding or stress reversals occur during the heating or pressure cycles.

3.2.5 Deformations

The effects of the thermal expansion of the structure do not appear severe or detrimental. Radially, the clearance between the engine and the structure will increase, and longitudinally the inner shell will tend to expand more than the outer one. If restrained, the latter induces some longitudinal stresses (ref. para. 3.2.2).

Buckling of the shell panels may occur between the supporting formers. The circular shape and the symmetry of the loading renders the panels quite stiff, but unless exceptional circumstances are encountered, this is not a problem.

Creep, however, could be a problem due to the long steady state heating times, but this is counterbalances by the low stress levels. The situation being in fact the reverse of what occurs under kinetic heating, where the loading is high but the heating times are low. In general, however, the requirements imposed by the other effects will produce a structure where creep is not important.

Due to the fact that most sections are inherently stable, creep buckling is an effect that should be checked, but seldom produces design requirements.

3.3 Design Consideration

In the design of a structure, several basic questions must be answered.. First, what conditions must be met? The answer to this question entails a critical examination of the heat inputs, the external loads and their combination. Secondly, what effects do these heat inputs and loads impose on the structure? This involves an exhaustive survey of the various effects; the important ones being singled out for further investigation. Finally, what is the best design? This is determined by calculations to seek the optimum combination of materials, geometry and methods to alleviate the design problems.

It follows, therefore, that a design is not merely the result of a mathematical equation, but rather a process which at every stage is greatly influenced by the emphasis placed on the various aspects of the overall problem, and by personal opinions.

With the foregoing points in mind, the following design considerations are advanced.

3.3.1 Loading

As the purposes of the structure are such that pressure and temperature are its main sources of loading, the investigation into the combinations which arise is simplified.

Assuming that basically three types of conditions can occur:

(a) Steady state

(b) Transients

(c) Special cases

The steady state represents the flight conditions which occur for instance at cruise or top speeds. The resulting combination of loads will tend to be such that for the cruise there is a high pressure combined with a low temperature, and vice-versa for top speed. The resulting stresses are low (ref. Figures 39 and 40), but due to the long exposure they must be considered in the creep and fatigue analyses.

The transient cases on the other hand, have higher stress levels and will mainly determine the required strength. Due to the uncertainties which occur in the determination of the pressure and temperature transients, hypothetical cases formed by taking the most sever combinations of pressure and temperature which can arise during instantaneous changes of condition are assumed. Finally, special cases are the conditions encountered during ground run up, or cases whose importance in relation to the previous ones depends very much on whether or not some special condition, such as negative pressure, is encountered.

3.3.2 Heat Input

Probably the best approach to alleviating the temperature effects on a structure is to reduce the net heat flow to the structure. This can be done in several ways. On the engine bay heating problem, it involves either decreasing the foil temperature or increasing the insulation resistance. The former can be accomplished by either increasing the effectiveness of the cooling air, or by decreasing the heat transfer from the engine. Increasing the cooling air flow is useful only to a point, since the drop in foil temperature is unlikely to compensate for the loss of thrust, due to the large bypass flow. Increasing insulation resistance is comparatively simple but would result in increased weight and reduced bypass area.

The most common practical method is to interpose a low emissivity radiation shield between the engine and the foil.

In general, steady state temperatures effects can be alleviated only if there is a heat sink available when it will be possible to either insulate from the heat source or provide high conducitivty material to take heat to the sink.

In the transient state, temperature effects can be modified by adding to the heat capacity of the structure, as well as by reducing the net flow of heat.

3.3.3 Structural Effects

From the discussion on thermal effects on the engine bay structure, it would appear that the decrease in material strength properties due to temperature and thermal stresses, were the only important factors to be considered. Within the range of conditions examined, it is always possible to chose a material such that its decrease in strength is acceptable. But as thermal stresses are a main component of the loading, the choice of material should be tempered by thermal stress considerations. For a given temperature distribution, the stress varies as the value of αE, and so a low value of this parameter is desirable. Another method for reducing thermal stresses is by using different materials for the shells and the formers with a view to reducing the differences between the thermal strains of the components.

Insulating the structure, reduces the thermal stresses and the working temperatures of the structure. But, on the the other hand, the insulation will impose new design problems.

The optimum combination of insulation and structure characteristics is determined from analyses which must also be based on the load level the structure has to carry.

As can be seen from the exact analysis (Figure 39), the junction between the structure components exhibits the highest thermal stresses. This localization is also further aggravated by stress concentrations due to riveting and joint conductance. The latter effect being particularly marked in the transient case, when a large drop of temperature can exist between the components, thereby raising the thermal stresses. Some measure of relief is afforded by the tensile pressure stresses (Figure 40) which decreases the compressive thermal stress of the shroud. These stresses will increased on the outer shell but the increase is minimized by the fact that this component is at the lowest temperature. Longitudinal bending stresses, however are all additive and their peaks occur at the joints. All this indicates that the principal design problem occurs at the structure joints.

Some thermal stress relief can also be obtained by changes in the geometry and structure. For instance, making the former of thicker material will allow it to respond more quickly to the heat flow from the shells. This results in a decrease in the average temperature of the shroud over the effective width region, and thereby reduces the thermal stress.

Structural and thermal characteristics of materials are difficult to determine accurately. There is reason to believe, for instance, that thermal conductivity can vary by 30% between two samples of the same alloy, and a great deal of experimental work needs to be done on material properties. Such quantities as the joint conductivity in a build up structure and surface emissivity are particularly difficult to determine, and all thermal characteristics vary with temperature. Also, in the case of a jet engine bay, the bypass flow and the temperature of the engine skin may be in error by as much as 10%.

Hence the results presented must be considered as approximate only, because of the nature of the input data and the assumptions made to produce and analyse a mathematical model.

Some refinement of the analysis is possible with more complex methods. However, such methods will not reduce the error, due to the uncertainty of physical data and this uncertainty can best be reduced by testing.

4. CONCLUSIONS

Within the range of conditions which were investigated, the engine bay problem can be resolved by the application of standard methods and ideas.

The approach to the heat transfer analysis, which assumes temperature independent material properties, and a simplified heat transfer model, appears justified when unknowns such as joint conductivity and scatter of material properties from sample to sample are considered. In general, cyclic heating and stress effects will not alter the thermal properties. Thus for given heating conditions, the temperature distribution will not change throughout the life of the aircraft.

Some of the concepts for stress analysis must be modified to account for the change in material behaviour, due to temperature. Room temperature methods, modified to include the effect of temperature on the material properties, can be used and give satisfactory results.

In design, the criterion is taken that the limit strength of the structure is not exceeded by the simultaneous application of two severe conditions of temperature and internal pressure. The fatigue and creep life problems are based on the long term endurance conditions, but the strength requirements produce a design which ensures practically limitless life. In the cases considered, the additional effect of temperature can be met by a change of material and/or increases in thickness.

Before more severe conditions can be confidently analysed (or present conditions more rigorously) it will be necessary to have a better understanding of the scatter in properties, high temperature behaviour of materials, thermal conductance of structural joints, and of engine bay heating conditions.

Acknowledgements

The authors wish to express their thanks to Avro Aircraft Limited for permission to publish this paper, and to their colleagues in the Technical Design Department for their assistance and suggestions.

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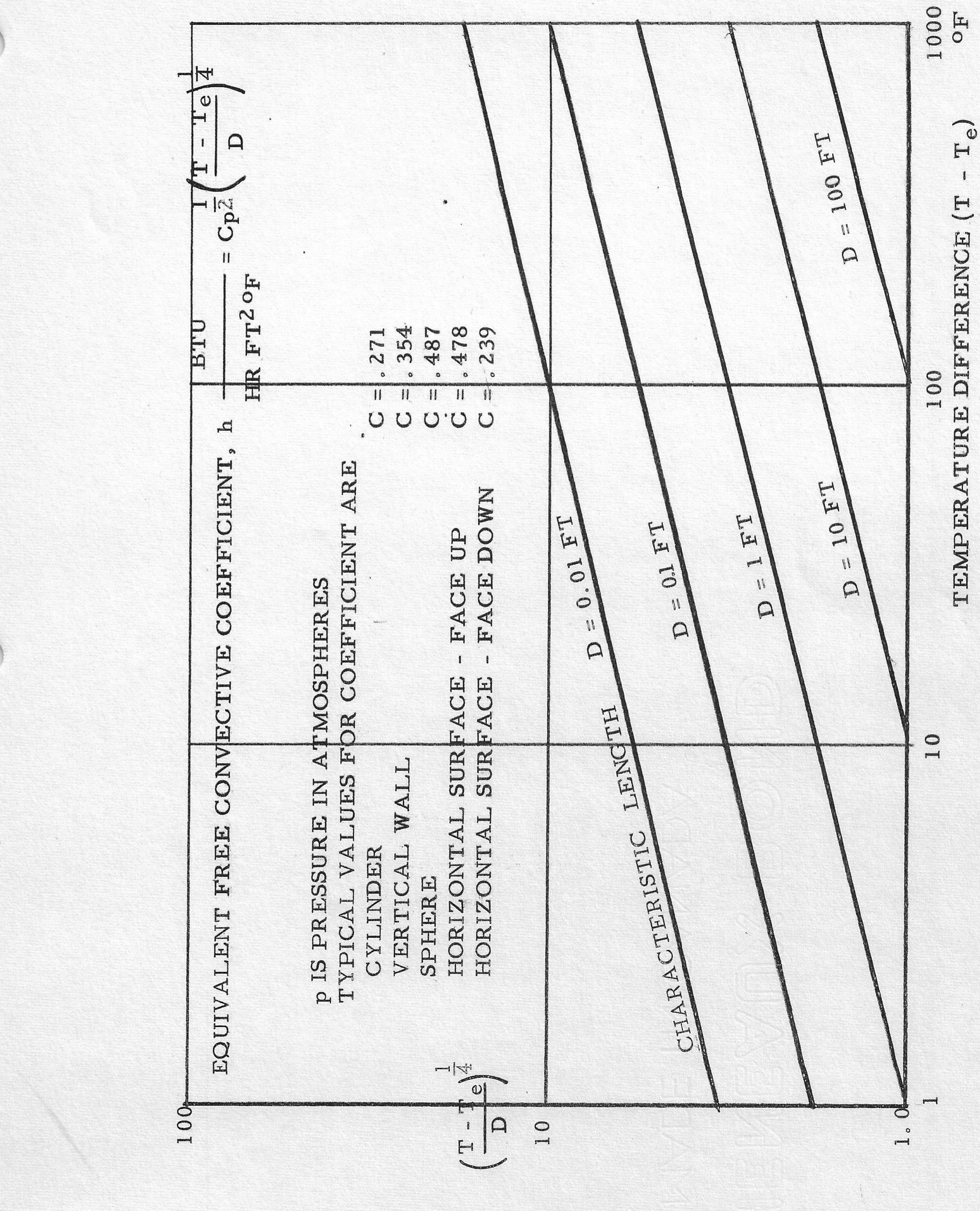


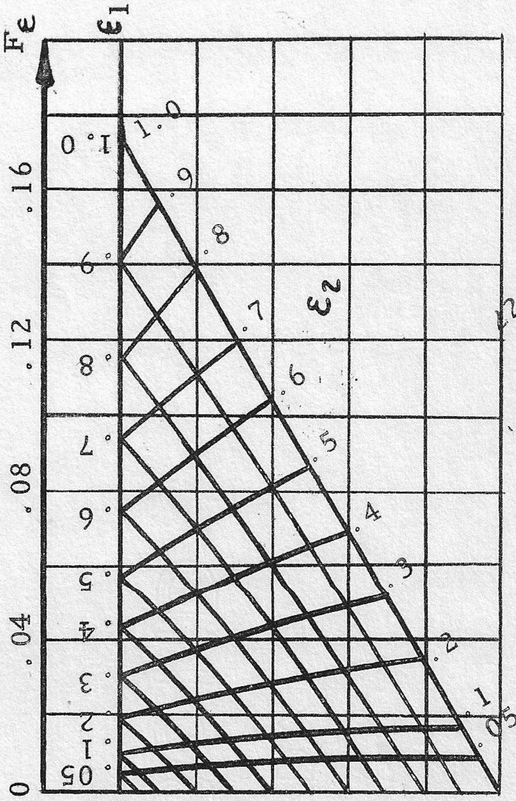
FIGURE 1 - EQUIVALENT FREE CONVECTIVE HEAT TRANSFER COEFFICIENT FOR AIR

EQUIVALENT RADIATIVE COEFFICIENT $h = F_\varepsilon F_T$

WITH

$$F_\varepsilon = \frac{0.174}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$

$$F_T = \frac{\left\{\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right\}}{T_1 - T_2}$$



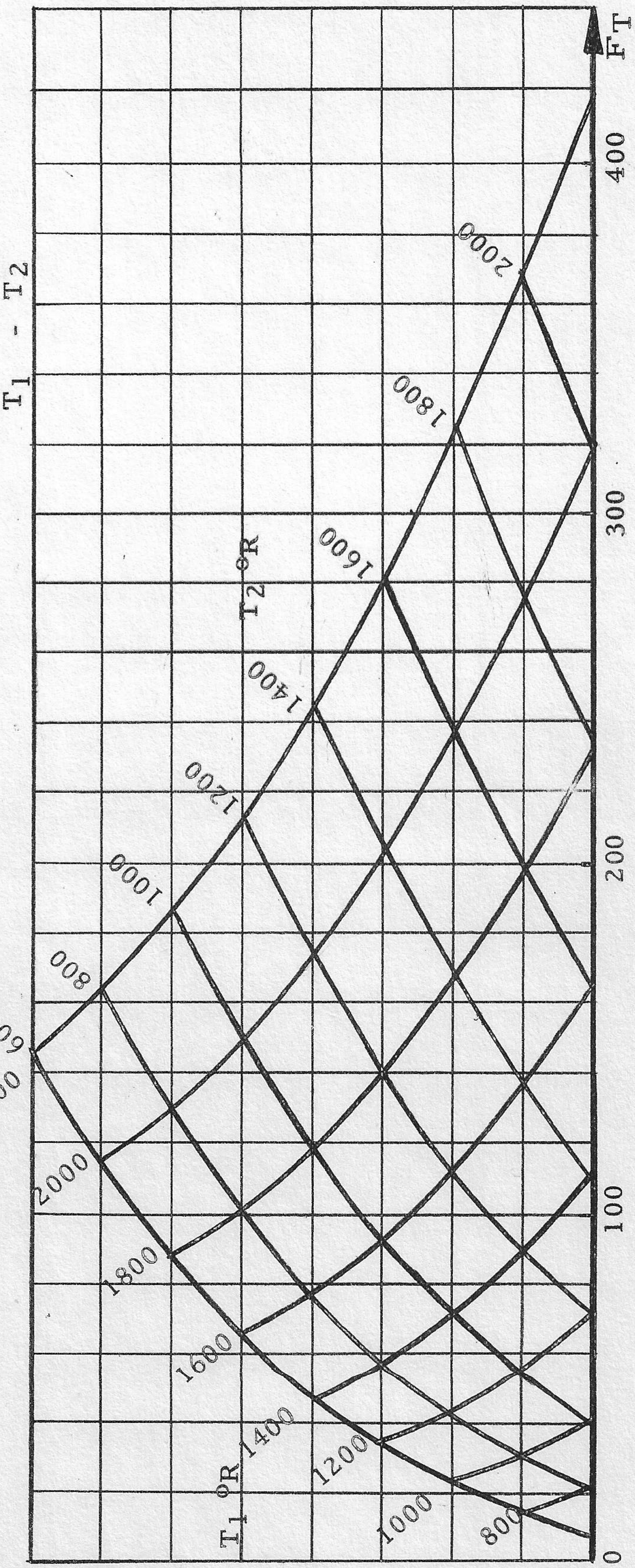
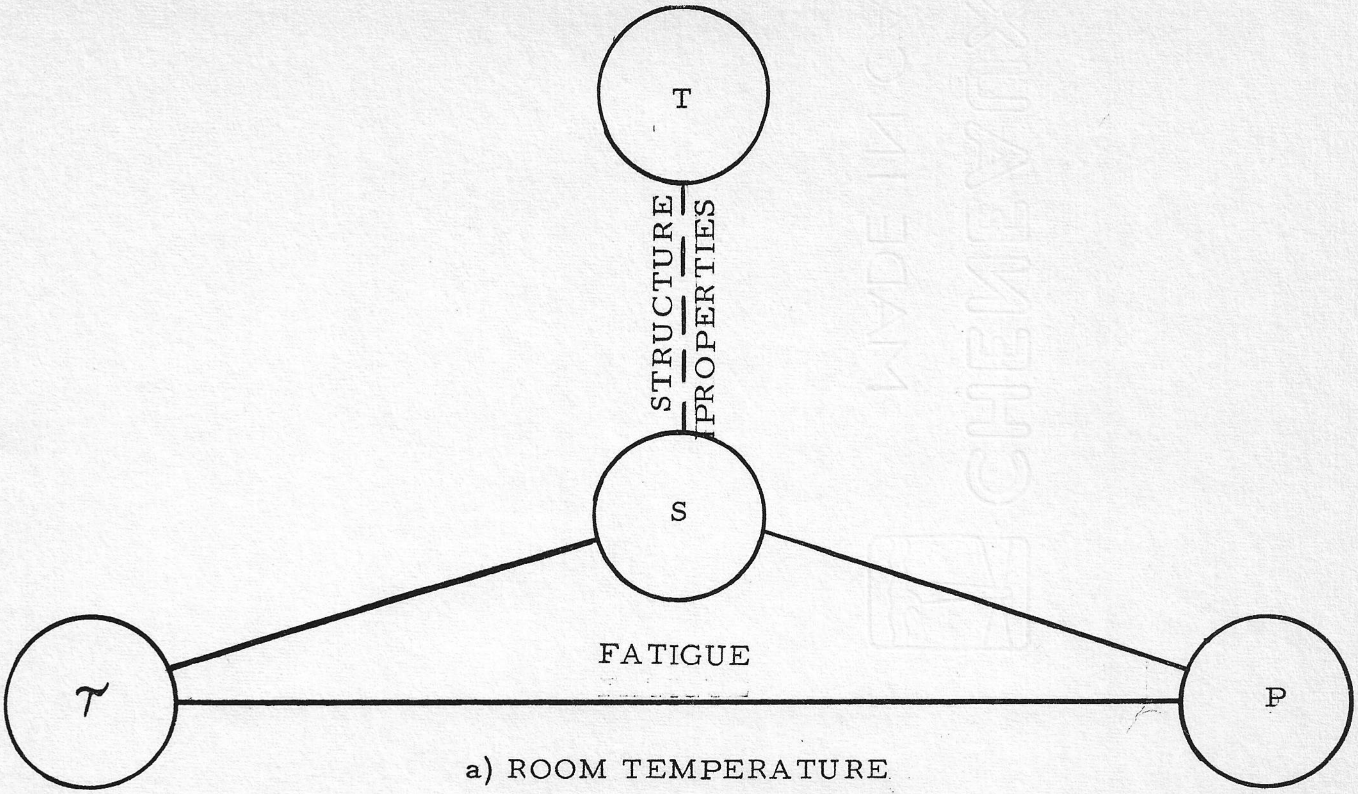
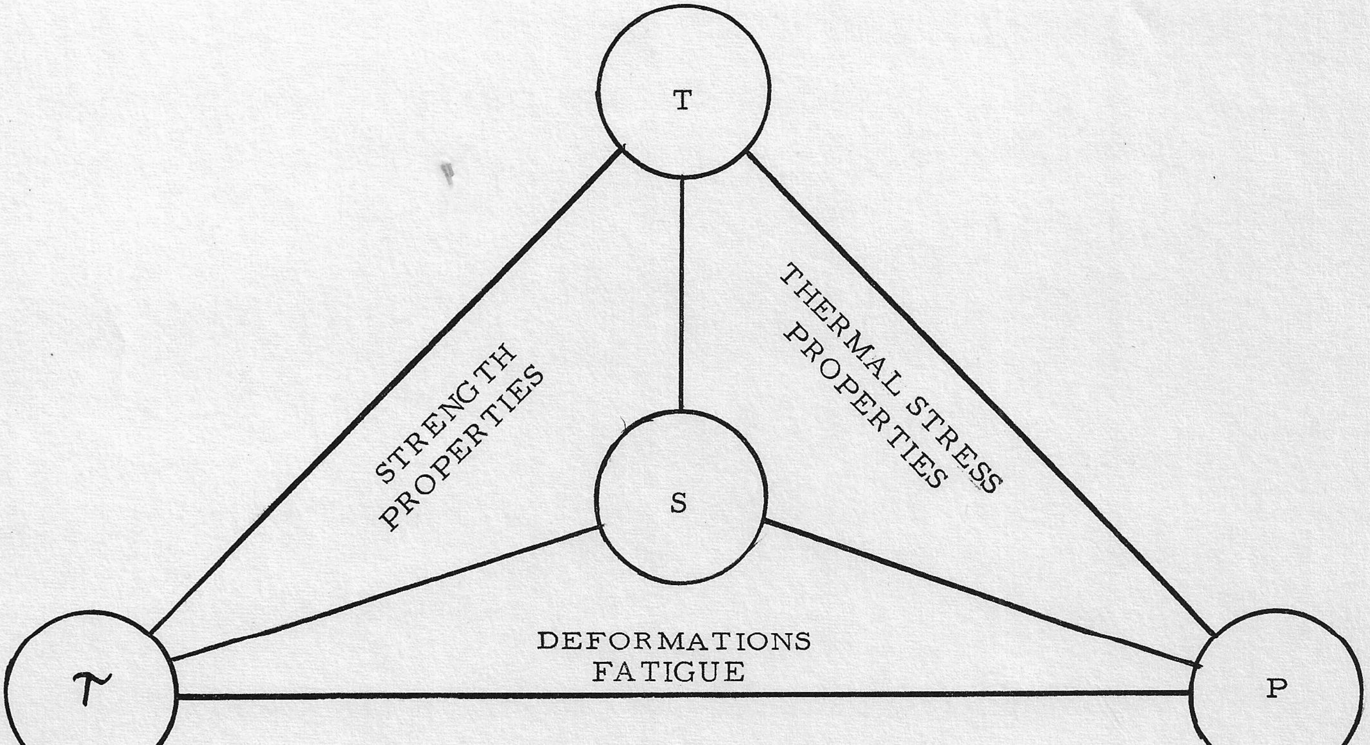


FIGURE 2 - EQUIVALENT RADIATIVE HEAT TRANSFER COEFFICIENT BETWEEN PARALLEL PLANES



a) ROOM TEMPERATURE

WHERE: S = STRUCTURE T = TEMPERATURE τ = TIME P = LOADING



b) ELEVATED TEMPERATURES

FIGURE 3 - INTERACTIONS ON THE STRUCTURE

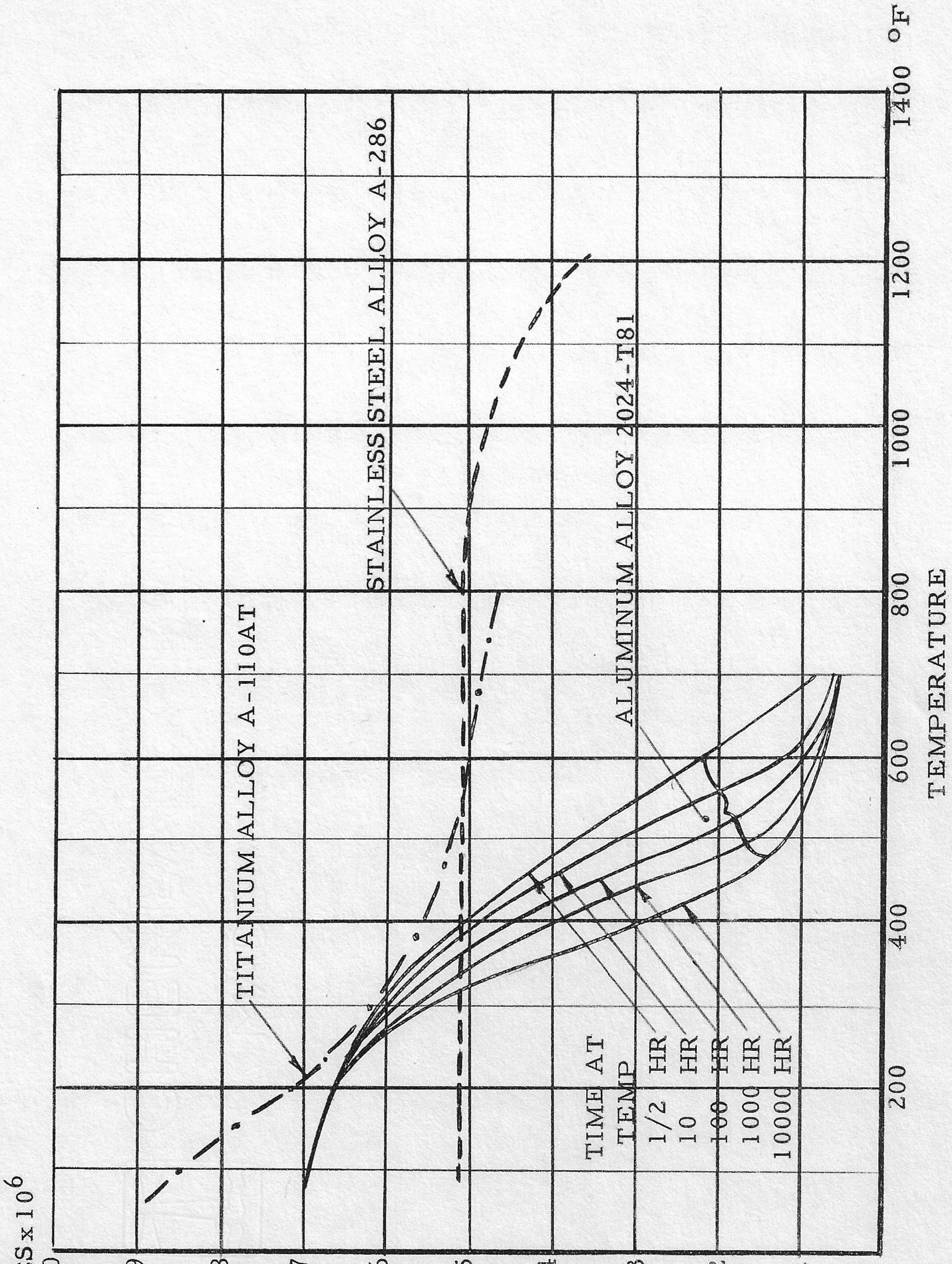


FIGURE 4 - VARIATION OF SPECIFIC ULTIMATE STRESS WITH TEMPERATURE

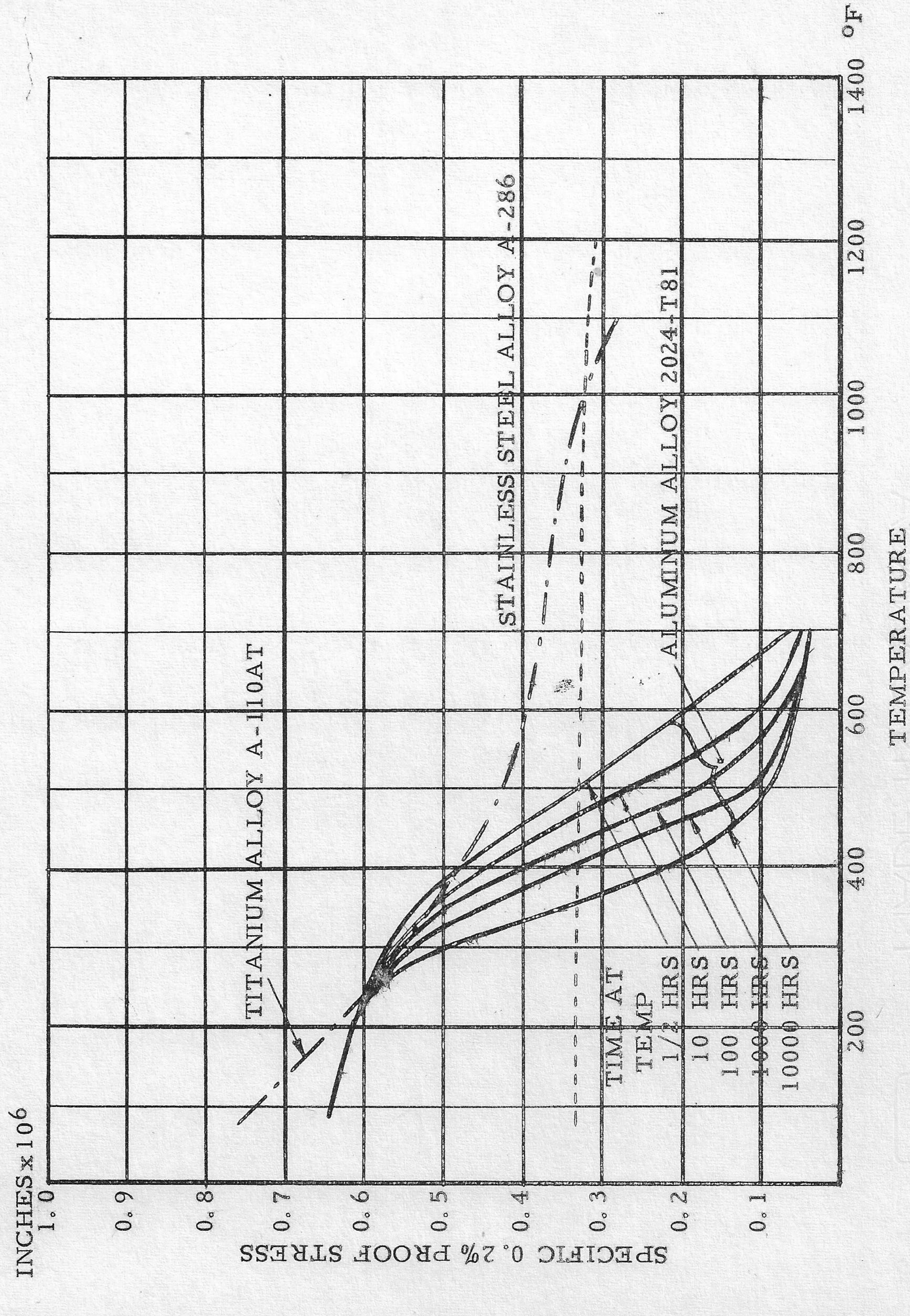


FIGURE 5 - VARIATION OF SPECIFIC 0.2% PROOF STRESS WITH TEMPERATURE

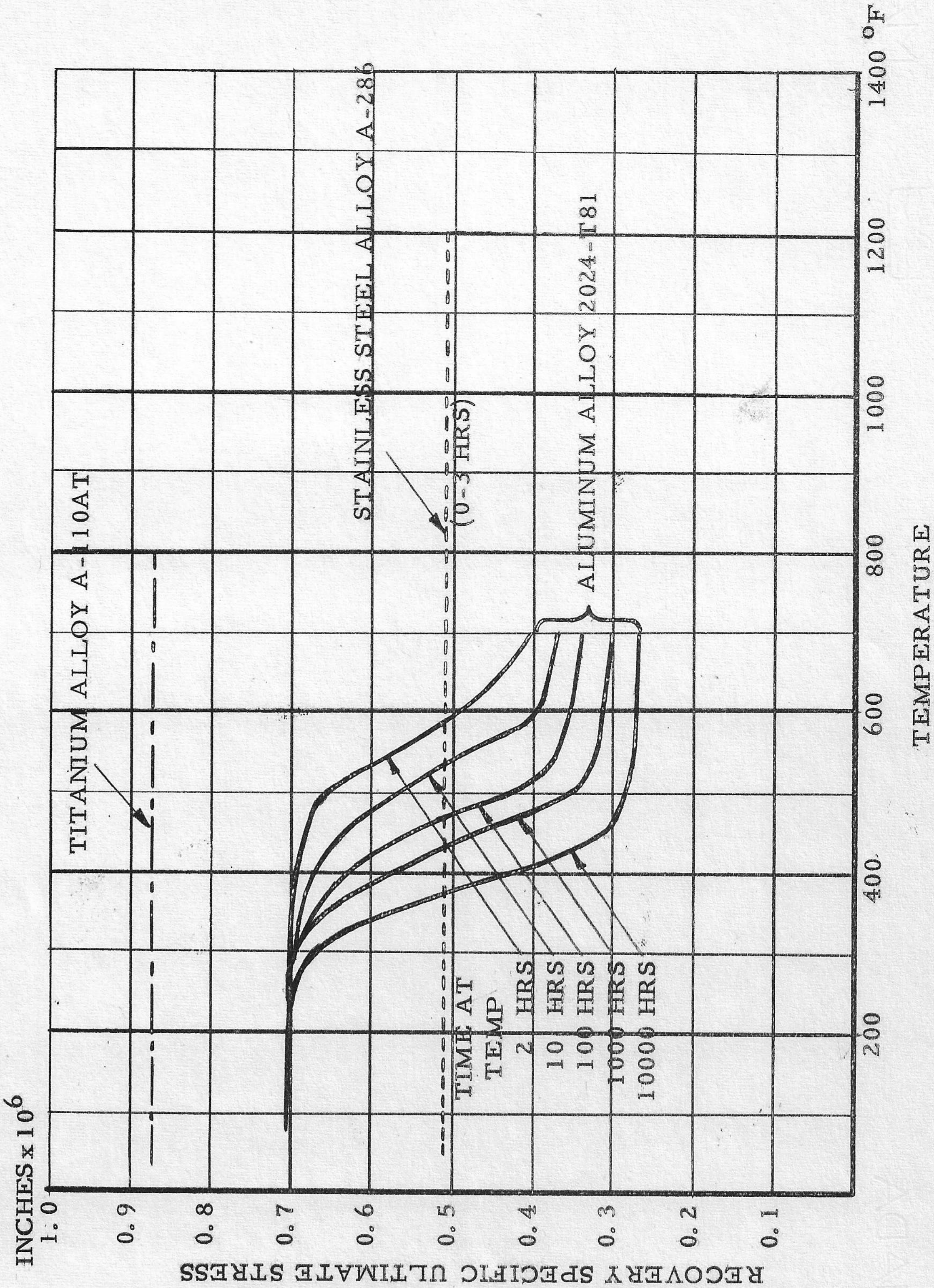


FIGURE 6 - VARIATION OF RECOVERY SPECIFIC ULTIMATE STRESS WITH TEMPERATURE

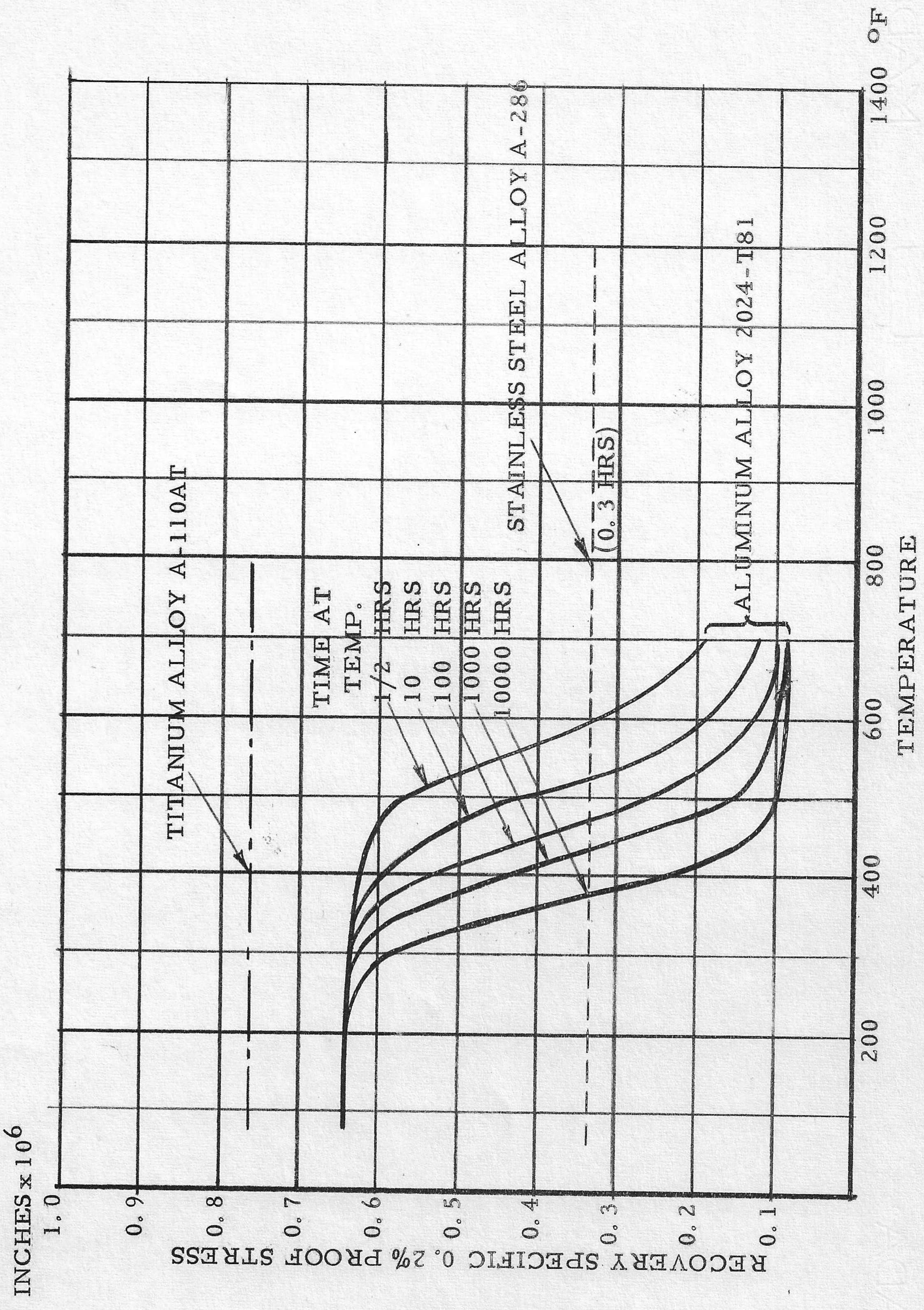


FIGURE 7 - VARIATION OF RECOVERY SPECIFIC 0.2% PROOF STRESS WITH TEMPERATURE

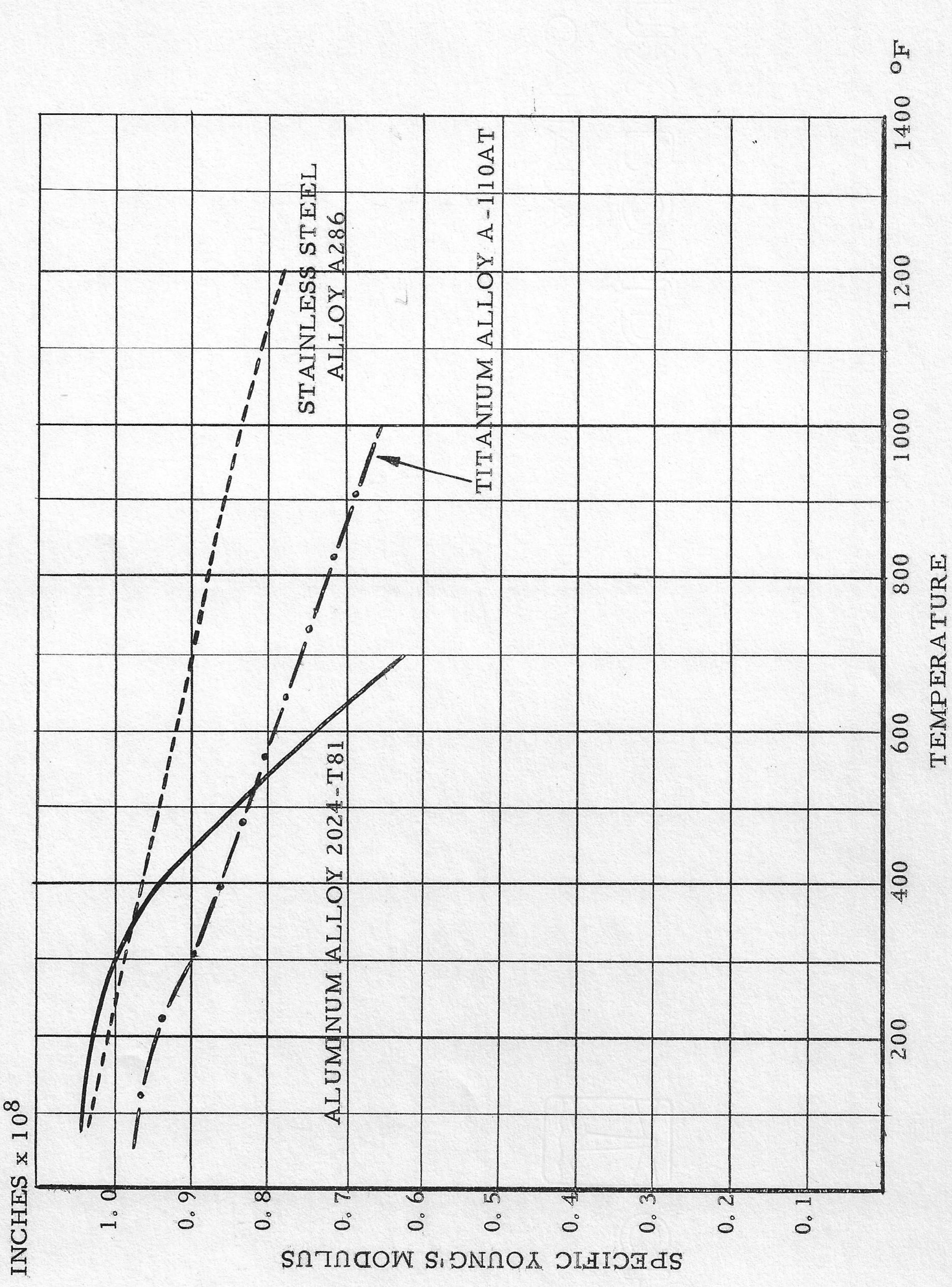


FIGURE 8 - VARIATION OF SPECIFIC YOUNG'S MODULUS WITH TEMPERATURE

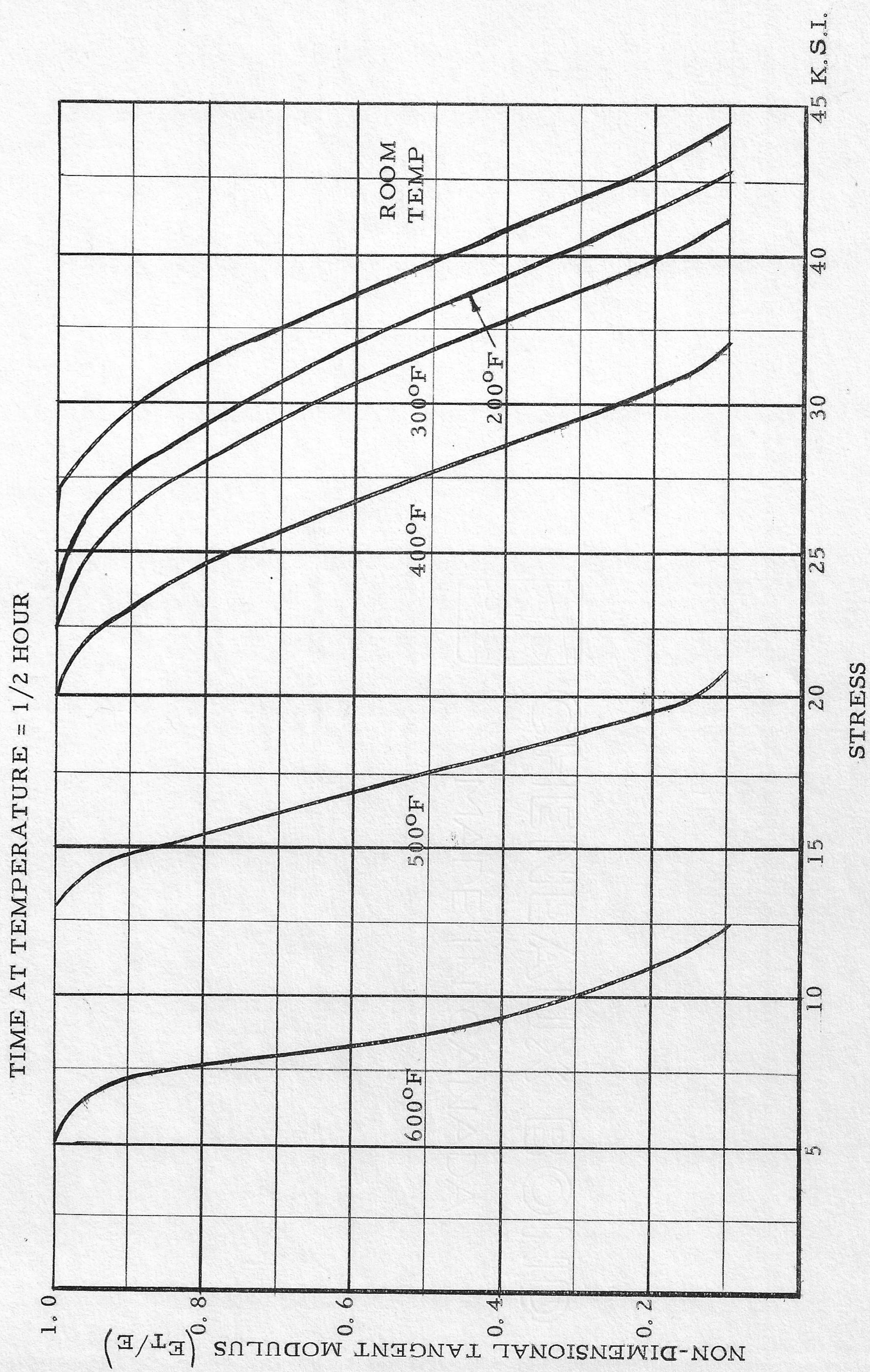


FIGURE 9 - VARIATION OF NON-DIMENSIONAL TANGENT MODULUS WITH TEMPERATURE AND STRESS FOR ALUMINUM ALLOY 2024-T81

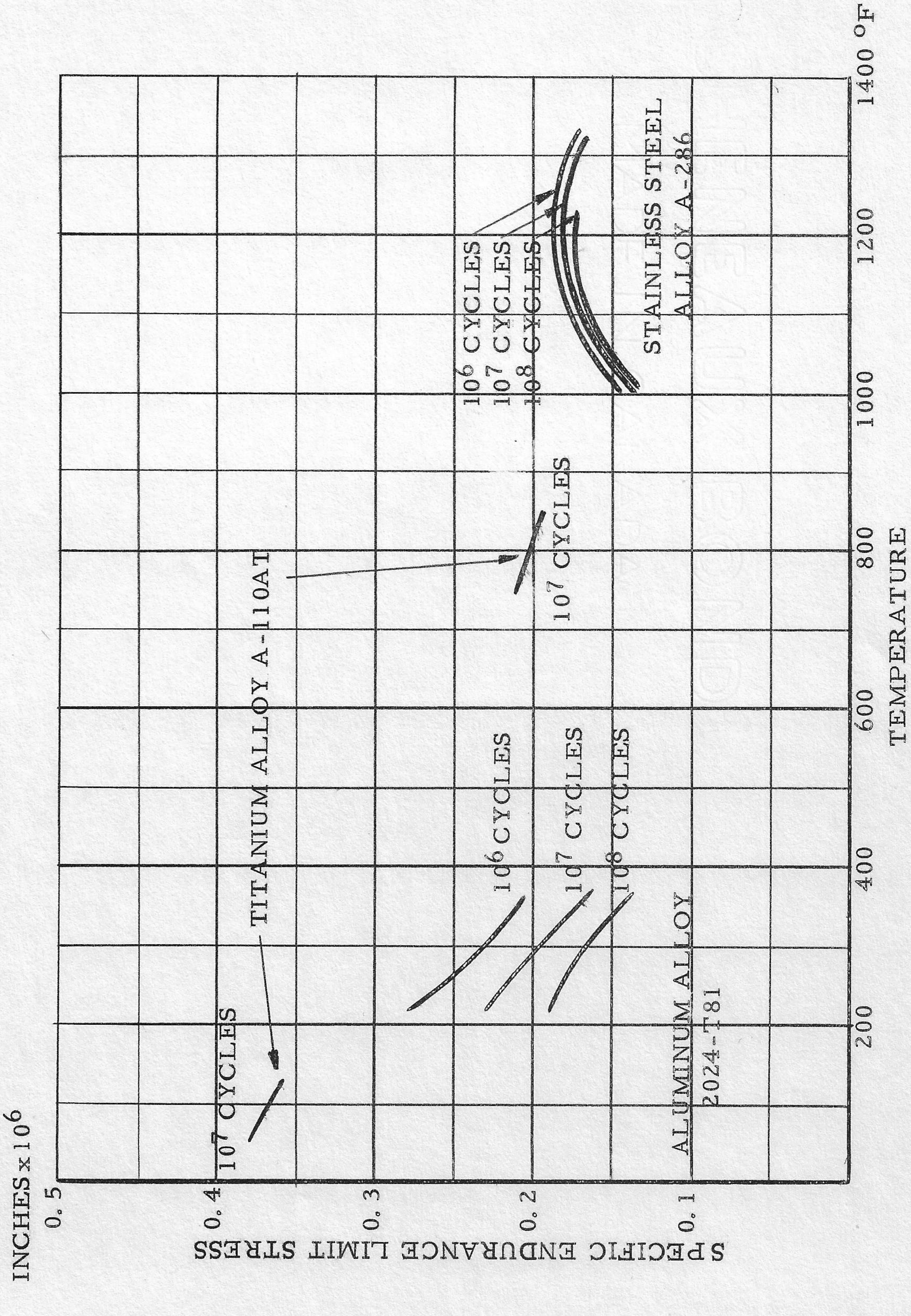


FIGURE 10 - EFFECT OF TEMPERATURE ON THE SPECIFIC ENDURANCE LIMIT STRESS

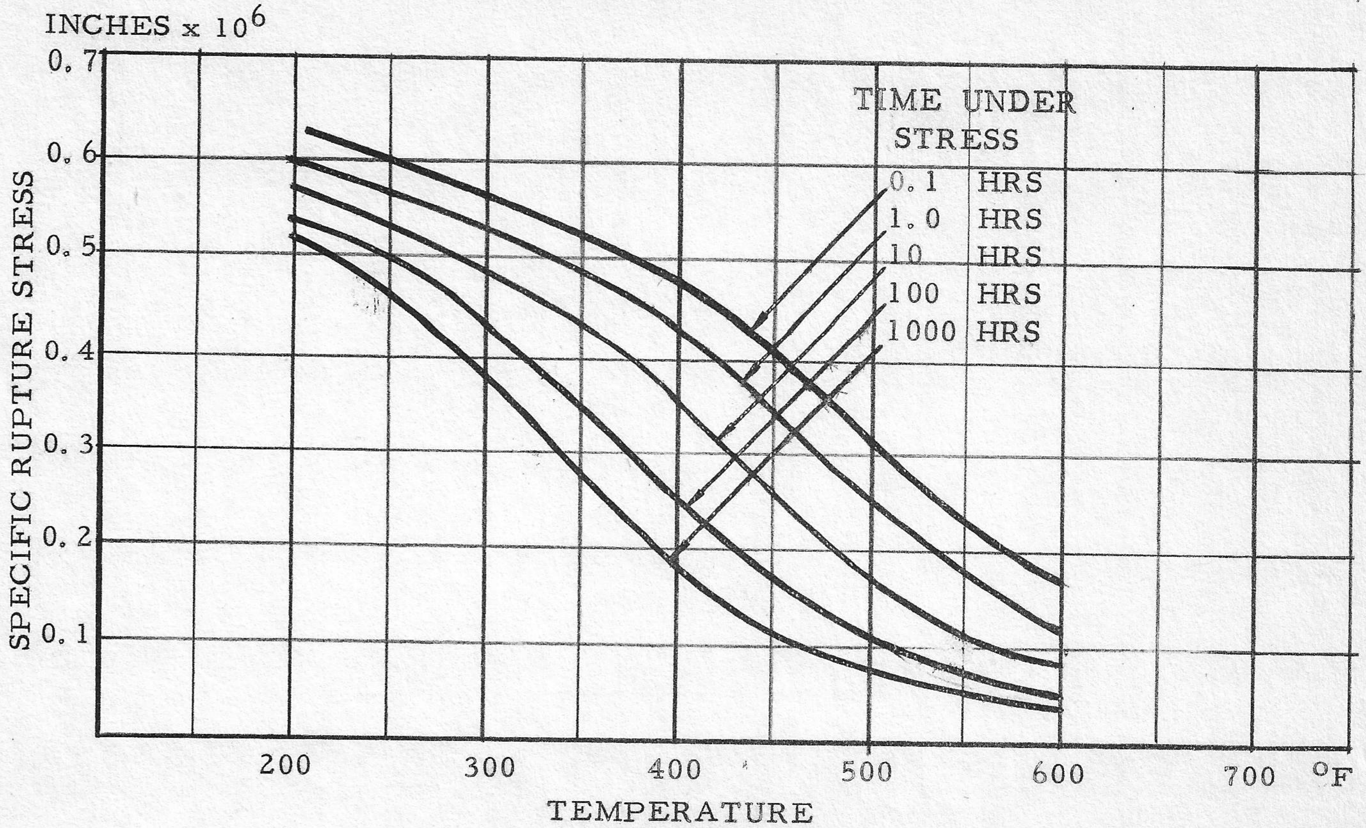


FIGURE 11 - VARIATION OF SPECIFIC RUPTURE STRESS WITH TEMPERATURE OF ALUMINUM ALLOY 2024-TB1

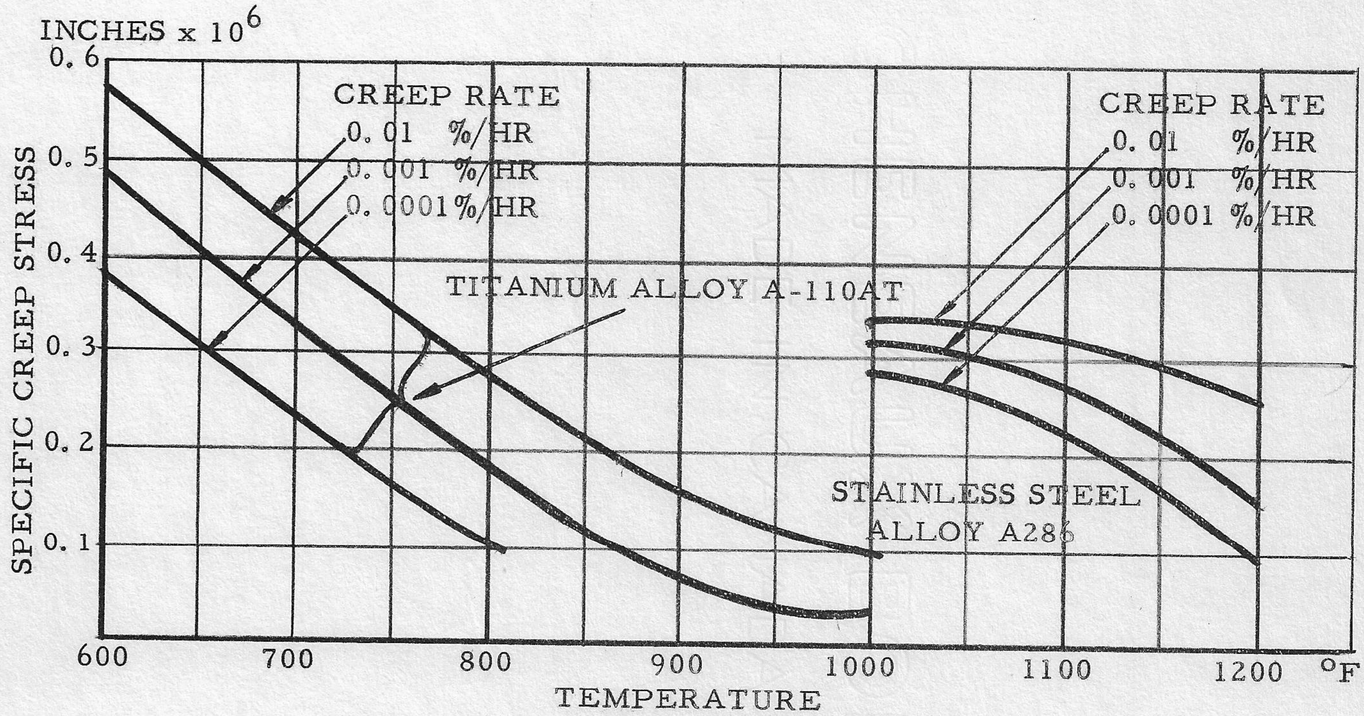


FIGURE 12 - VARIATION OF SPECIFIC CREEP STRESS WITH TEMPERATURE FOR VARIOUS CREEP RATES

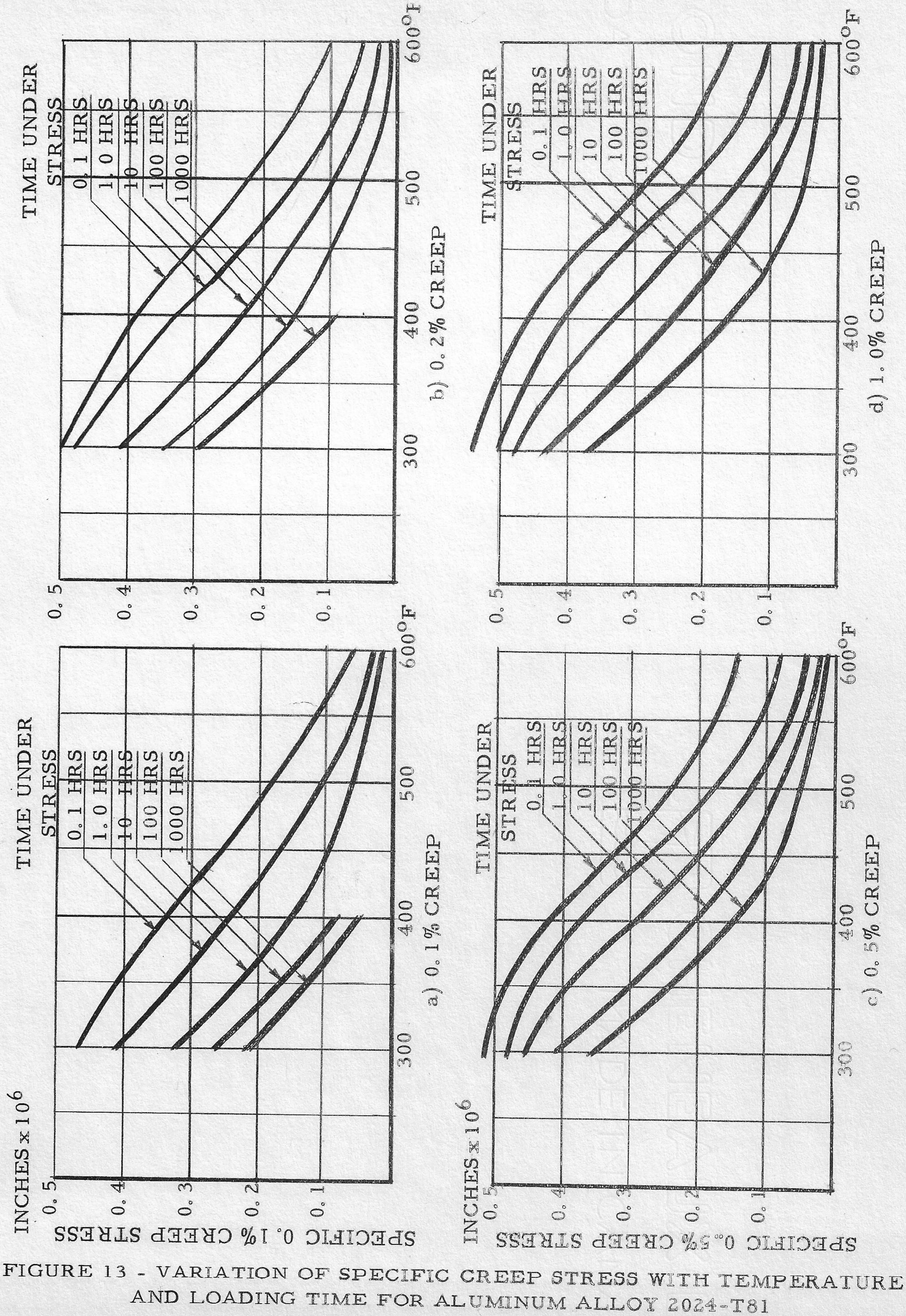


FIGURE 13 - VARIATION OF SPECIFIC CREEP STRESS WITH TEMPERATURE AND LOADING TIME FOR ALUMINUM ALLOY 2024-T81

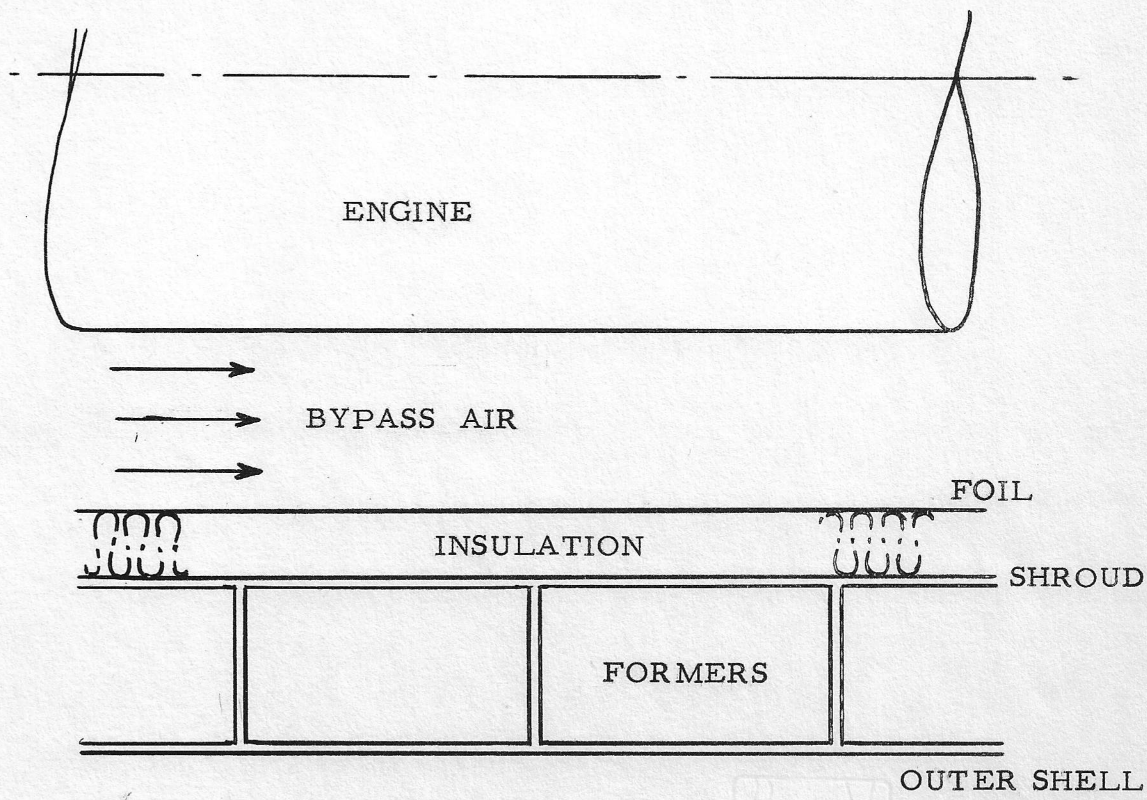


FIGURE 14 - TYPICAL ENGINE BAY STRUCTURE

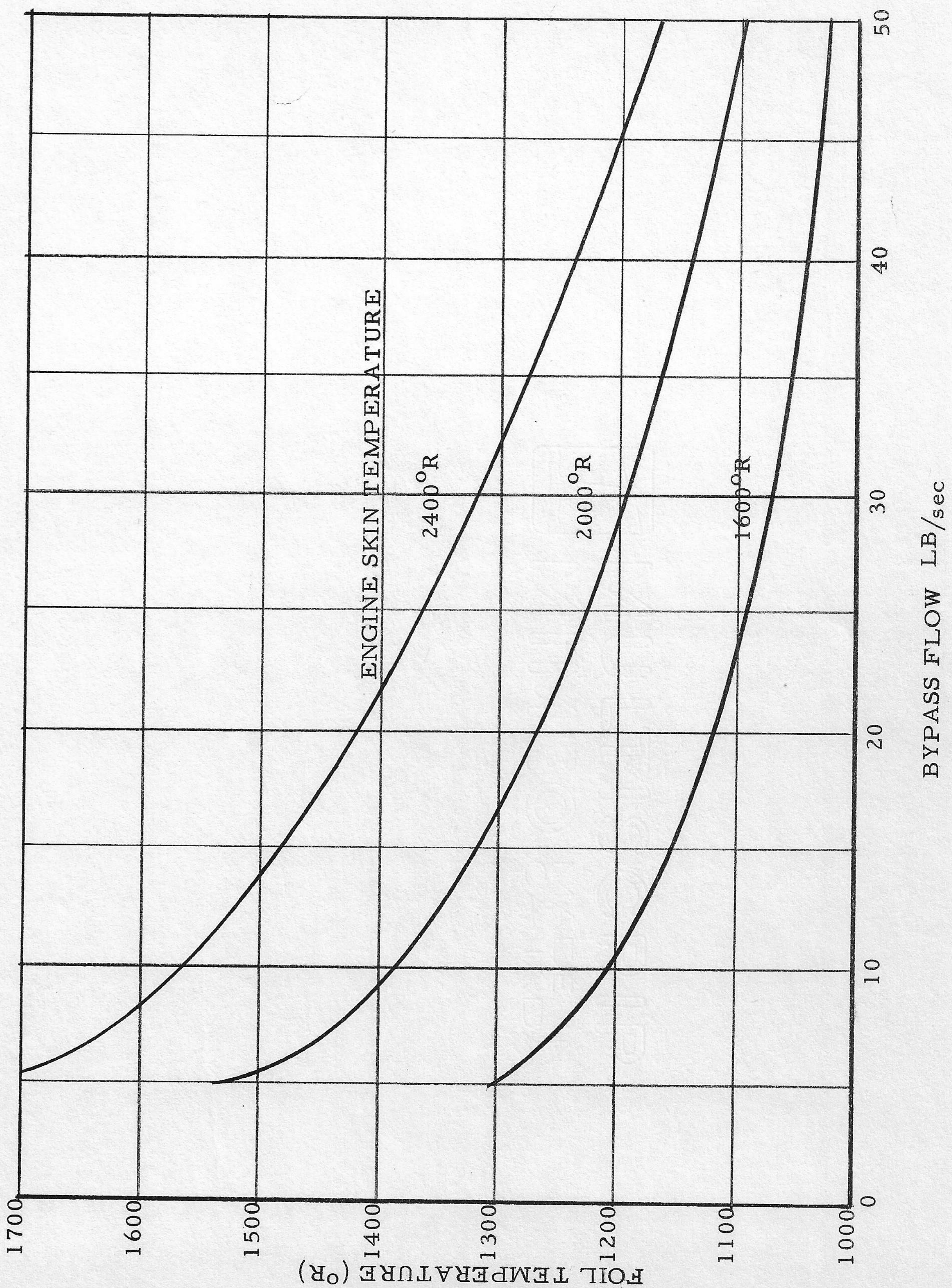


FIGURE 15 - TYPICAL VARIATION OF FOIL TEMPERATURE WITH ENGINE SKIN TEMPERATURE AND BYPASS FLOW

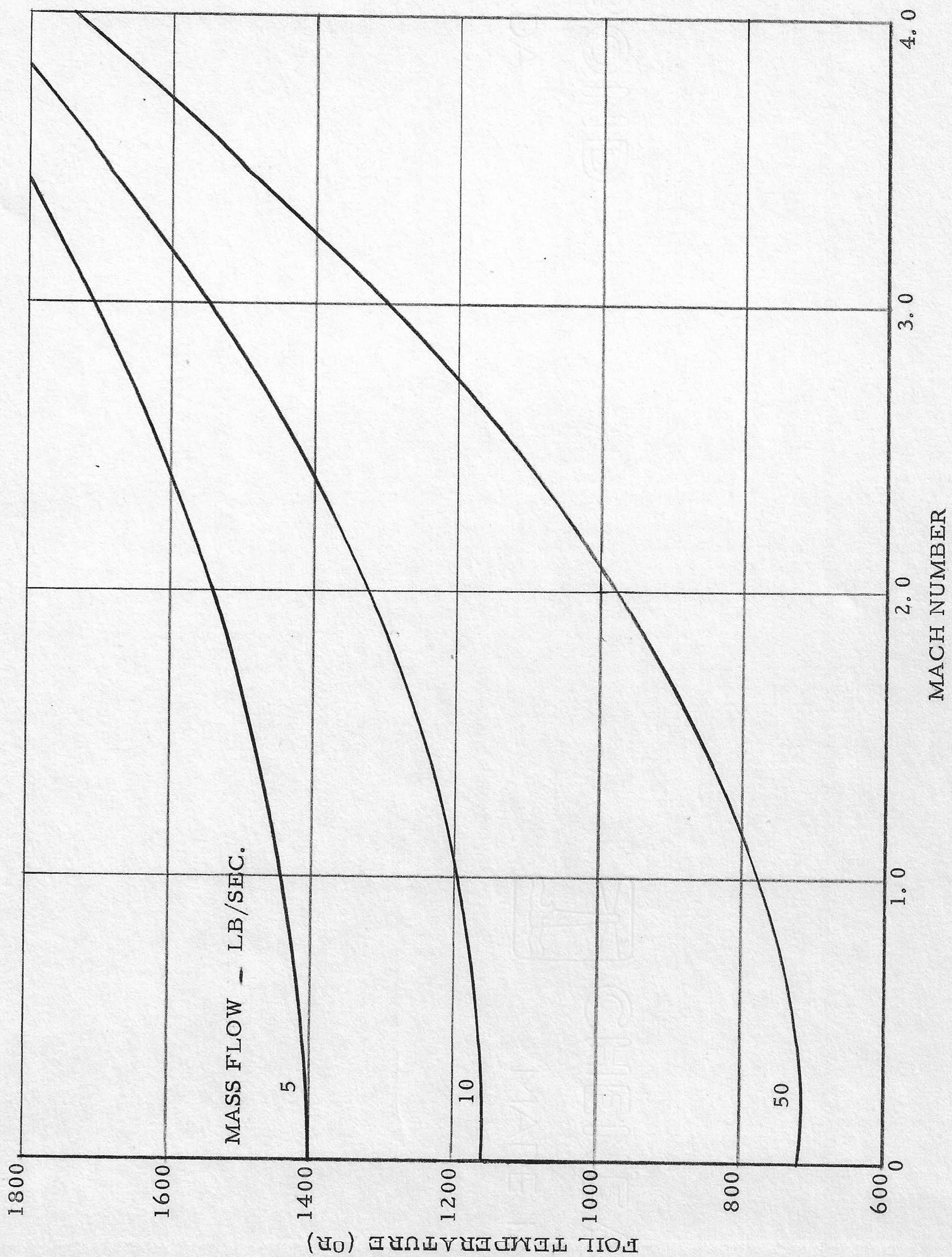
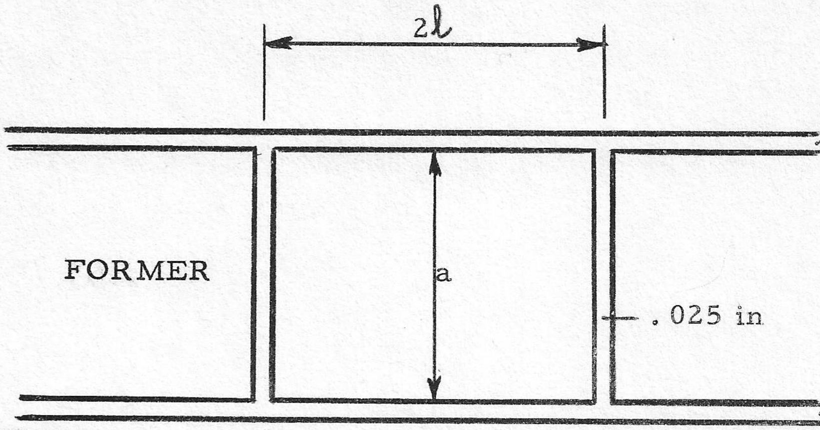


FIGURE 16 - TYPICAL VARIATION OF FOIL TEMPERATURE WITH BYPASS FLOW AND MACH NUMBER



MATERIALS

STRUCTURE FOR	SHROUD	FORMER	OUTER SHELL
M = 2	TITANIUM	ALUMINUM	ALUMINUM
M = 3	TITANIUM	TITANIUM	TITANIUM
M = 4	STEEL	STEEL	STEEL

FIGURE 17 - TYPICAL ENGINE BAY STRUCTURE

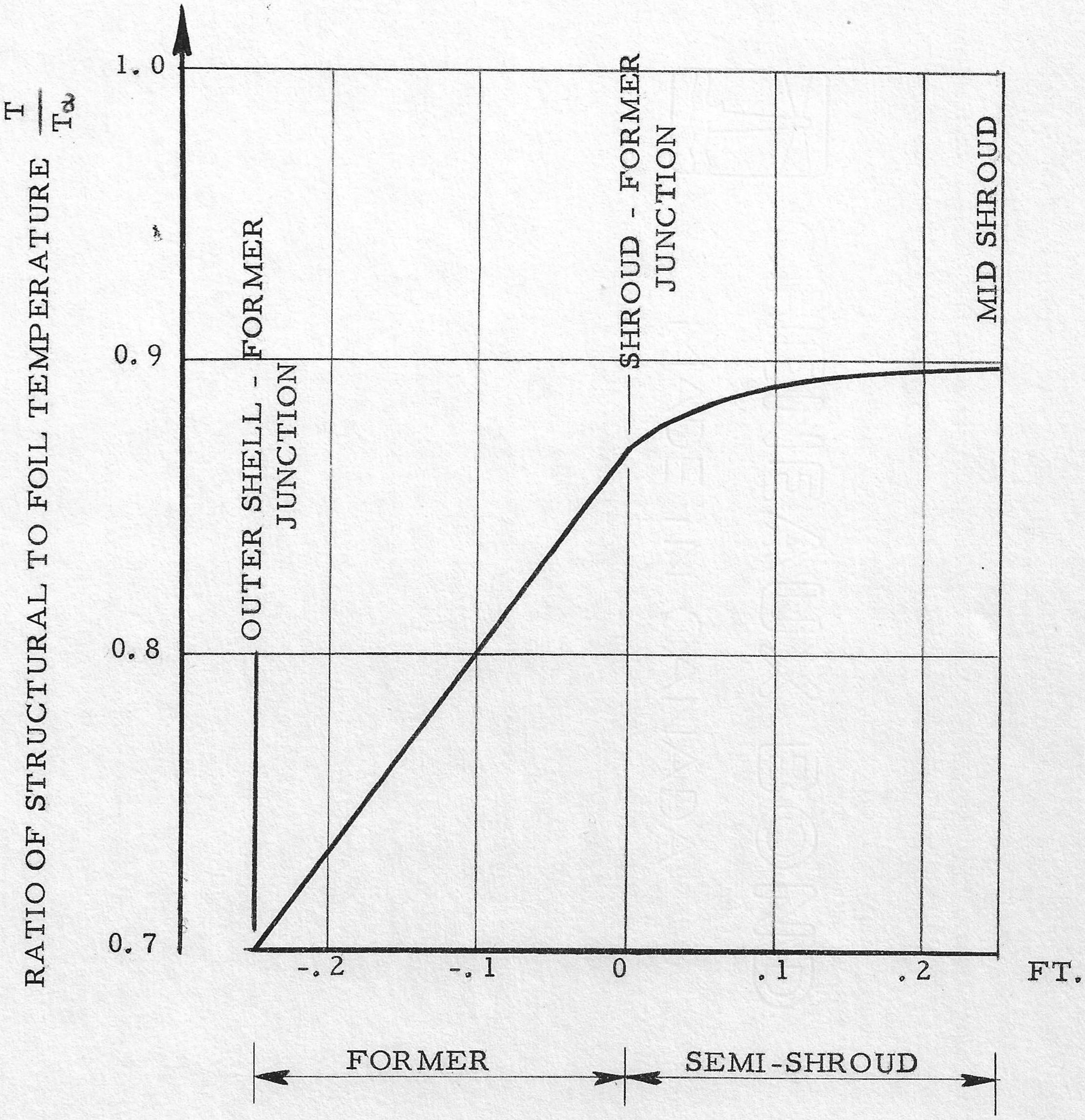


FIGURE 18 - STEADY STATE TEMPERATURE DISTRIBUTION FOR TYPICAL MACH 3 STRUCTURE

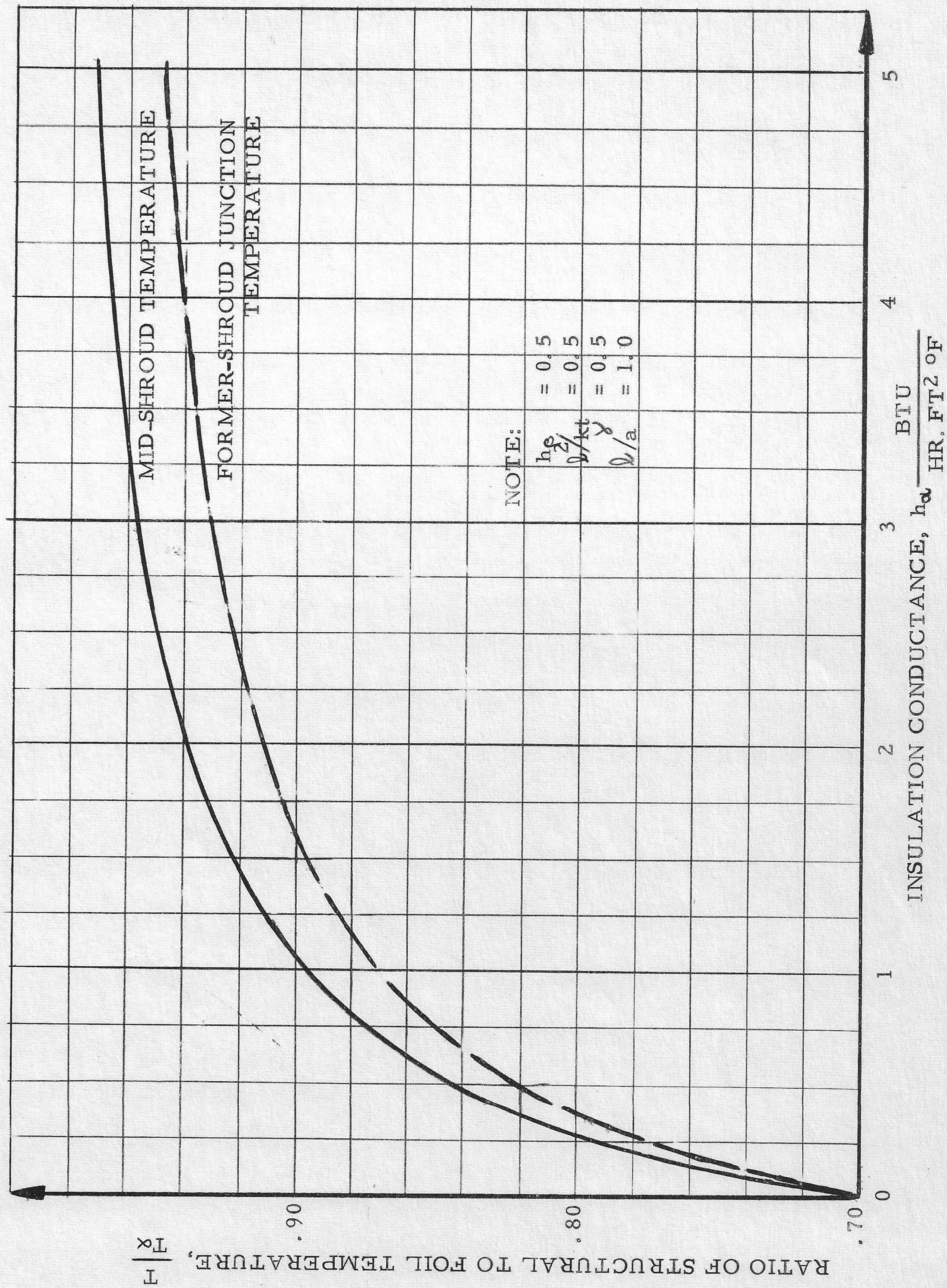


FIGURE 19 - VARIATION OF STEADY STATE STRUCTURAL TEMPERATURE WITH INSULATION CONDUCTANCE, h_a

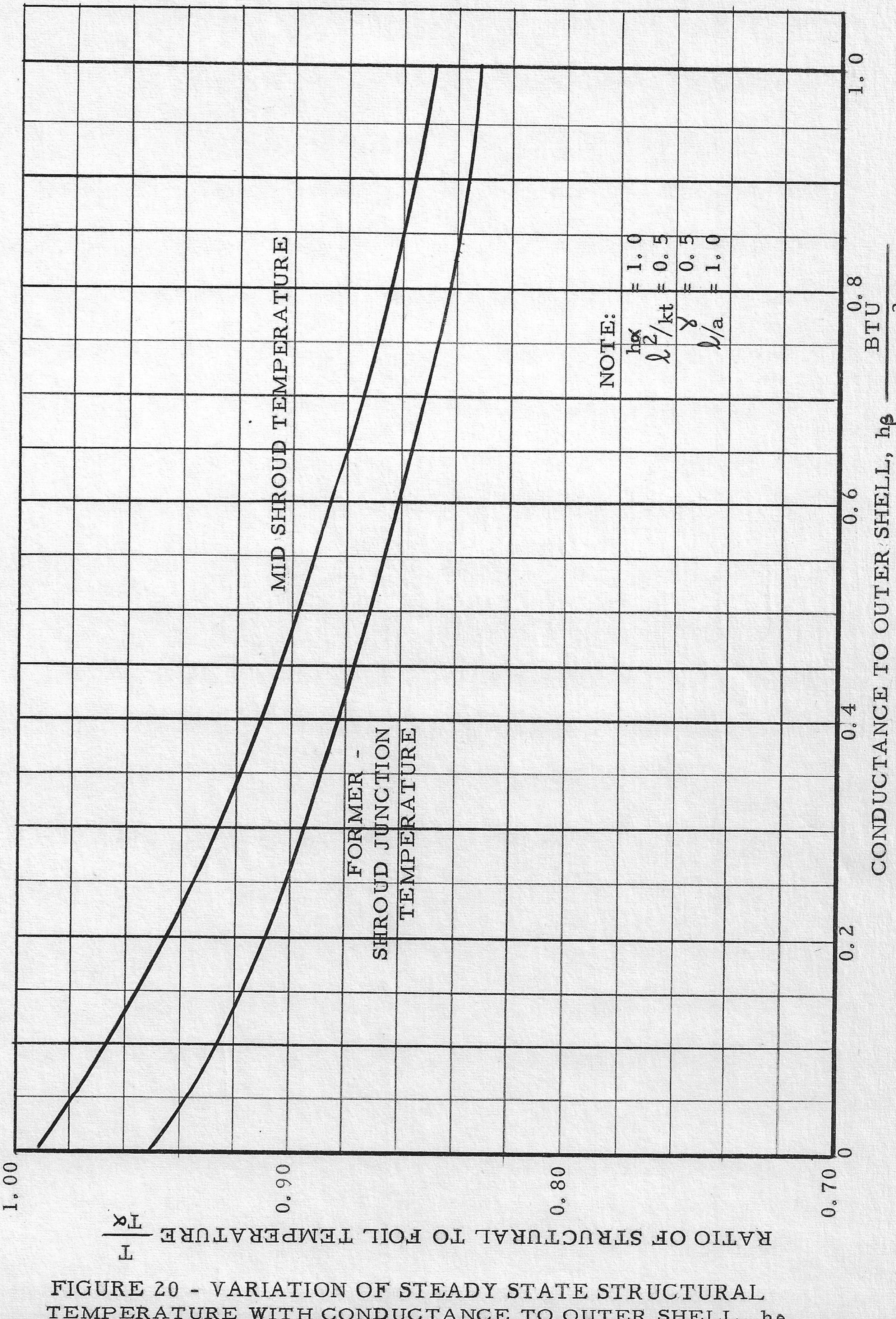


FIGURE 20 - VARIATION OF STEADY STATE STRUCTURAL TEMPERATURE WITH CONDUCTANCE TO OUTER SHELL, h_β

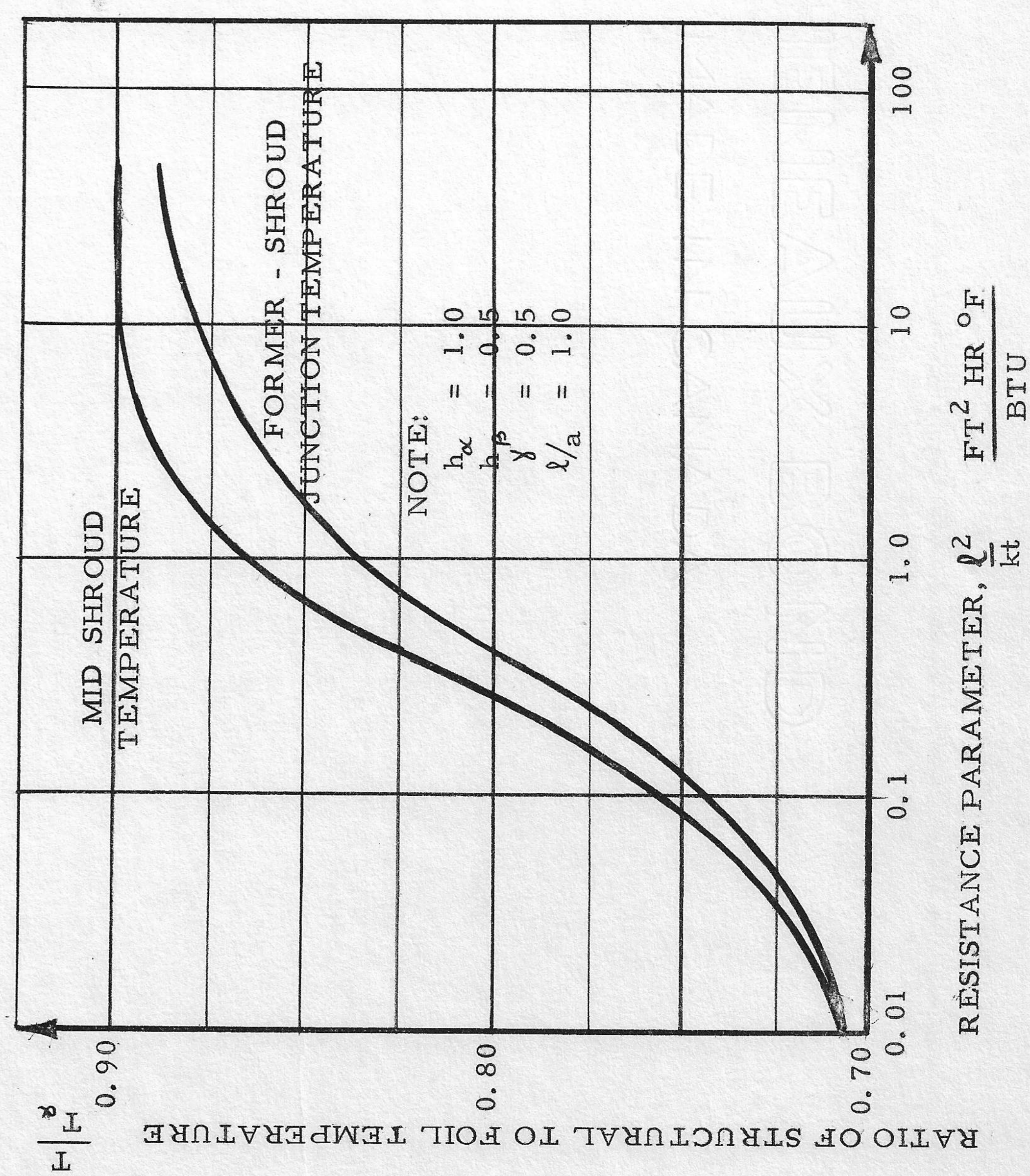


FIGURE 21 - VARIATION OF STEADY STATE STRUCTURAL TEMPERATURE WITH RESISTANCE PARAMETER, $\frac{\ell^2}{kt}$.

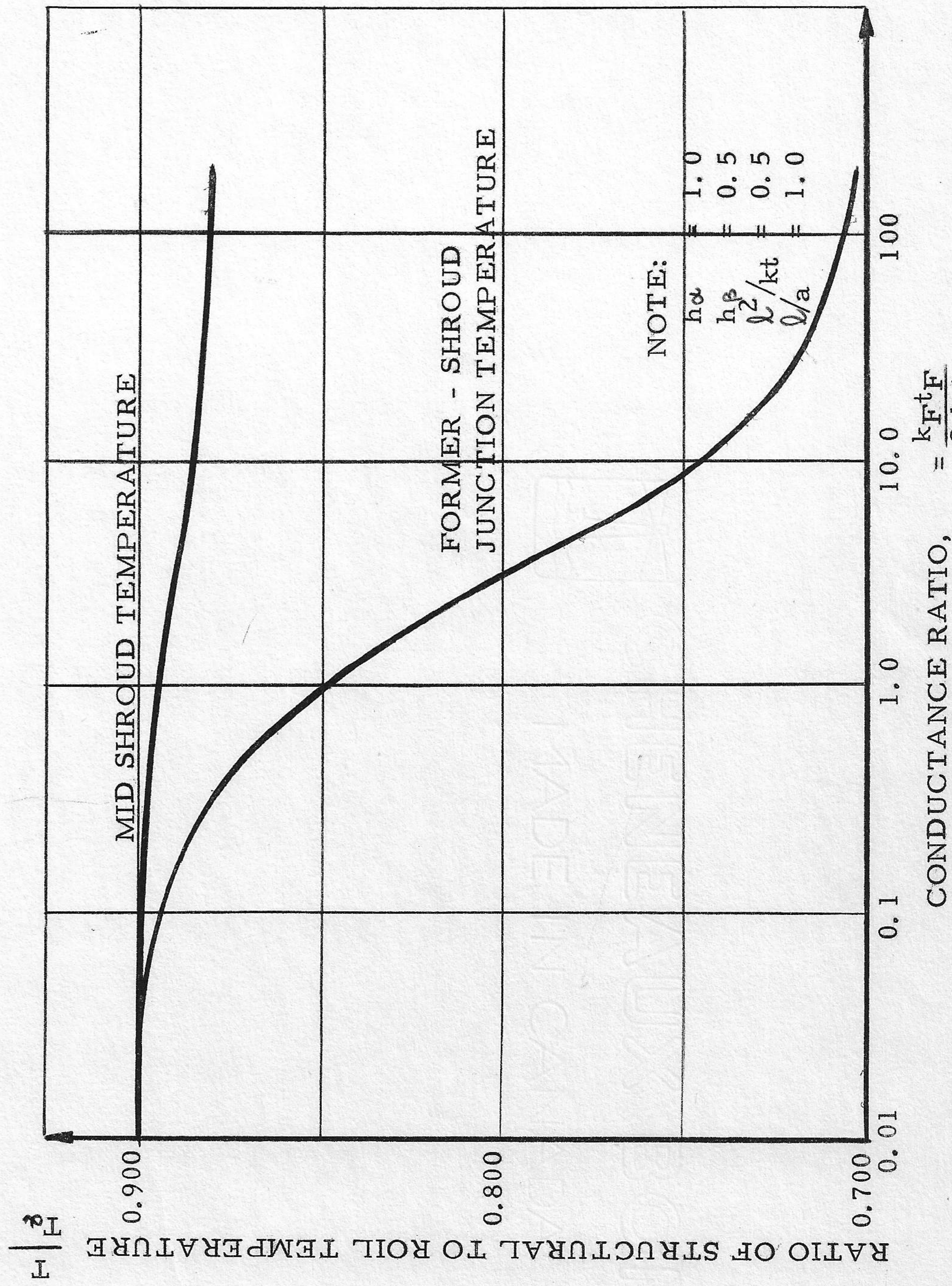


FIGURE 22 - VARIATION OF STEADY STATE STRUCTURAL TEMPERATURE WITH CONDUCTANCE RATIO, $= \frac{k_F t_F}{2kt}$

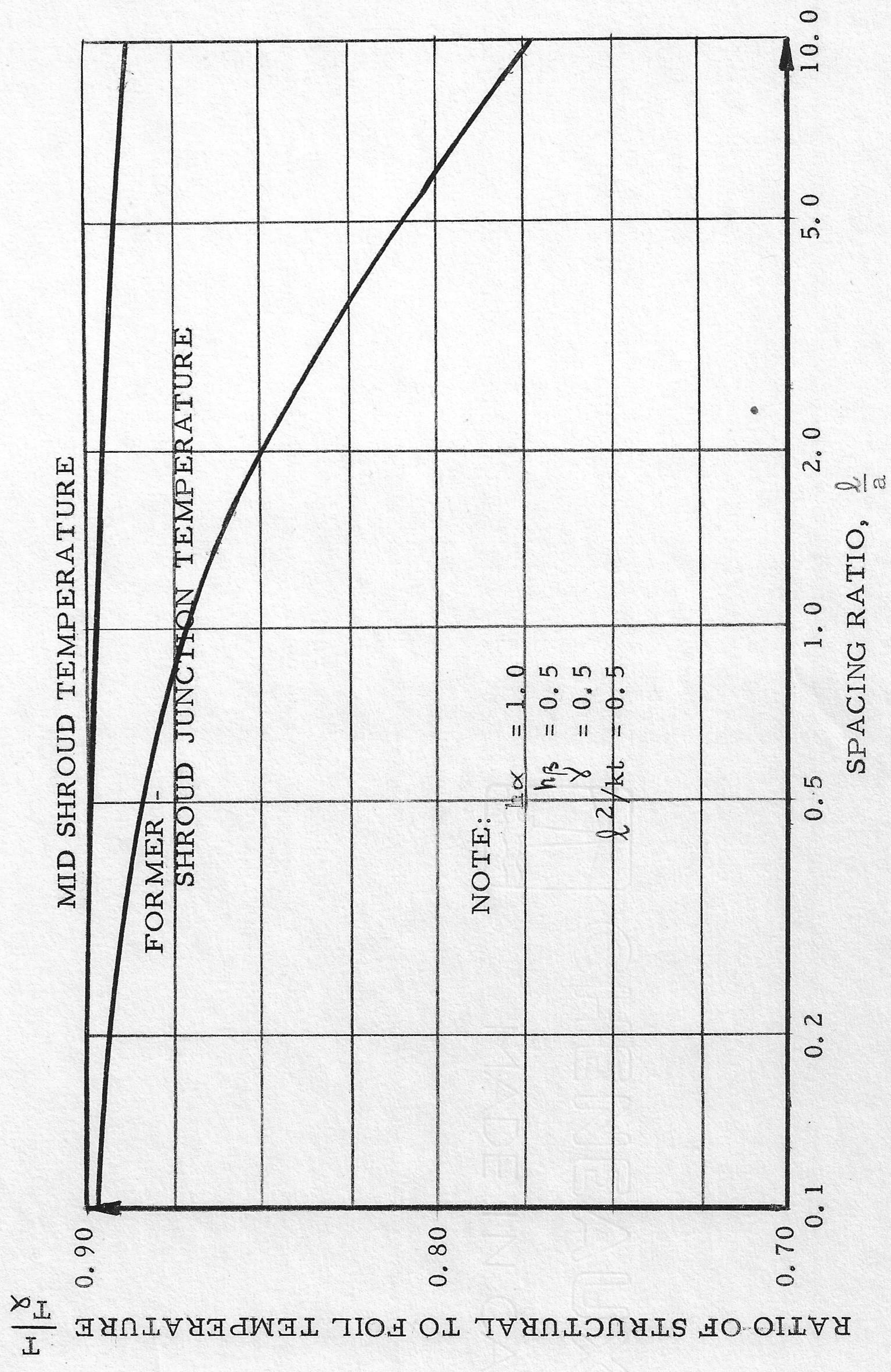


FIGURE 23 - VARIATION OF STEADY STATE STRUCTURAL TEMPERATURE WITH SPACING RATIO, $\frac{\ell}{a}$

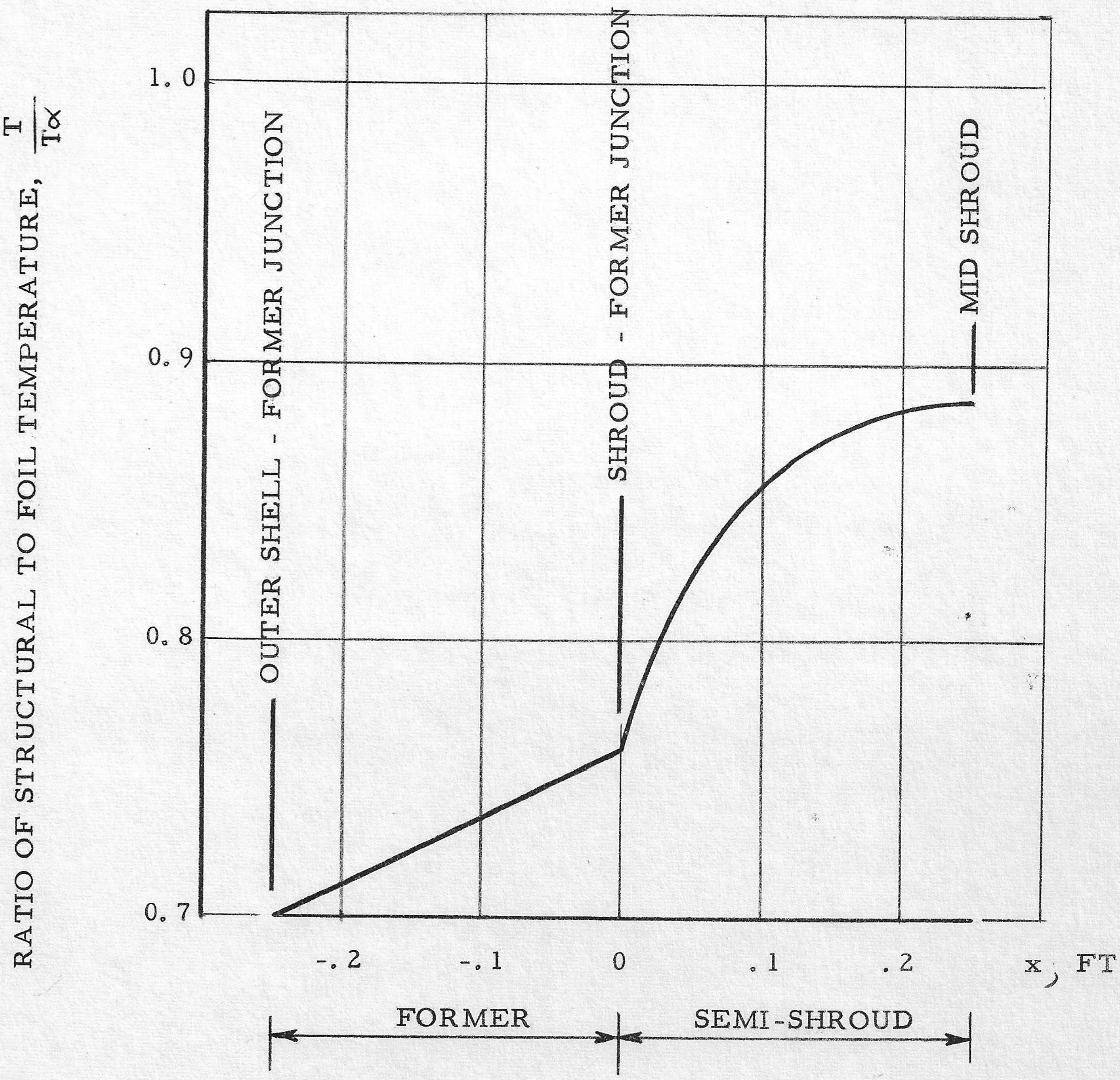


FIGURE 24 - STEADY STATE TEMPERATURE DISTRIBUTION FOR TYPICAL MACH 2 STRUCTURE

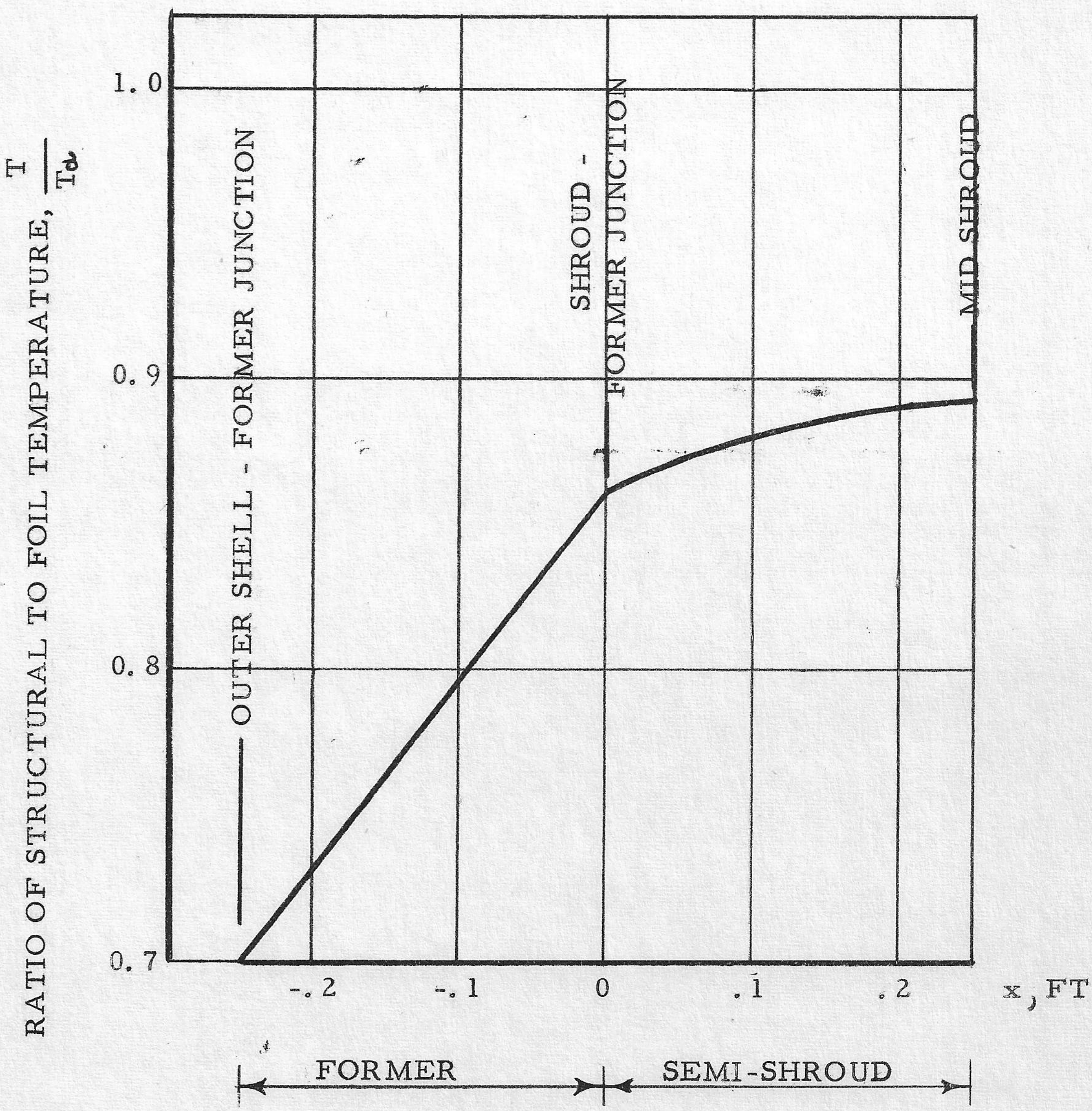


FIGURE 25 - STEADY STATE TEMPERATURE DISTRIBUTION FOR TYPICAL MACH 4 STRUCTURE

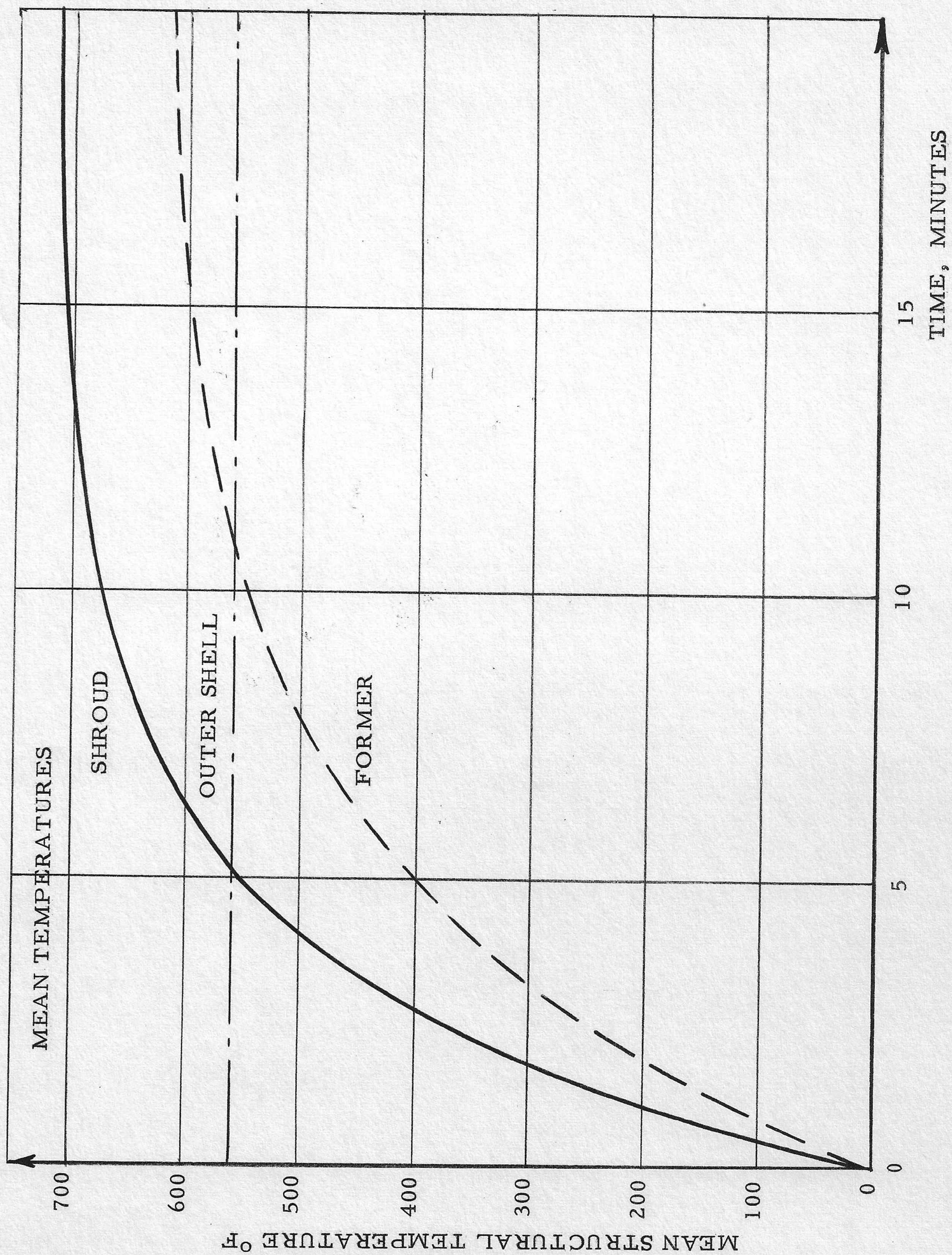


FIGURE 26 - APPROXIMATE TRANSIENT HEATING
OF TYPICAL MACH 3 STRUCTURE

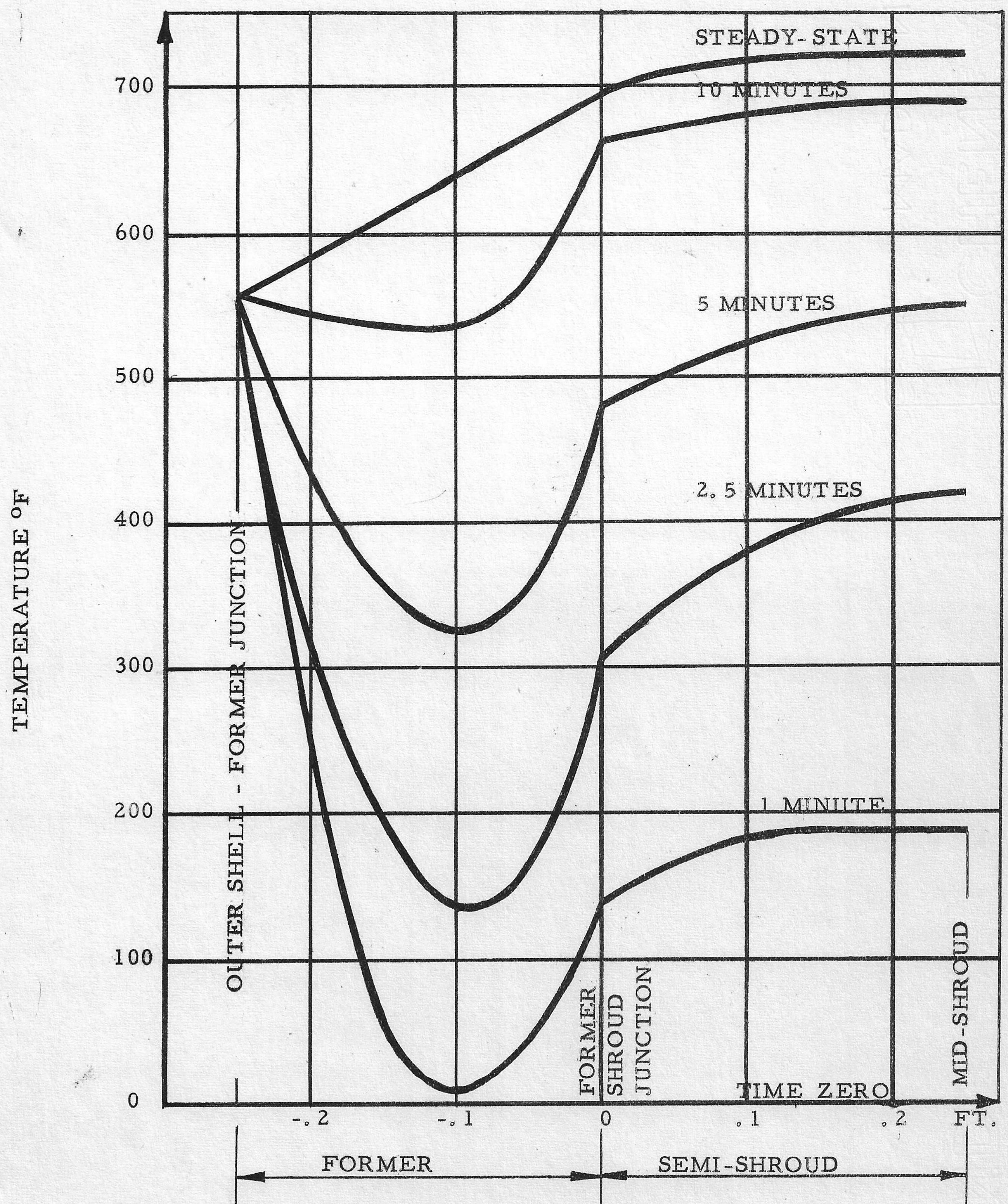


FIGURE 27 - TEMPERATURE DISTRIBUTION IN A TYPICAL MACH 3 STRUCTURE

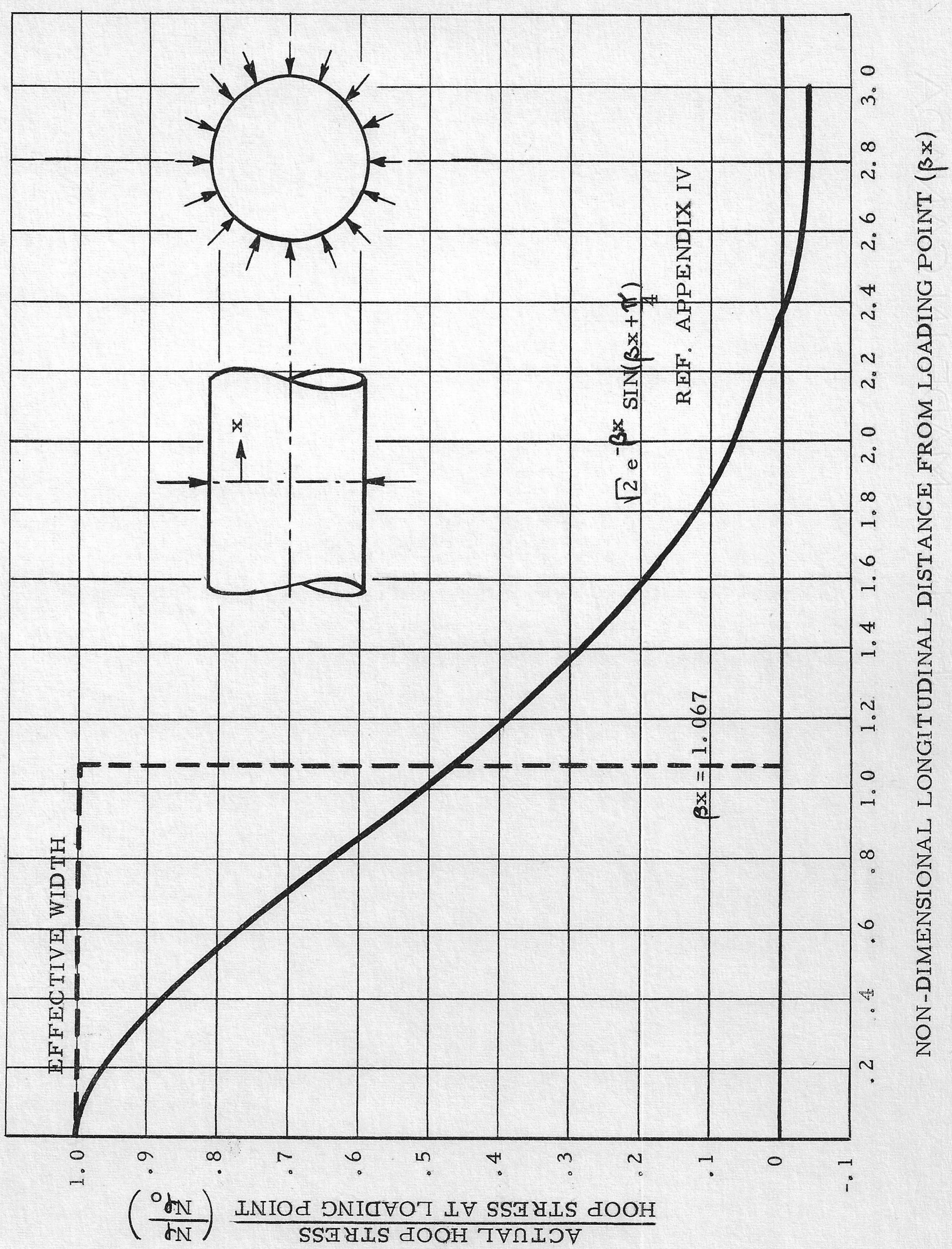


FIGURE 28 - SHELL HOOP STRESSES DIE-AWAY

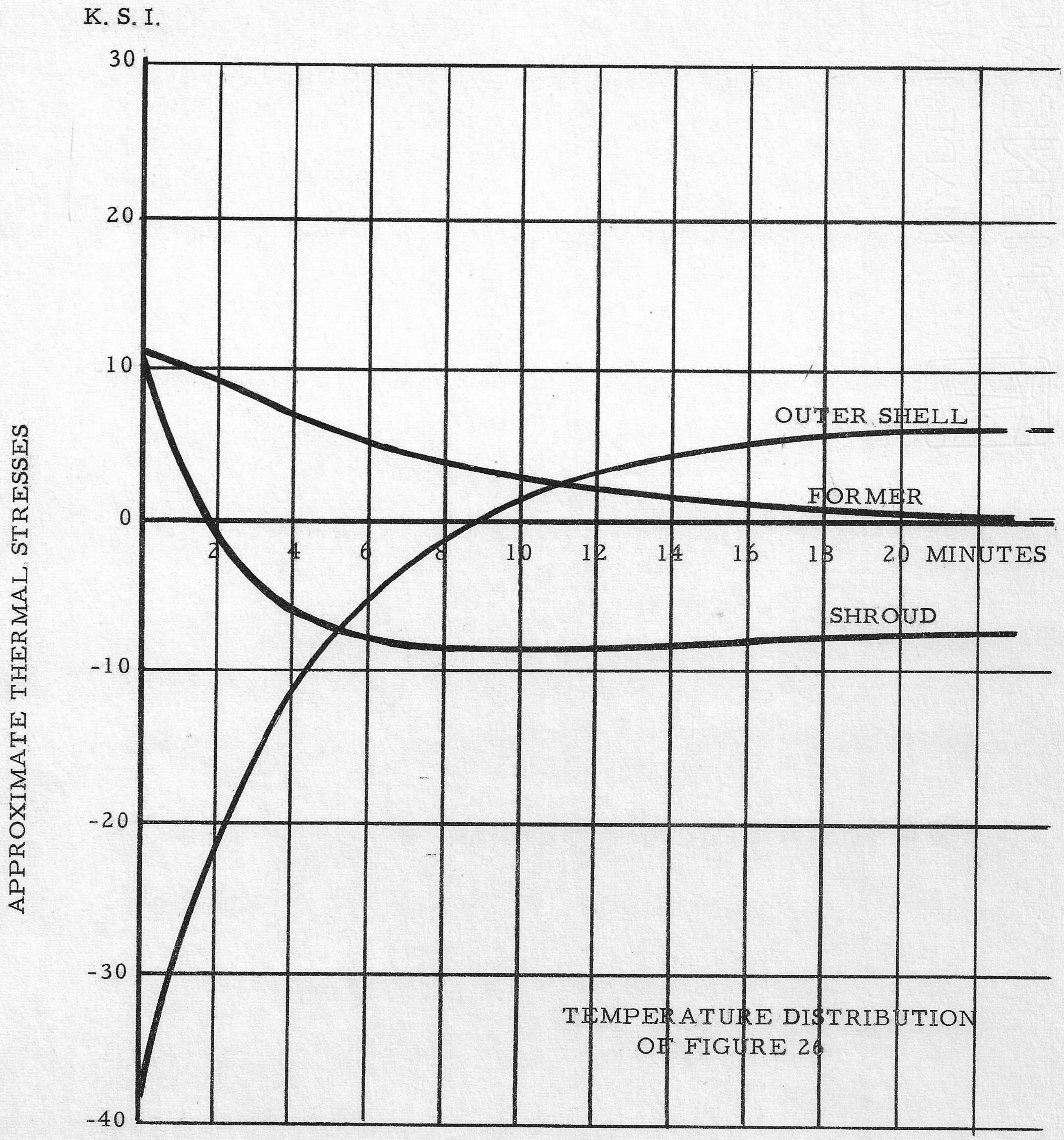


FIGURE 29 - APPROXIMATE THERMAL STRESS HISTORY FOR TYPICAL MACH 3 STRUCTURE, BASED ON APPROXIMATE TEMPERATURE DISTRIBUTION

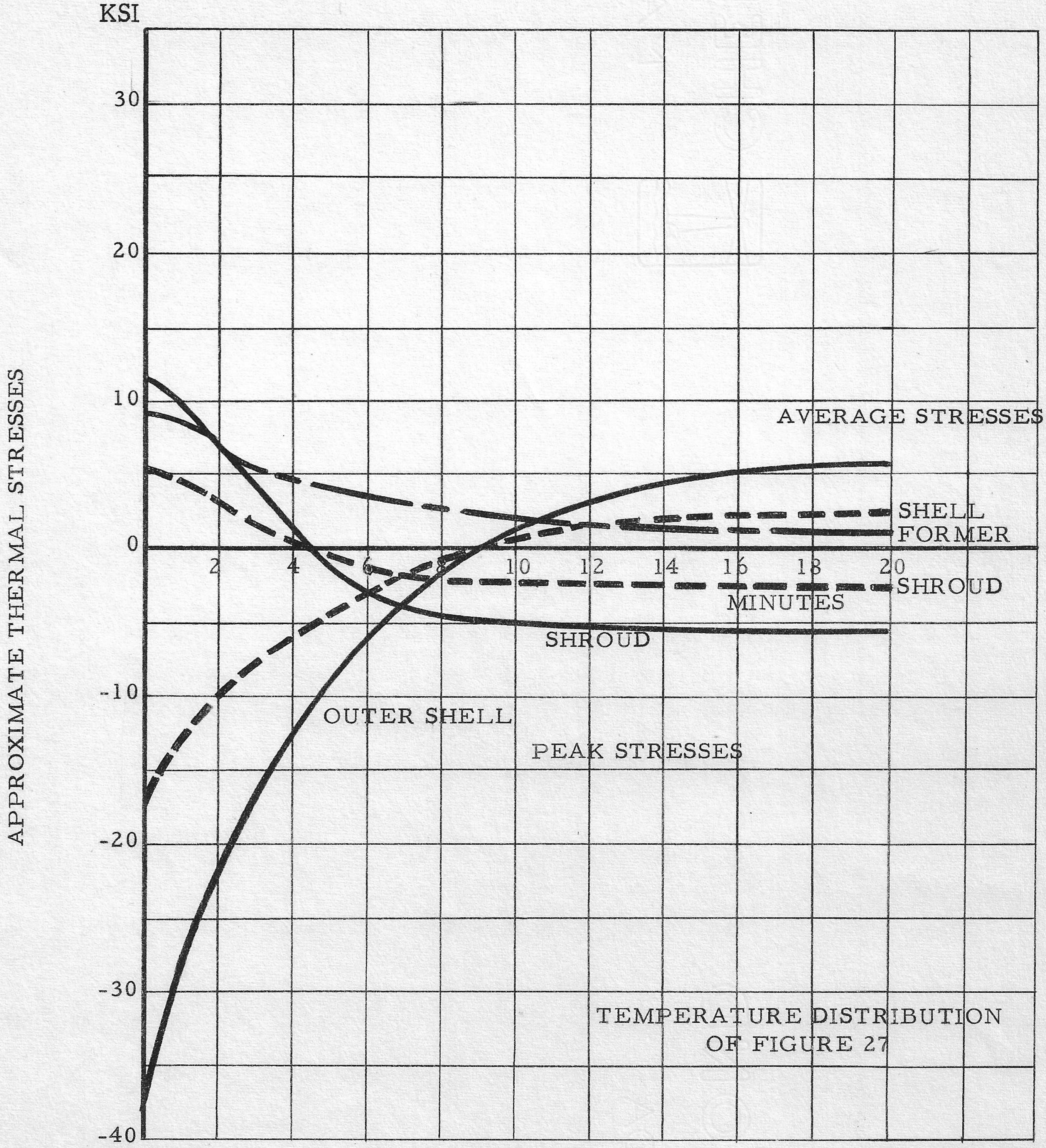


FIGURE 30 - APPROXIMATE THERMAL STRESS HISTORY FOR TYPICAL MACH 3 STRUCTURE, BASED ON EXACT TEMPERATURE DISTRIBUTION

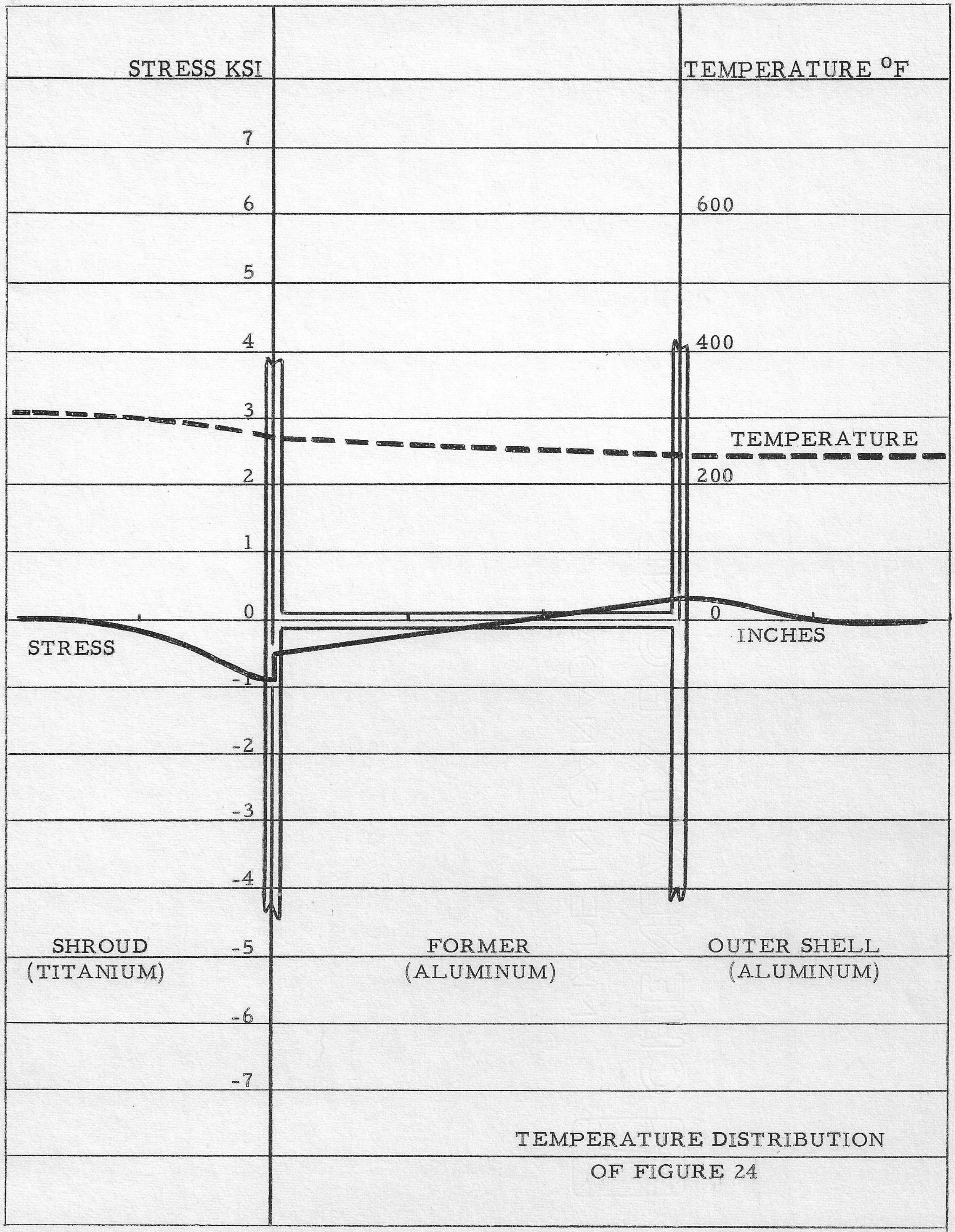


FIGURE 31 - ENGINEERS THERMAL STRESSES ON A TYPICAL M=2 ENGINE BAY STRUCTURE (STEADY STATE)

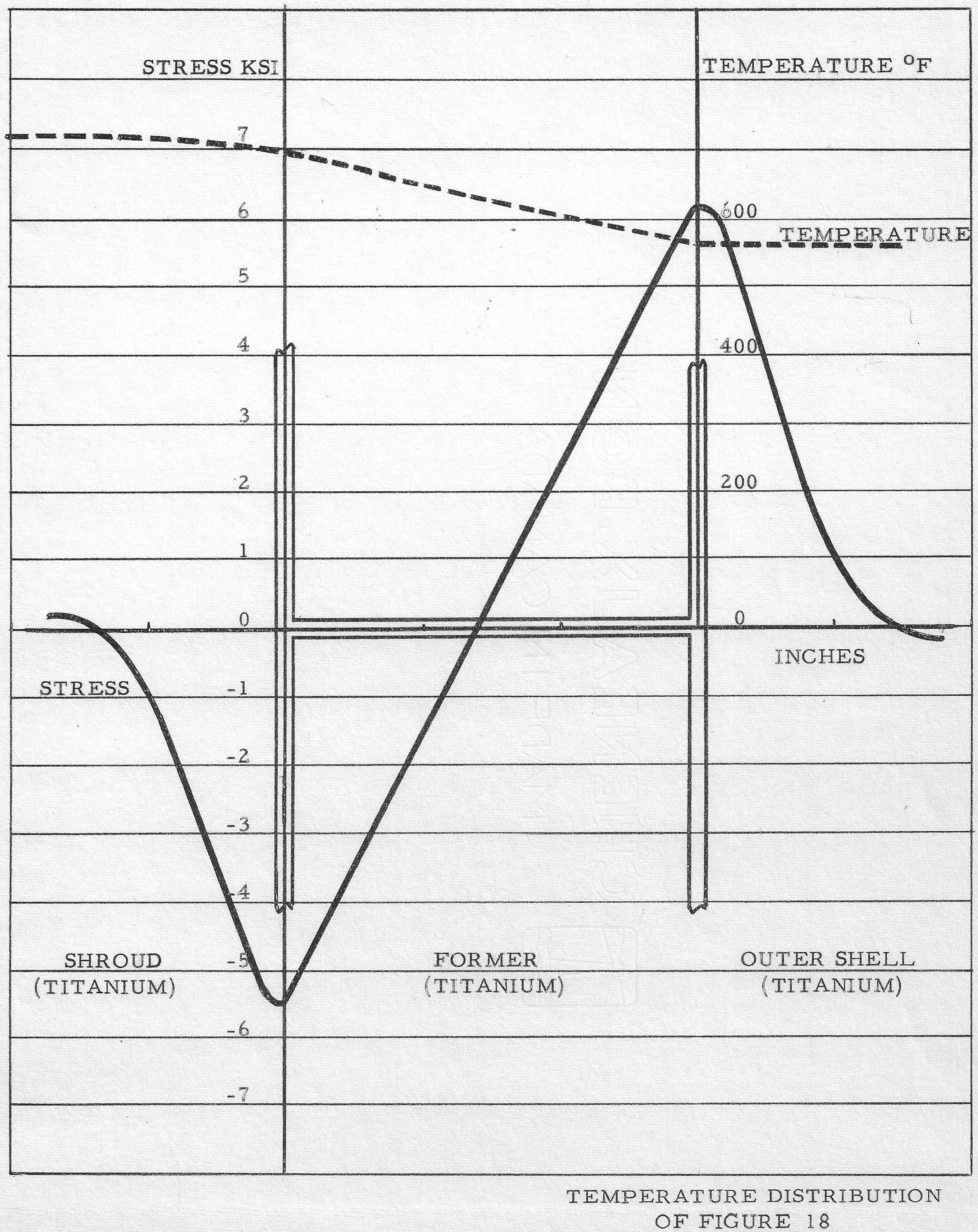


FIGURE 32 ENGINEER'S THERMAL STRESSES ON A TYPICAL M=3 ENGINE BAY STRUCTURE (STEADY STATE)

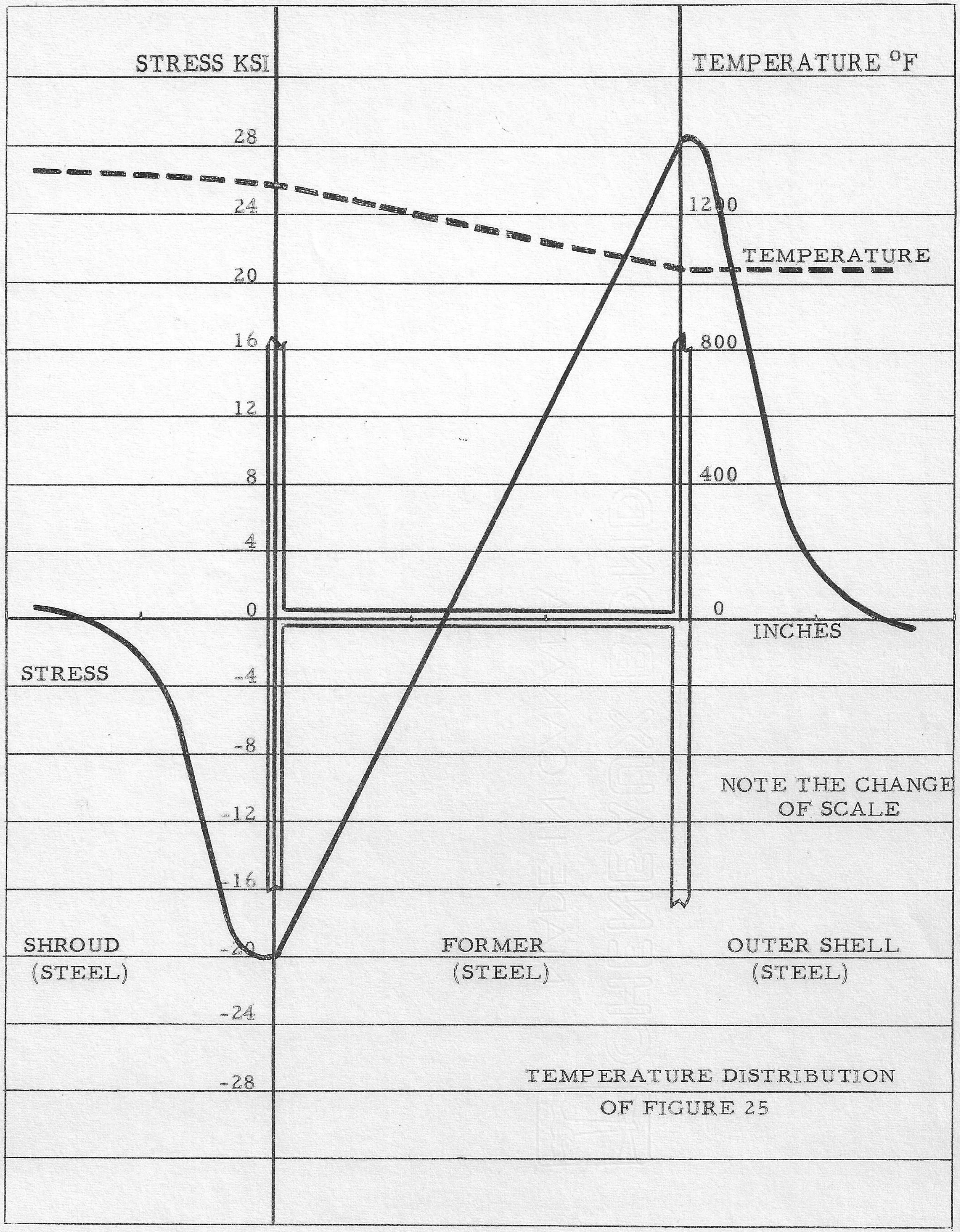


FIGURE 33 - ENGINEERS THERMAL STRESS ON A TYPICAL M=4 ENGINE BAY STRUCTURE (STEADY STATE)

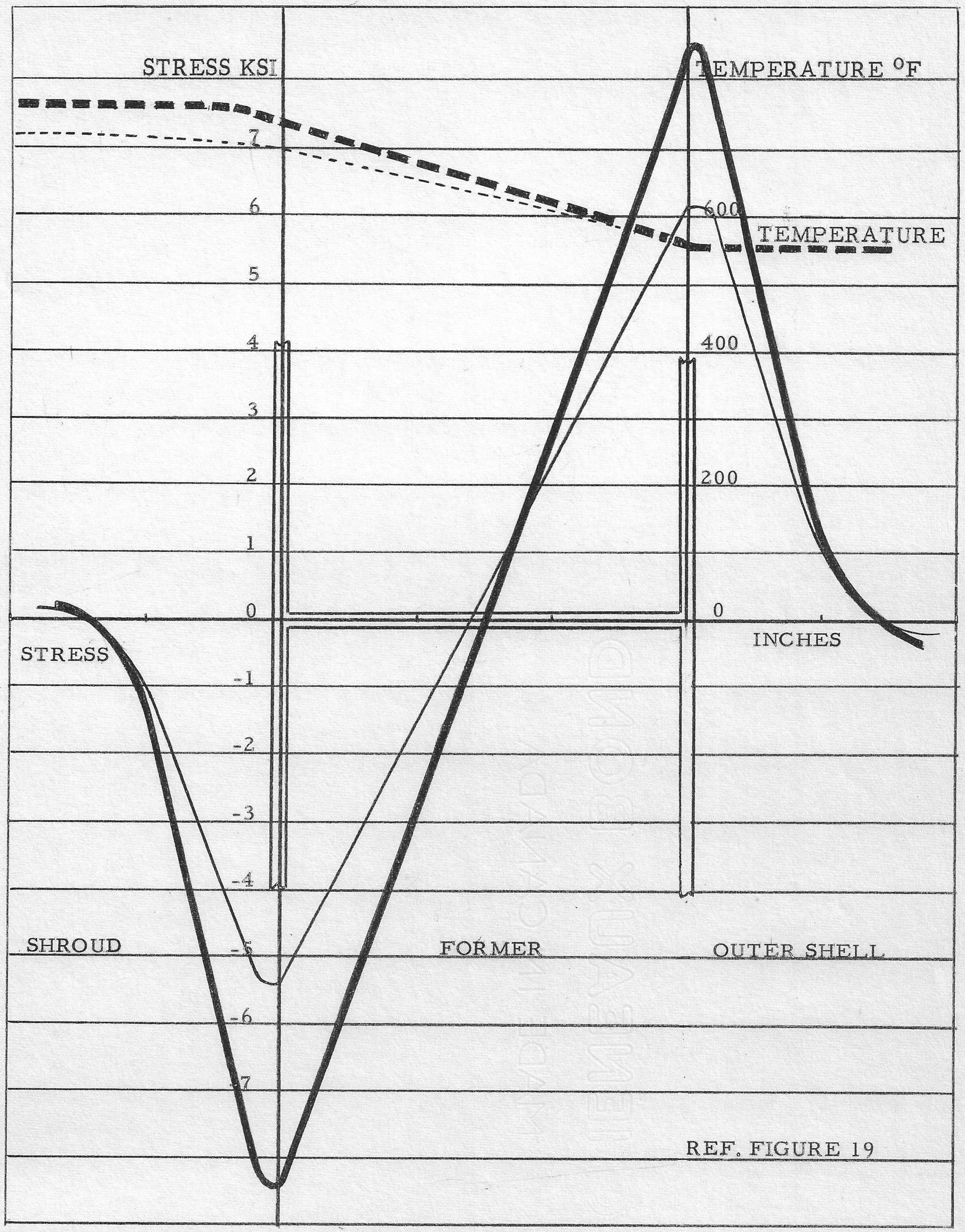
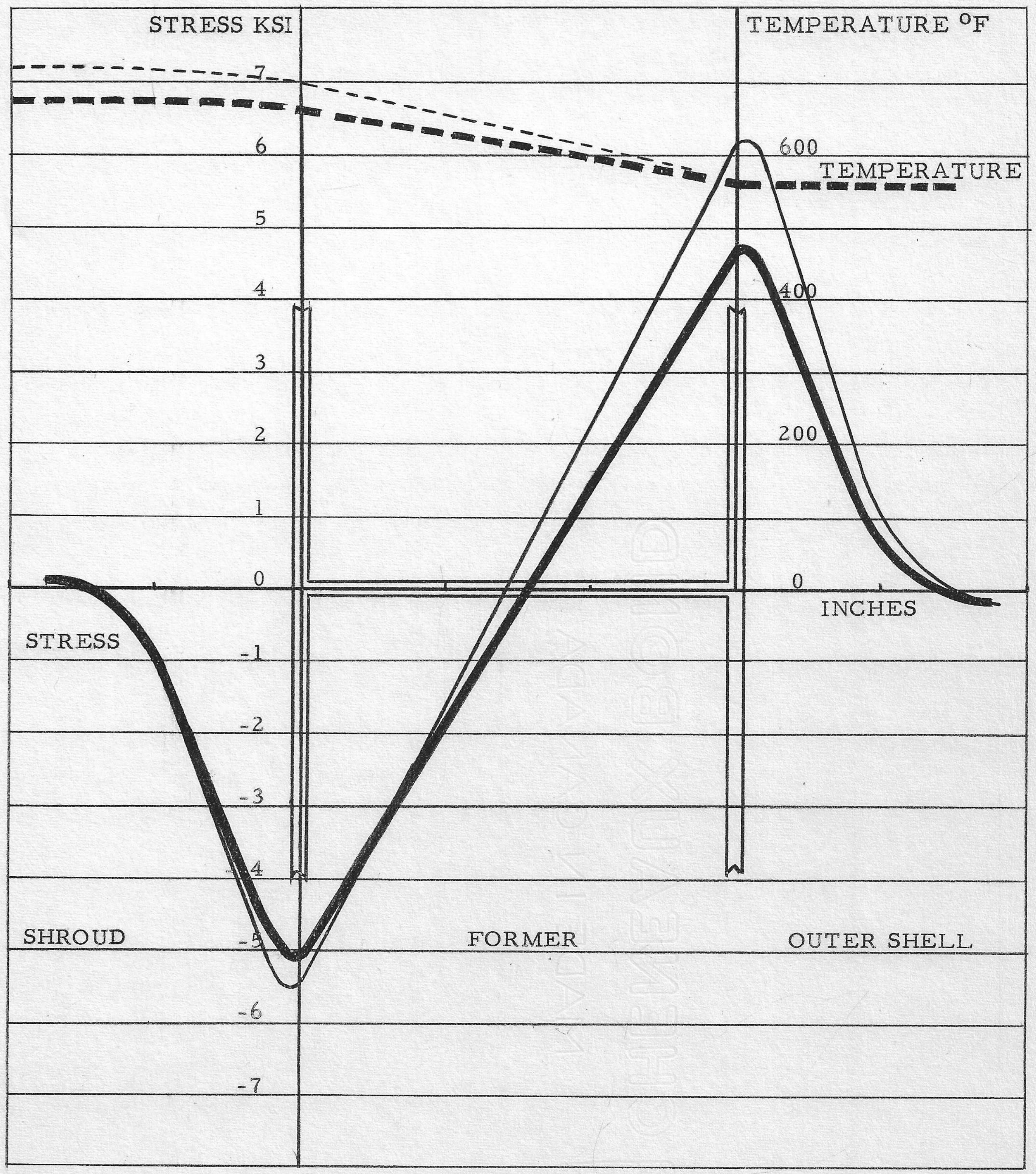


FIGURE 34 - EFFECT OF CHANGING INSULATION CONDUCTANCE, h_a FROM 1.0 TO 4.0 - TYPICAL CASE (M=3 - STEADY STATE)



REF. FIGURE 20

FIGURE 35 - EFFECT OF CHANGING CONDUCTANCE TO OUTER SHELL, h_β, FROM 0.5 TO 1.0, TYPICAL CASE (M=3.0 STEADY STATE)

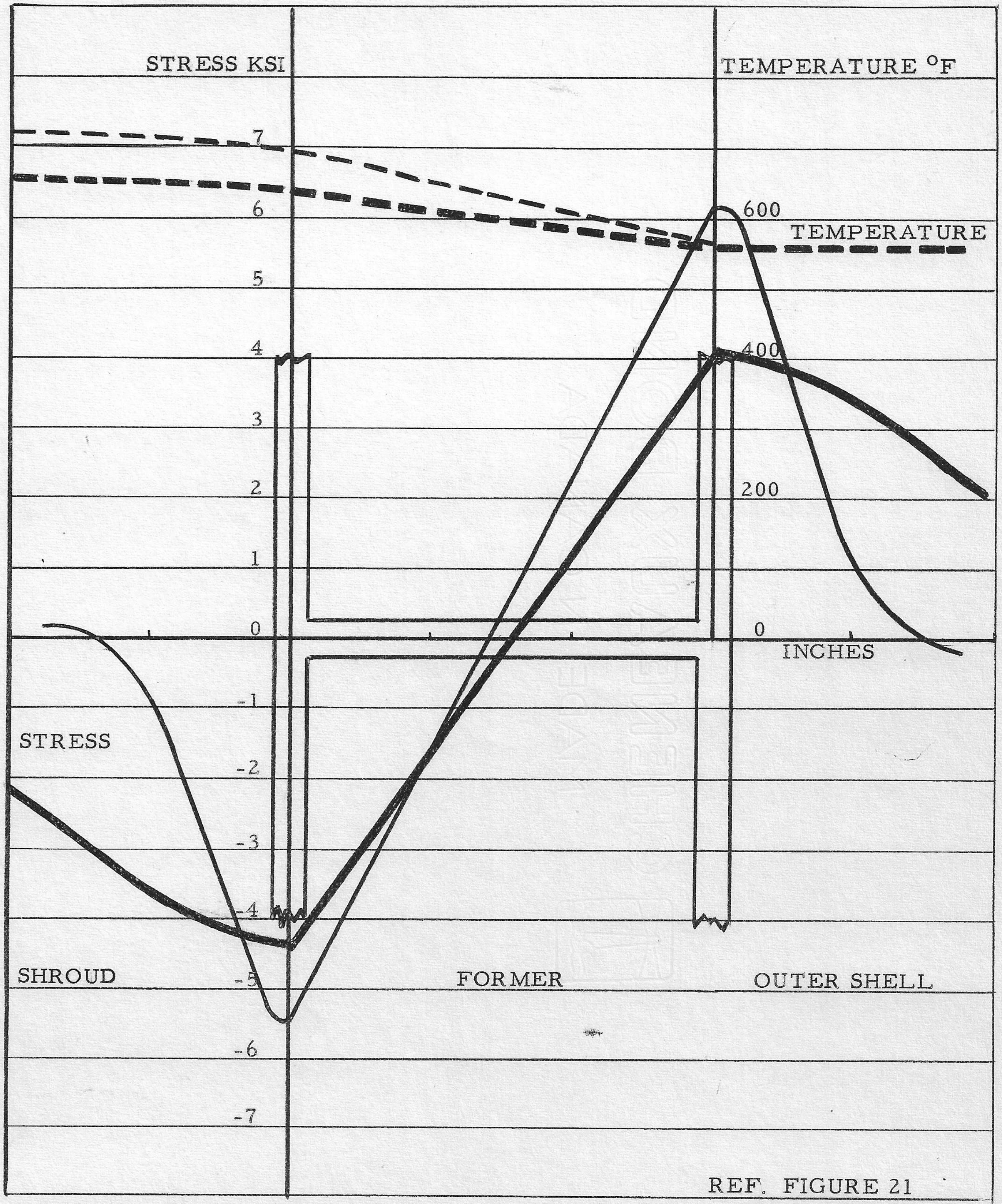


FIGURE 36 - EFFECT OF CHANGING RESISTANCE PARAMETER, ℓ^2/kt, FROM 0.5 TO 5.0. TYPICAL CASE (M=3 - STEADY STATE)

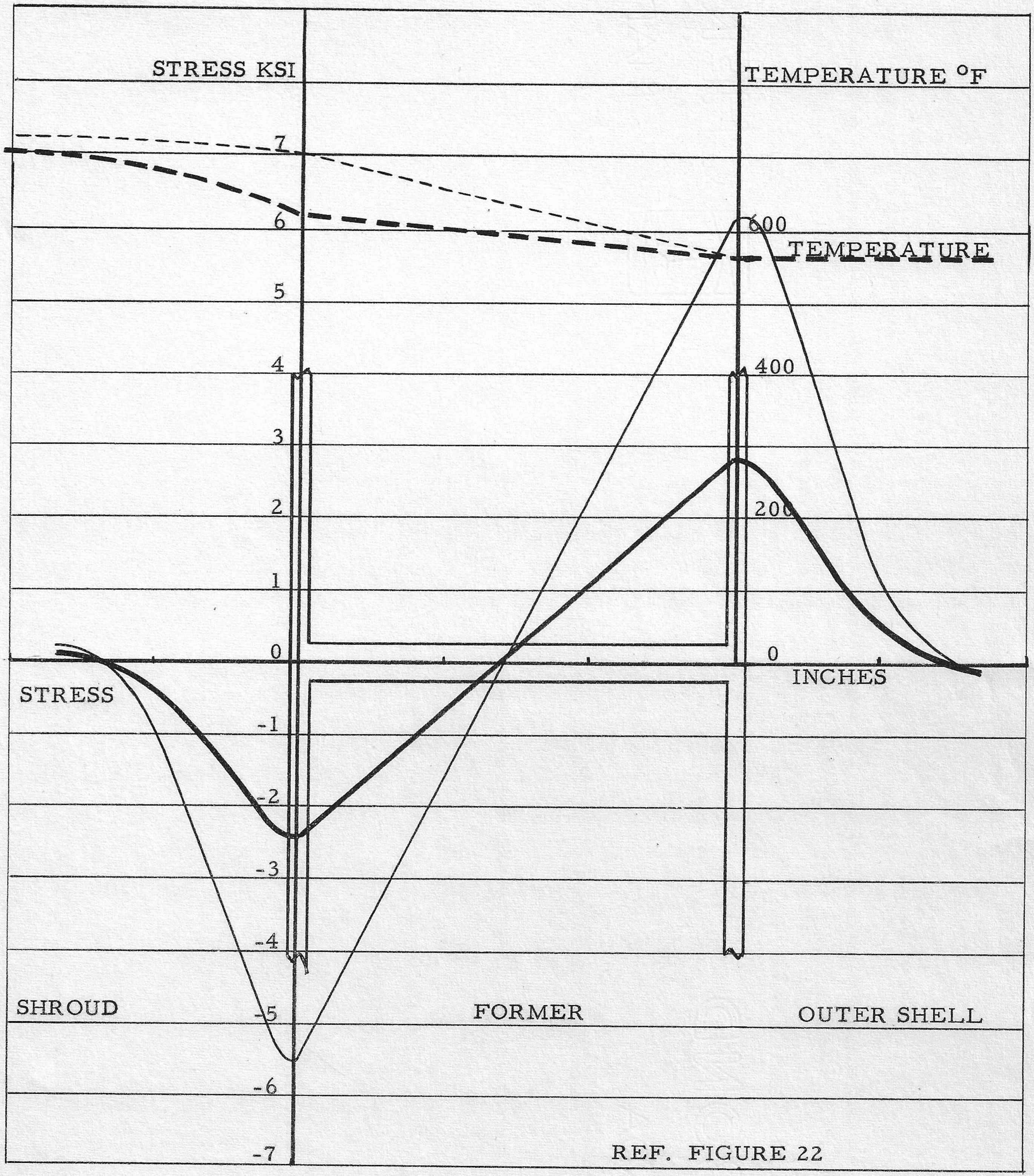


FIGURE 37 - EFFECT OF CHANGING CONDUCTANCE RATIO, γ, FROM 0.5 TO 5.0.TYPICAL CASE (M=3 - STEADY STATE)

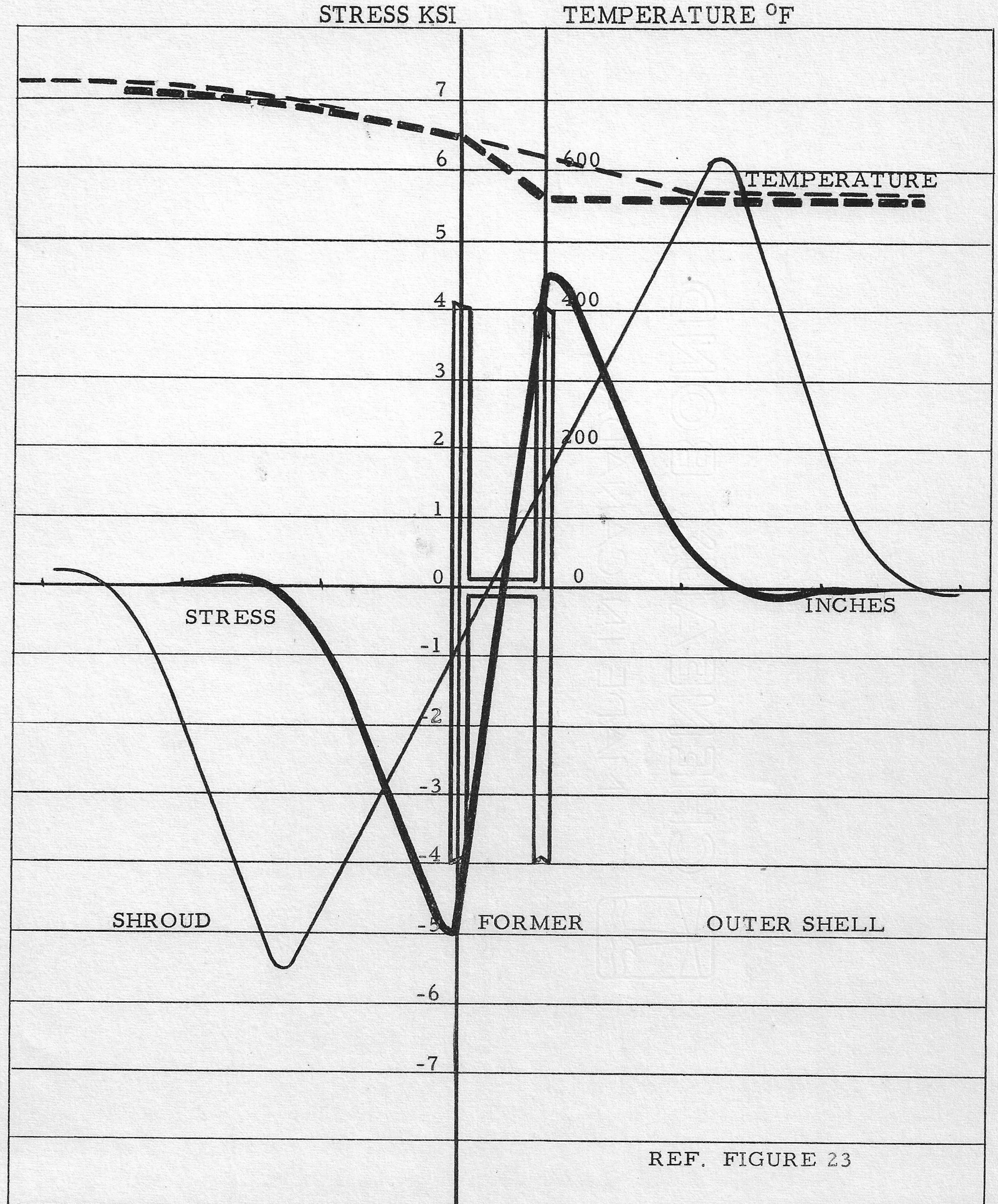


FIGURE 38 - EFFECT OF CHANGING SPACING RATIO, l/a, FROM 1.0 TO 5.0.TYPICAL CASE (M=3 - STEADY STATE)

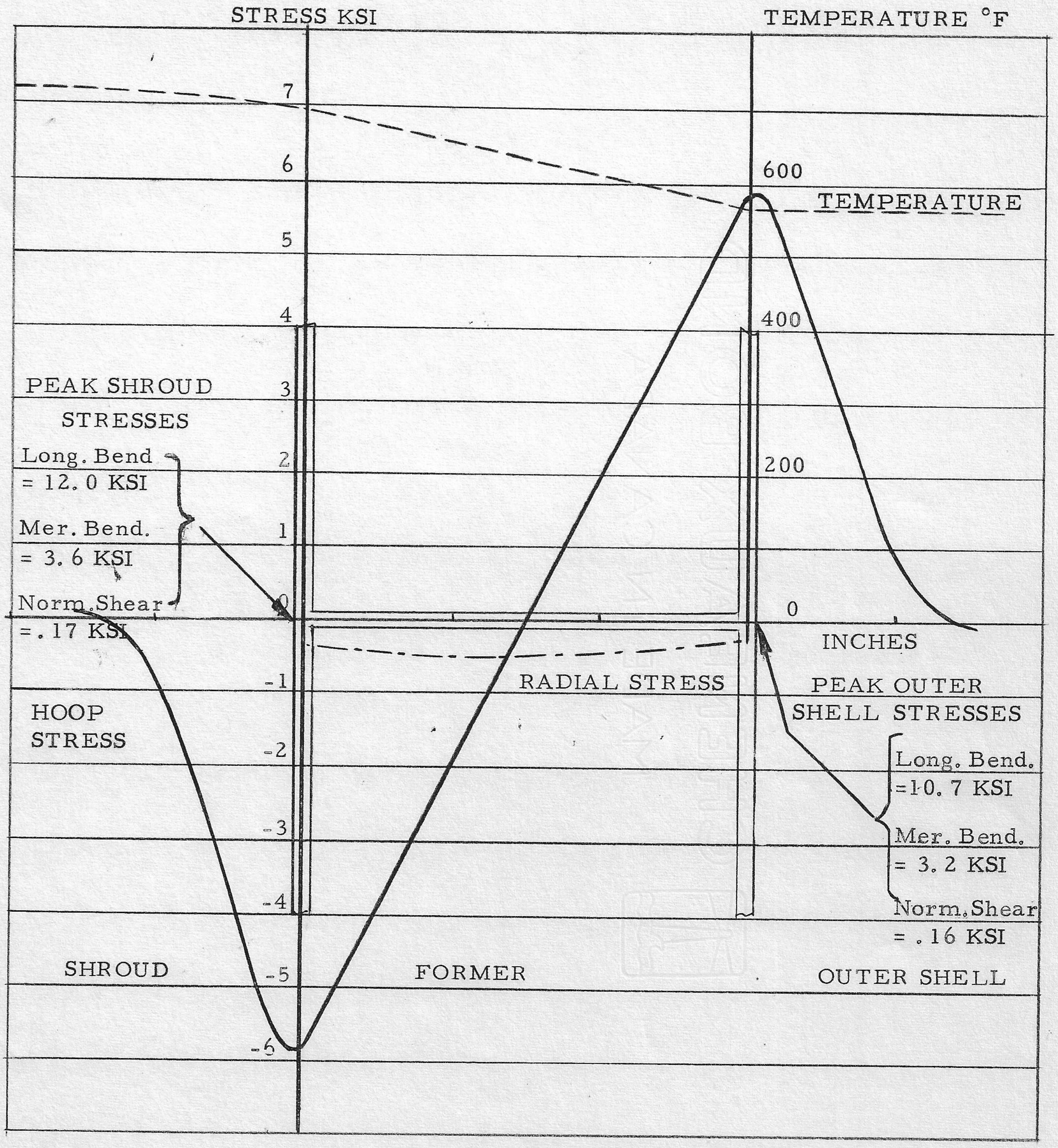


FIGURE 39 - EXACT THERMAL STRESSES FOR TYPICAL M=3 STRUCTURE
(STEADY STATE)

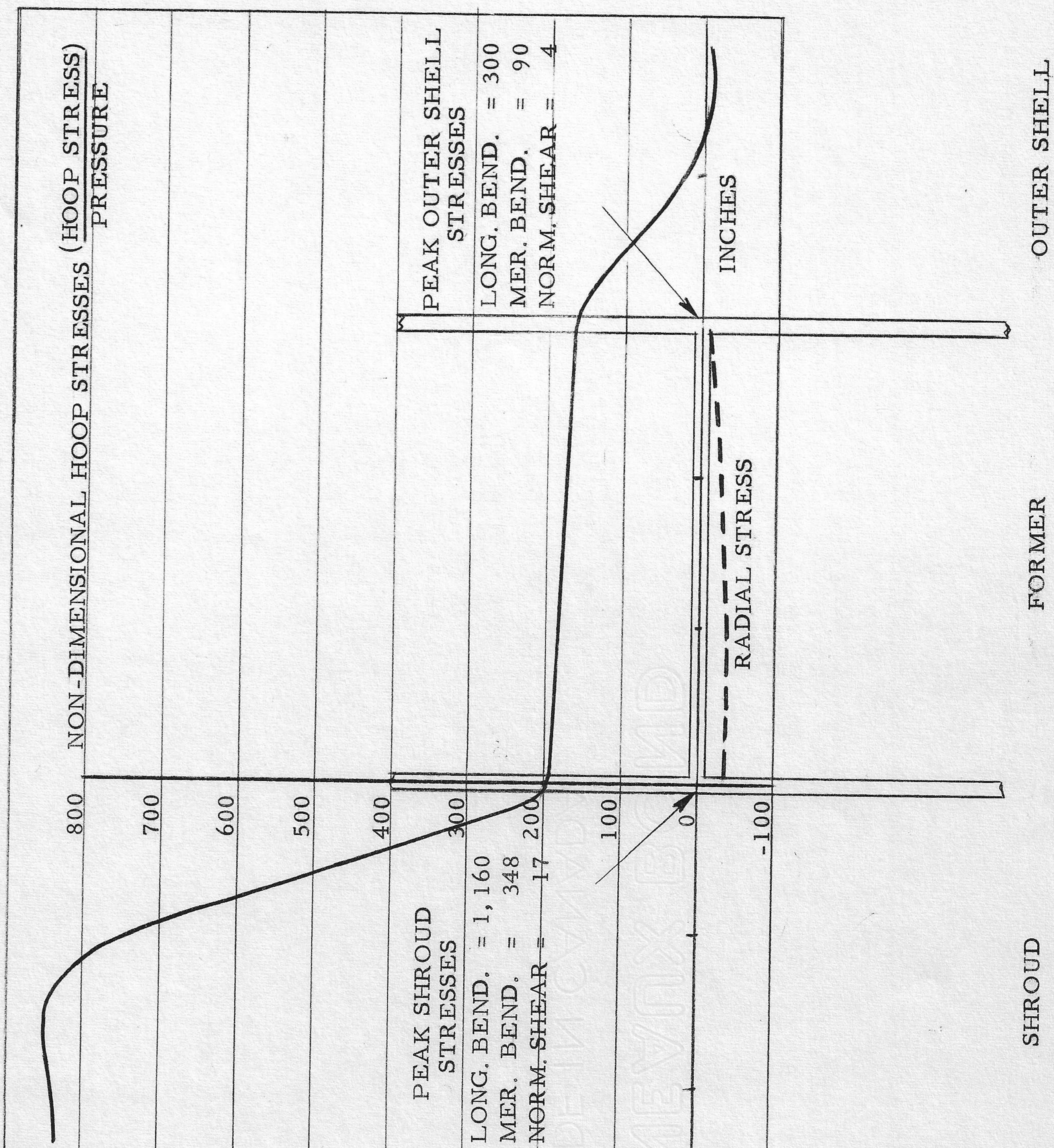


FIGURE 40 - NON-DIMENSIONAL PRESSURE STRESSES
FOR TYPICAL M=3 STRUCTURE

APPENDIX I

A simple expression for the Heat Transfer Coefficient in an Annular Duct

Appendix I

CONTENTS

Notation

N	Dimensionless ratio
ρ	Density
V	Velocity
ℓ	A characteristic length
A	Cross sectional area of duct
m	Mass flow through duct
C	Total perimeter of duct
k	Thermal conductivity
$D_{1,2}$	Inside and outside wall diameters
μ	Viscosity

Subscripts

PR	Prandtl
RE	Reynolds
NU	Nusselt

Summary

A simple form of the heat transfer coefficient in an annular duct has been developed. For a particular flight vehicle, the coefficient can be expressed in terms of a single temperature dependent parameter, the duct cross sectional area and mass flow.

Analysis

Take for the equation governing heat transfer

$$N_{NU} = 0.027\, N_{RE}^{0.8}\, N_{PR}^{1/3}$$

now $N_{RE} = \dfrac{\rho V l}{\mu}$

and $\rho V A = m$ $\therefore\ \rho V = \dfrac{m}{A}$

Hence $\dfrac{hl}{k} = 0.027 \left(\dfrac{m}{A} \dfrac{l}{\mu} \right)^{0.8} N_{PR}^{1/3}$

and for l in an annular region we take

$$l = (D_2 - D_1) = \frac{4a}{c}$$

Hence

$$h = \frac{0.027}{4^{0.2}} \frac{m}{A}^{0.8} c^{0.2} \frac{k N_{PR}^{1/3}}{\mu^{0.8}} = \tau \frac{m}{A}^{0.8}$$

With $\tau = 0.0204\, c^{0.2} \left(\dfrac{k N_{PR}^{1/3}}{\mu^{0.8}} \right)$

For any particular flight vehicle, $C^{0.2}$ will vary only slightly, and $\dfrac{k N_{PR}^{1/3}}{\mu^{0.8}}$ can be evaluated as a function of temperature.

Conclusions

For any particular flight vehicle it is possible to express heat transfer coefficient in the bypass duct in terms of a single temperature dependent parameter, the duct area and the mass flow.

APPENDIX II

STEADY STATE SOLUTION FOR ENGINE BAY

TEMPERATURE DISTRIBUTION

CONTENTS

LIST OF FIGURES

Notation

T	Temperature (of shroud if no subscript)
h	Heat transfer coefficient
k	Thermal conductivity
t	Thickness
2ℓ	Former spacing
a	Former Depth
λ	$\left(\frac{h_\alpha + h_\beta}{kt}\right)^{1/2}$
A, B	Constants
γ	$\frac{k_F t_F}{2kt}$
x	Distance along shroud from shroud-former junction

Subscripts

α	Pertaining to foil temperature or heat transfer to shroud
β	Pertaining to outer shell temperature or heat transfer from shroud
0	Shroud-former junction
∞	Corresponding to infinite former spacing
F	Former

1. Summary

The steady state temperature distribution in an engine bay structure of the type shown in Figure 1, has been determined for a typical case of engine bay heating. The form of the temperature distribution is shown in Figure 2.

2. Introduction

Given a structure as shown in Figure 1, heated by a foil at a constant temperature, the temperature distribution through the structure can be determined if the following assumptions are made:

(1) The capability of the boundary layer for removal of heat is so great that the outer shell is maintained at a constant temperature.

(2) No heat is lost from the web.

(3) Heat transfer to the outer shell can be approximated by an overall heat transfer coefficient.

3. Analysis

The equation governing the temperature distribution in the shroud is

$$\frac{d^2T}{dx^2} = \lambda^2 T - \lambda^2 T_\infty \qquad (1)$$

with

$$\lambda^2 = \left(\frac{h_\alpha + h_\beta}{kt}\right) \qquad T_\infty = \left(\frac{h_\alpha T_\alpha + h_\beta T_\beta}{h_\alpha + h_\beta}\right)$$

A solution of (1) is

$$T = Ae^{\lambda x} + Be^{-\lambda x} + T_\infty \qquad (2)$$

The boundary conditions are

$$T = T_0 \quad @ \quad x = 0$$

$$T = T_0 \quad @ \quad x = 2l$$

Solving for A and B and substituting in (2) the temperature distribution in the shroud is

$$T = T_\infty + (T_0 - T_\infty)\frac{\{\sinh \lambda x + \sinh \lambda(2l - x)\}}{\sinh 2\lambda l} \qquad (3)$$

And heat flow, Q, from the end of the shroud at $x=0$ is $Q = kt \left.\frac{dT}{dx}\right|_{x=0}$

Differentiating (3) and simplifying

$$Q = kt\lambda \tanh \lambda l \,(T_\infty - T_0)$$

Double this amount of heat passes through each former hence

$$2kt\lambda \tanh \lambda l\,(T_\infty - T_0) = \frac{k_F t_F}{a}(T_0 - T_\beta)$$

or

$$a\lambda \sinh \lambda l\,(T_\infty - T_0) = \gamma \cosh \lambda l\,(T_0 - T_\beta)$$

hence

$$T_0 = T_\beta + \frac{(T_\infty - T_\beta)\, a\lambda \sinh \lambda l}{a\lambda \sinh \lambda l + \gamma \cosh \lambda l}$$

Substituting which in (3) gives for the shroud temperature distribution

$$T_{SHROUD} = T_{\infty} - \frac{(T_{\infty} - T_{\beta})\,\gamma \cosh \lambda (l - x)}{a\lambda \sinh \lambda l + \gamma \cosh \lambda l}$$

and taking x negative along the former the temperature distribution in the former is

$$T_{FORMER} = T_{\beta} + \frac{(T_{\infty} - T_{\beta})(a + x)\,\lambda \sinh \lambda l}{a\lambda \sinh \lambda l + \gamma \cosh \lambda l}$$

4. Results

The temperature distributions determined above will apply to an annular former if the radius of curvature is large. If the radius of curvature is small, a similar analysis can be carried out with

$$r_s \ln\left(\frac{r_o}{r_s}\right) \quad \text{replacing} \quad a$$

in which r_o and r_s are outer shell and shroud radii respectively.

A somewhat more detailed analysis can be carried out without making assumption 1 if this is desired.

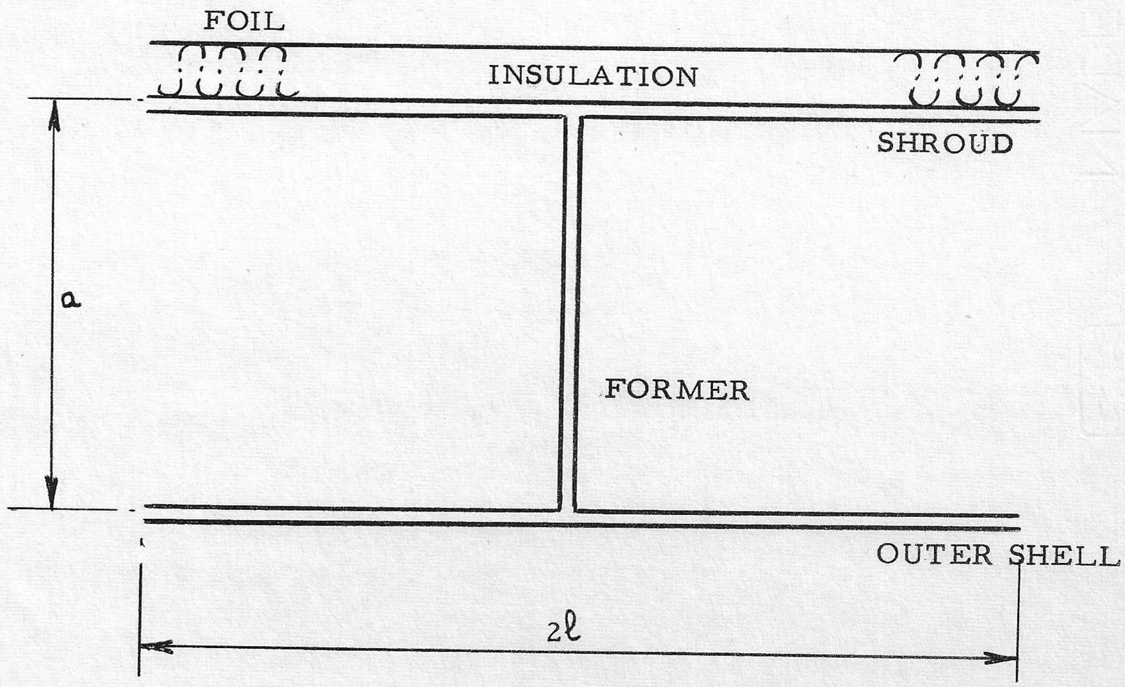


FIGURE 1 - TYPICAL ENGINE BAY STRUCTURE

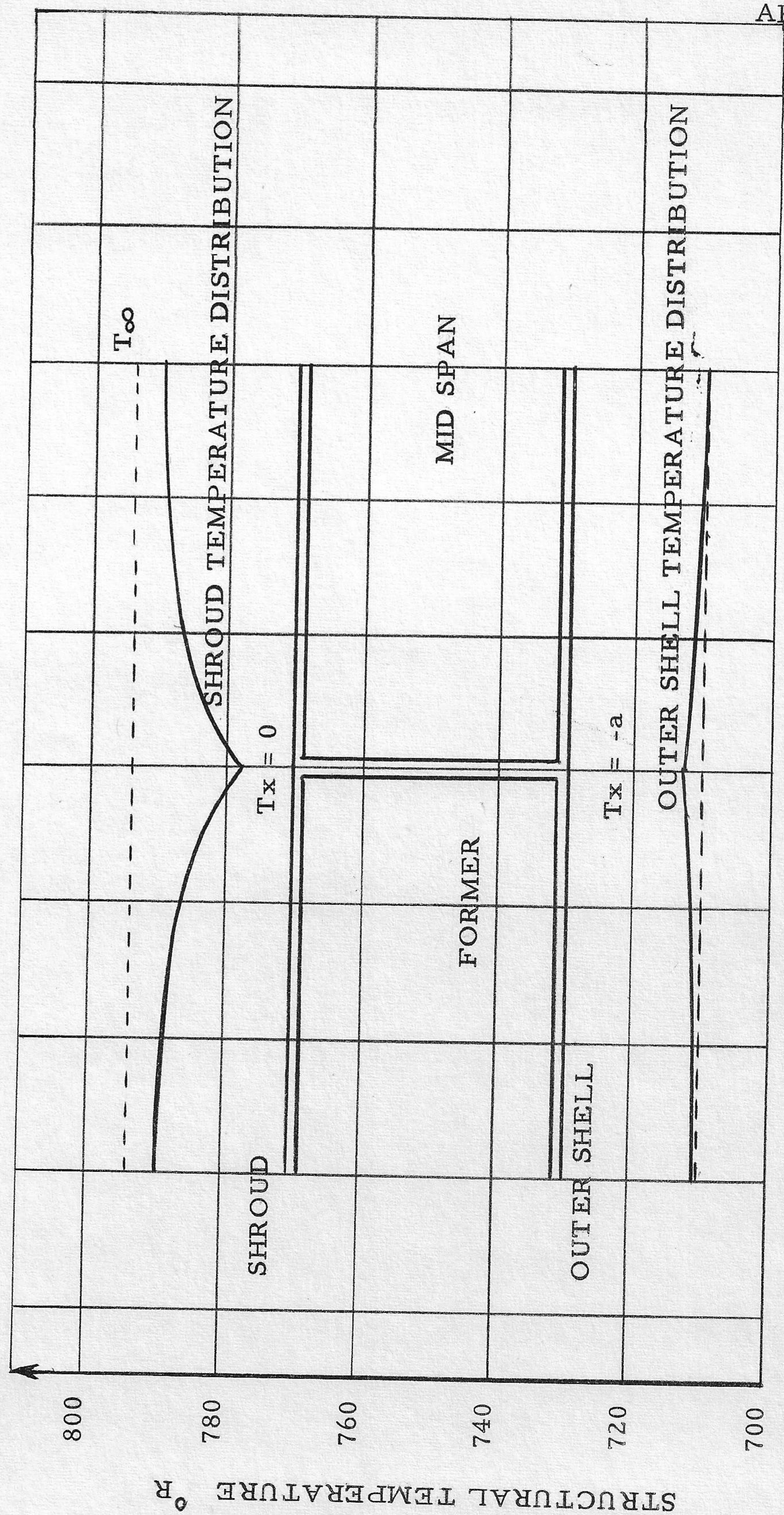


FIGURE 2 - CHARACTERISTIC TEMPERATURE DISTRIBUTION

APPENDIX III

AN ANALYTICAL METHOD FOR DETERMINING TRANSIENT STATE TEMPERATURE RESPONSE IN AN ENGINE BAY STRUCTURE

CONTENTS

LIST OF FIGURES

Notation

c	Specific Heat
ρ	Density
k	Thermal Conductivity
K	Diffusivity
θ	Temperature
h	Heat Content $= c\rho\theta$
V	Thermal Potential
D	Dissipation Function
Q	Thermal Force
H	Heat Flow Vector
$r, p,$	Distances Defined in Figure 1
y	Distance Co-ordinate
ϕ	Net Heat Transfer Coefficient to Shroud
τ	$\frac{b c \rho}{\phi}$
t	Time
b	Shroud Thickness
b_1	Semi Former Thickness
a	Former Length
ℓ	Semi Shroud Length
γ	$\frac{b_1}{b}$

Subscripts

1	Shroud Former Junction
2	Mid Former
3	Location of Zero Former Heat Flow
β	Outside Shell
∞	Effective Source
S	Shroud
F	Former

1. Summary

The method of Biot[1] is a powerful tool in heat flow analysis which gives very close approximations for temperature distribution. Its application to most structural heating problems yields a considerable simplification from the rigorous method.

2. Analysis

A solution has been developed for the heating of a slab with a step in the temperature of one end. This can be extended to the case in which end temperature does not jump, and in which the position of zero heat flow varies with time.

Consider a slab of thickness r. The face at y = 0 is at a temperature θ_1, the face at $y = \ell$ is insulated and reaches a temperature θ_3.

If the heat content is $h = c\rho\theta$, then the distribution is

$$h = (h_1 - h_3)\left(1 - \frac{y}{r}\right)^2 + h_3 \qquad (1)$$

Conservation of energy requires

$$-\frac{dH}{dy} = h \qquad (2)$$

Integrating (2) with H = 0 at y = r gives

$$H = h_3(r - y) + (h_3 - h_1)\left\{y - \frac{y^2}{r} + \frac{y^3}{3r^2} - \frac{r}{3}\right\}$$

Treating r as variable we can evaluate $\dot{H}$ and hence

$$D = \frac{1}{2K}\int_0^r (\dot{H}^2)\, dy \qquad (3)$$

Differentiating the result obtained from (3) with respect to $\dot{h}_3$ we find

$$\frac{\partial D}{\partial \dot{h}_3} = \frac{r^2}{420K}\left\{\frac{2r}{3}(136\dot{h}_3 + 32\dot{h}_1) + \dot{r}(136h_3 + 39h_1)\right\} \qquad (4)$$

Now $$V = \frac{1}{2\rho c}\int_0^r h^2\, dy \qquad (5)$$

and $$\frac{\partial V}{\partial h_3} = \frac{2r}{15\rho c}(4h_3 + h_1) \qquad (6)$$

Finally the generalized force Q_3 is derived by considering the variation of H at y = 0

$$Q_3\,\delta h_3 = \theta_1\,\delta H = \frac{2}{3}\theta_1 r\,\delta h_3$$

hence $$Q_3 = \frac{2}{3}\theta_1 r = \frac{2}{3}\frac{h_1}{\rho c} r \qquad (7)$$

Now the differential equation for h_3 is

$$\frac{\partial V}{\partial h_3} + \frac{\partial D}{\partial \dot{h}_3} = Q_3 \qquad (8)$$

which becomes

$$(h_1 - h_3) = \frac{r}{224K}\left\{\frac{2r}{3}(136\dot{h}_3 + 32\dot{h}_1) + \dot{r}(136 h_3 + 39 h_1)\right\}$$

or, changing from heat content to temperature

$$(\theta_1 - \theta_3) = \frac{r}{224K}\left\{\frac{2r}{3}(136\dot{\theta}_3 + 32\dot{\theta}_1) + \dot{r}(136\theta_3 + 39\theta_1)\right\} \qquad (9)$$

Now let us consider the system shown in Figure 1 which represents a typical former and shroud of the engine bay structure. The transit time for the half former is given by:

$$t_1 = \frac{13}{147}\left(\frac{a}{2}\right)^2 \times \frac{1}{K}$$

The penetration of heat proceeds at a rate such that the penetration depth is

$$\mu = \sqrt{\frac{147Kt}{13}}$$

Hence for the period up to $t = t_1$ the temperature distributions have been shown to be:

for the outer segment of the former $\theta_F = \theta_\beta\left(\frac{y}{\mu}\right)^2$

for the inner segment of the former $\theta_F = \theta_1\left(\frac{y}{\mu}\right)^2$

for the shroud $\theta_S = \theta_2 + (\theta_1 - \theta_2)\left(\frac{y}{\mu}\right)^2$

with
$$\theta_1 = \theta_{00}\left(1 - e^{-t/(1+\nu)\tau}\right)$$
$$\theta_2 = \theta_{00}\left(1 - e^{-t/\tau}\right) \qquad (10)$$

Equation (10) will continue to describe θ_2

until $t = t_2 = \frac{13}{147} \cdot \frac{\ell^2}{K}$

During this second phase we can equate the net heat flow to the shroud to that stored in the structure.

Now with $\theta_F = \theta_3 + (\theta_1 - \theta_3) \frac{y^2}{r^2}$

$$H = H_2 + (H_1 - H_2) \frac{y^2}{p^2}$$

and $H_2 = ac\rho\,\theta_2$ $\qquad \int_0^{p} H\,dy = kc\rho \int_0^{p} \theta_s\,dy + k_1\rho c \int_0^{r} \theta_F\,dy$

hence $H_1 = k\rho c\,\theta_1 + k_1 \rho c \frac{r}{p} (2\theta_3 + \theta_1)$

Now the dissipation function

$$D = \frac{\tau}{2k\rho c} \int_0^{p} \dot{H}^2\,dy$$

becomes in this case

$$D = \frac{\tau}{k\rho c} \left\{ \frac{\dot{H}_2^2 p}{2} + \frac{\dot{H}_2(\dot{H}_1 - \dot{H}_2) p}{3} + \frac{p(\dot{H}_1 - \dot{H}_2)^2}{10} \right\} \qquad (11)$$

Differentiating (11) with respect to $\dot{\theta}_1$ and simplifying gives

$$\frac{\partial D}{\partial \dot{\theta}_1} = \tau k\rho c \left(1 + \frac{\gamma r}{p}\right) p \left\{ \frac{\dot{\theta}_2}{3} + \frac{(\dot{\theta}_1 - \dot{\theta}_2)}{5} + \frac{\gamma r}{5p} (2\dot{\theta}_3 + \dot{\theta}_1) + \frac{\gamma (2\theta_3 + \theta_1)}{5p^2} (p\dot{r} - r\dot{p}) \right\}$$

Nor the thermal force is

$$Q_1 = \int_0^{p} (\theta_\infty - \theta_s) \frac{\partial H}{\partial \theta_1}\,dy = k\rho c \left(1 + \frac{\gamma r}{p}\right) p \left\{ \frac{(\theta_\infty - \theta_2)}{3} - \frac{(\theta_1 - \theta_2)}{5} \right\}$$

hence the differential equation for θ_1,

$$Q_1 = \frac{\partial D}{\partial \dot{\theta}_1}$$

becomes

$$(\theta_\infty - \theta_1) = \tau \left(1 + \frac{\gamma r}{p}\right) \dot{\theta}_1 + \frac{2\tau\gamma r}{p} \dot{\theta}_3 + \frac{\tau\gamma}{p^2} (2\theta_3 + \theta_1)(p\dot{r} - r\dot{p}) \qquad (12)$$

Another equation of the form of (9) can be written expressing the relationship between θ_3 , θ and $(a - r)$

Hence the three equations (9), (12) and (13) can be solved simultaneously to give the time variations of θ_1 , θ_3 and r starting the solution with

$$\left.\begin{aligned} \theta_3 &= 0 \\ r &= a/2 \\ \theta_1 &= \theta_\infty\left(1 - e^{-t/(1+\gamma)\tau}\right) \end{aligned}\right\} \quad @\ t = t_1$$

For times greater than t_2 the value of θ_2 will no longer be given by (10)

Let us define θ_4 as the value, shroud temperature would assume in the absence of formers.

Thus

$$\theta_4 = \theta_2 \qquad\qquad t \leq t_2$$

$$\theta_4(t) = \theta_\infty\left(1 - e^{-t/\tau}\right) \qquad\qquad t > t_2$$

Further we can apply the result of (9) to the shroud temperature decrease below θ_4 hence.

$$(\theta_4 - \theta_1) - (\theta_4 - \theta_2) = \frac{l^2}{336K}\left\{136\frac{d}{dt}(\theta_4 - \theta_2) + 32\frac{d}{dt}(\theta_4 - \theta_1)\right\}$$

and expressing l^2/K in terms of the transit time this becomes

$$(\theta_2 - \theta_1) = \frac{7t_2}{26}\left(21\dot{\theta}_4 - 17\dot{\theta}_2 - 4\dot{\theta}_1\right) \qquad (14)$$

The heat balance equation now becomes

$$\int_0^l H\,dy = b\rho c\int_0^l \theta_s\,dy + b_1\rho c\int_0^r \theta_F\,dy$$

and following a similar approach to that taken in deriving equation (12) gives

$$(\theta_\infty - \theta_1) = \tau\left(1 + \frac{\gamma r}{l}\right)\dot{\theta}_1 + \frac{2\tau\gamma r}{l}\dot{\theta}_3 + \frac{\tau\gamma\dot{r}}{l}(2\theta_3 + \theta_1) + 2\left\{\tau\dot{\theta}_2 - (\theta_\infty - \theta_2)\right\}$$

Thus a phase III solution for $t > t_2$ can be carried out by solving equations (9) (13) (14) and (15) for the time variations of θ_1, θ_2, θ_3 and r.

3. Conclusions

Using the method of Biot[1] a set of equations has been derived, the solution of which gives the transient response for the heating of a typical engine bay structure.

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Biot, M. A. New Methods in Heat Flow Analysis with Application to Flight Structures.
J. Aero Sci. Vol. 24 No. 12
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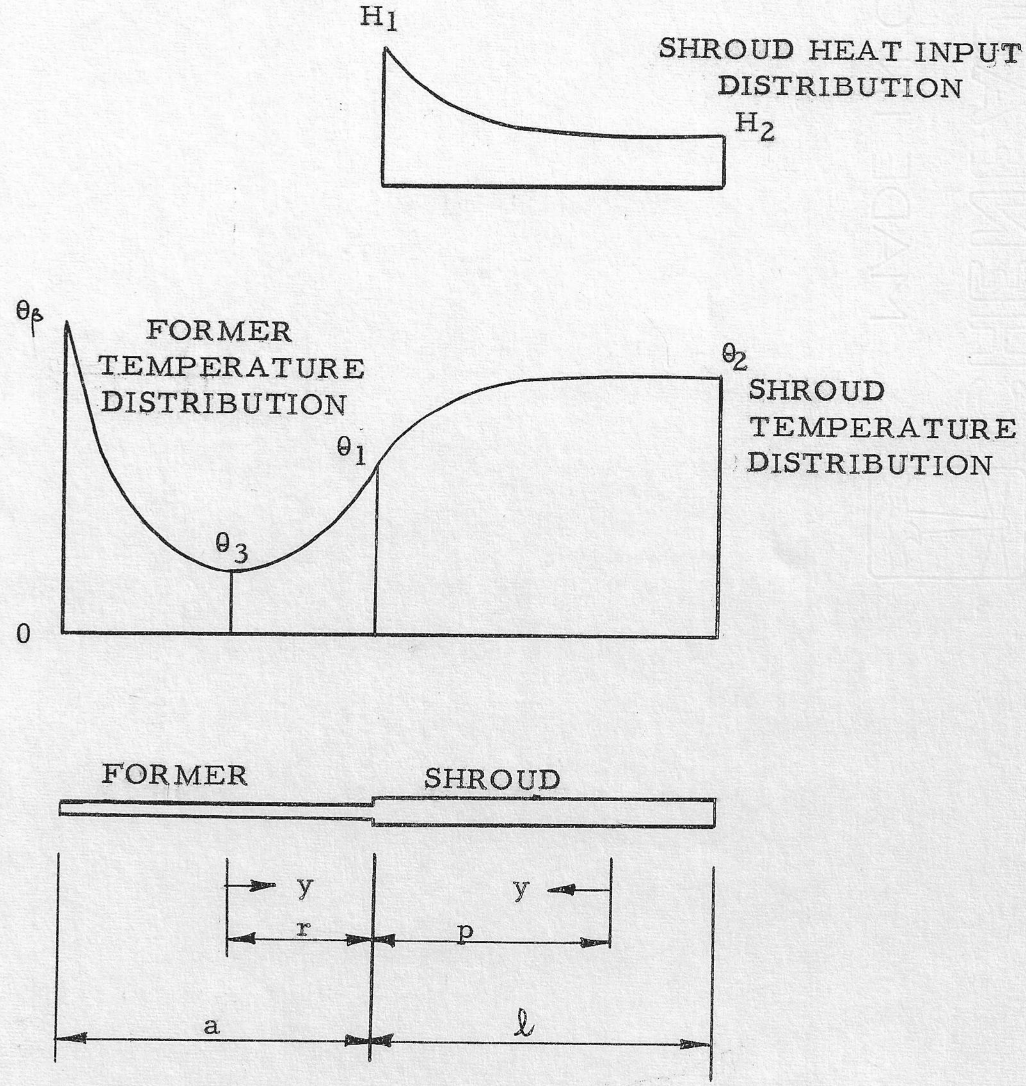

FIGURE 1 - FORM OF ASSUMED TEMPERATURE DISTRIBUTIONS IN ENGINE BAY STRUCTURE

APPENDIX IV

THERMAL STRESSES IN ENGINE BAY STRUCTURE

CONTENTS

LIST OF FIGURES

Notation

N_x , N_y	Unit Membrance Forces in x and y directions respectively (lb/in)
Q_x	Unit Shear Force Parallel to z axis (lb/in)
M_x, M_y	Unit Bending Moments Perpendicular to Shell and Meridian respectively lb in/in)
W	Radial Displacement (in)
P	Unit Pressure Force between Shells and Former (lb/in)
x, y	Co-ordinates along shell axis or periphery
r, θ	Polar co-ordinates
β	Parameter $\beta^4 = \frac{3(1-\nu^2)}{a^2h^2}$
D	Flexural Rigidity of the Shell $D = \frac{Eh^3}{12(1-\nu^2)}$
ν	Poisson's Ratio
E	Young's Modulus (lb/in^2)
h	Thickness (in)
a, b	Inner and Outer Radii respectively (in)
α	Coefficient of Thermal Expansion (in/in ^{o}F)
T	Temperature (^{o}F)
σ_r, σ_θ	Radial and Tangential Stresses in a Former (lb/in^2)
l	Semi-Former Spacing

Suffixes

o	Outer Surface of Shell or Former
i	Inner Surface of Shell or Former
f	Former
c	Position on the Shell at $x = 0$
l	Position on the Shell at $x = l$

1. Introduction

A short analysis of the exact thermal stresses on the engine bay structure is recapitulated. The method follows the line of attack of reference.

The effect of adjacent formers has not been considered due to the very localized nature of the disturbance on the shells. However, should these formers by spaced very closely, the modification to the analysis is very simple.

2. <u>Analysis</u>

a) Stresses in a cylinder due to a load uniformly distributed along a circular section. From reference[2], page 395 - 397, we have the following basic equations. See also Figures 1 and 2.

@ x	@ $x=0$
$Q_x = -\frac{P}{2} e^{-\beta x} \cos(\beta x)$	$Q_{x_0} = -\frac{P}{2}$
$N_x = 0$	
$N_\varphi = -\frac{\sqrt{2}}{2} P a \beta e^{-\beta x} \sin\left(\beta x + \frac{\pi}{4}\right)$	$N_{\varphi_0} = -\frac{P a \beta}{2}$
$M_x = -\frac{\sqrt{2} P}{4\beta} e^{-\beta x} \sin\left(\beta x - \frac{\pi}{4}\right)$	$M_{x_0} = \frac{P}{4\beta}$
$M_\varphi = \nu M_x$	
$w = \frac{\sqrt{2} P}{8\beta^3 D} e^{-\beta x} \sin\left(\beta x + \frac{\pi}{4}\right)$	$w_0 = \frac{P}{8\beta^3 D}$

b) Stresses in a cylinder due to an axial temperature distribution. This distribution being symmetrical about the x-axis of parabolic. See Figure 2. Modifying the equations of reference[2] , pages 423-426 we obtain:

Temperature distribution $= F(x) = \left\{ T_l - (T_l - T_c)\left(1 - \frac{x}{l}\right)^2 \right\}$

Hence particular integral $= f(x) = -\alpha a \left\{ T_l - (T_l - T_c)\left(1 - \frac{x}{l}\right)^2 \right\}$

$$\therefore\ w = e^{-\beta x}\left(C_3 \cos\beta x + C_4 \sin\beta x\right) - \alpha a \left\{ T_l - (T_l - T_c)\left(1 - \frac{x}{l}\right)^2 \right\}$$

Boundary conditions

At $x = 0$ $\quad Q_0 = 0 \quad \therefore \left(\frac{d^3 w}{dx^3}\right)_0 = 0 \quad \& \quad \left(\frac{dw}{dx}\right)_0 = 0$

which gives $C_3 = -C_4 = -\frac{\alpha a}{\beta l}(T_l - T_c)$ substituting gives

At x | At x = 0

$$Q_x = -D\frac{d^3w}{dx^3} = -\frac{4\alpha a\beta^2 D}{l}(T_l - T_c)e^{-\beta x}\sin\beta x \qquad Q_0 = 0$$

$$N_\phi = -\frac{Ehw}{a} + \sigma_\phi h = \frac{\alpha Eh}{\beta l}(T_l - T_c)e^{-\beta x}(\cos\beta x - \sin\beta x) \qquad N_{\phi 0} = \frac{\alpha Eh}{\beta l}(T_l - T_c)$$

$$M_x = -D\frac{d^2w}{dx^2} = -\frac{2\alpha a D}{l}(T_l - T_c)\left\{\frac{1}{l} - \beta e^{-\beta x}(\cos\beta x + \sin\beta x)\right\} \qquad M_0 = -\frac{2\alpha a D}{l^2}(1 - \beta l)(T_l - T_c)$$

$$w = -\alpha a\left\{\frac{(T_l - T_c)}{\beta l}e^{-\beta x}(\cos\beta x - \sin\beta x) + T_l - (T_l - T_c)\left(1 - \frac{x}{l}\right)^2\right\} \qquad w_0 = -\frac{\alpha a}{\beta l}\left\{T_l - T_c(1 - \beta l)\right\}$$

The above will apply to the shroud which has a parabolic temperature distribution. The outer shell however is at constant temperature and so we have: $w_0 = -\alpha a T$ & $M_0 = N_{\phi 0} = 0$

c) Stresses in a frame due to temperature. From reference[3], pages 364-366 the basic equations can be modified as follows:

$$\sigma_r = \frac{C_1}{2} + \frac{C_2}{r^2} - \frac{\alpha E}{r^2}\int_a^r Tr\,dr \qquad \sigma_r = 0 \ @\ r = 0 \ \&\ r = b$$

Hence $C_1 = \dfrac{2\alpha E}{(b^2 - a^2)}\displaystyle\int_a^b Tr\,dr$ $\qquad C_2 = -\dfrac{\alpha E a^2}{(b^2 - a^2)}\displaystyle\int_a^b Tr\,dr$

and $$\sigma_r = \alpha E\left\{\frac{(r^2 - a^2)}{r^2(b^2 - a^2)}\int_a^b Tr\,dr - \frac{1}{r^2}\int_a^r Tr\,dr\right\}$$

We can then show that $$\sigma_\theta' = \alpha E\left\{\frac{\nu T}{(1-\nu)} + \frac{1}{r^2}\int_a^r Tr\,dr + \frac{(r^2 + a^2)}{r^2(b^2 - a^2)}\int_a^b Tr\,dr\right\}$$

and $$\sigma_\theta = \alpha E\left\{-T + \frac{1}{r^2}\int_a^r Tr\,dr + \frac{(r^2 - a^2)}{r^2(b^2 - a^2)}\int_a^b Tr\,dr\right\}$$

But $w = \dfrac{r}{E}(\sigma_\theta - \nu\sigma_r) + r\alpha T$ $\quad \therefore\ w = \alpha r\left\{\dfrac{(1+\nu)}{r^2}\displaystyle\int_a^r Tr\,dr + \dfrac{[r^2(1-\nu) + a^2(1+\nu)]}{r^2(b^2 - a^2)}\displaystyle\int_a^b Tr\,dr\right\}$

Thus $w_i = \dfrac{2\alpha a}{(b^2 - a^2)}\displaystyle\int_a^b Tr\,dr$ $\quad \&\quad w_0 = \dfrac{2\alpha b}{(b^2 - a^2)}\displaystyle\int_a^b Tr\,dr$

d) Stresses in a frame due to uniform pressure on inner and outer surfaces.

Reference[3] pages 55-57 give the basic equations

$$\sigma_r = \frac{a^2 b^2 (P_o - P_i)}{r^2 h (b^2 - a^2)} + \frac{(P_i a^2 - P_o b^2)}{h (b^2 - a^2)}$$

$$\sigma_\theta = \frac{-a^2 b^2 (P_o - P_i)}{r^2 h (b^2 - a^2)} + \frac{(P_i a^2 - P_o b^2)}{h (b^2 - a^2)}$$

As $w = \frac{r}{E}(\sigma_\theta - \nu \sigma_r)$

$$w = -\frac{r}{E}\left\{ \frac{a^2 b^2 (P_o - P_i)(1+\nu)}{r^2 h (b^2 - a^2)} - \frac{(P_i a^2 - P_o b^2)(1-\nu)}{h (b^2 - a^2)} \right\}$$

Thus

$$w_i = \frac{-a}{E h (b^2 - a^2)}\left\{ b^2 (1+\nu)(P_o - P_i) - (1-\nu)(P_i a^2 - P_o b^2) \right\}$$

$$w_o = \frac{-b}{E h (b^2 - a^2)}\left\{ a^2 (1+\nu)(P_o - P_i) - (1-\nu)(P_i a^2 - P_o b^2) \right\}$$

From the equation for the displacements of the shells and the former, we can determine the unknown pressures, see Figure 4. After this, the computation of the stresses is simply a matter of routine.

Displacement of outer shell $= W_{co} = -\left\{ \frac{P_o}{8\beta_o^3 D_o} + \alpha_o b T_o \right\}$

Displacement of shroud $= W_{ci} = \left\{ \frac{P_i}{8\beta_i^3 D_i} - \frac{\alpha_i a}{\beta_i l}\left[T_l - T_c (1 - \beta_i l) \right] \right\}$

Displacements of former;

Outer $$W_{fo} = \frac{b}{E_f h_f (b^2 - a^2)}\left\{ a^2 (1+\nu_f)(P_o - P_i) - (1-\nu_f)(P_i a^2 - P_o b^2) \right\} - \frac{2\alpha_f b}{(b^2 - a^2)} \int_a^b T_f r \, dr$$

Inner $$W_{fi} = \frac{a}{E_f h_f (b^2 - a^2)}\left\{ b^2 (1+\nu_f)(P_o - P_i) - (1-\nu_f)(P_i a^2 - P_o b^2) \right\} - \frac{2\alpha_f a}{(b^2 - a^2)} \int_a^b T_f r \, dr$$

Equating the displacements gives $W_{co} = W_{fo}$ & $W_{ci} = W_{fi}$

Note that the sign of the former displacements has been changed to agree with the convention used for the shells.

Note:

This method can readily be extended to the case of internal pressure on the shroud, as follows:

$$W_{co} = -\left\{\frac{P_o}{8\beta_o^3 D_o}\right\} \qquad W_{ci} = \left\{\frac{P_i}{8\beta_i^3 D_i} - \frac{p a^2}{E_i h_i}\right\}$$

where p = internal pressure (p.s.i)

The former displacements are as given on page D-5.

The final hoop stress of the shroud is the algebraic sum of the pressure stress and the restraint load stress. Therefore, shroud total hoop stress = $\left(\frac{p a}{h_i} + N_\varphi\right)$

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3. Timoshenko, S. — Theory of Elasticity - 1st edition - McGraw - Hill Book Company, 1934.

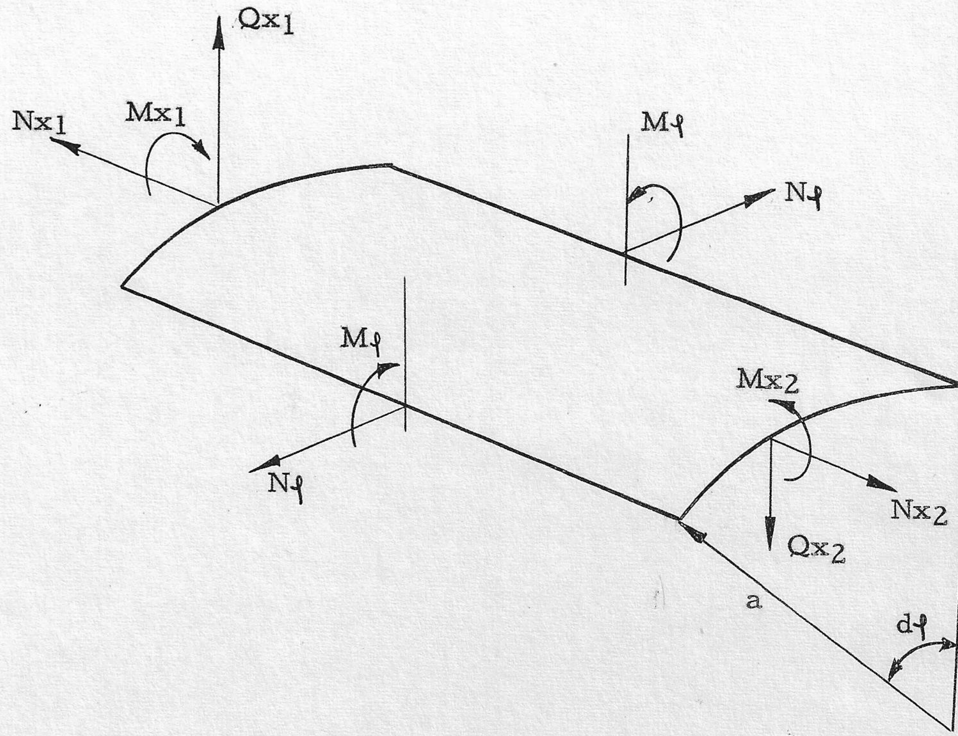

FIGURE 1 - INTERNAL FORCES ON AN ELEMENT OF THE CYLINDER

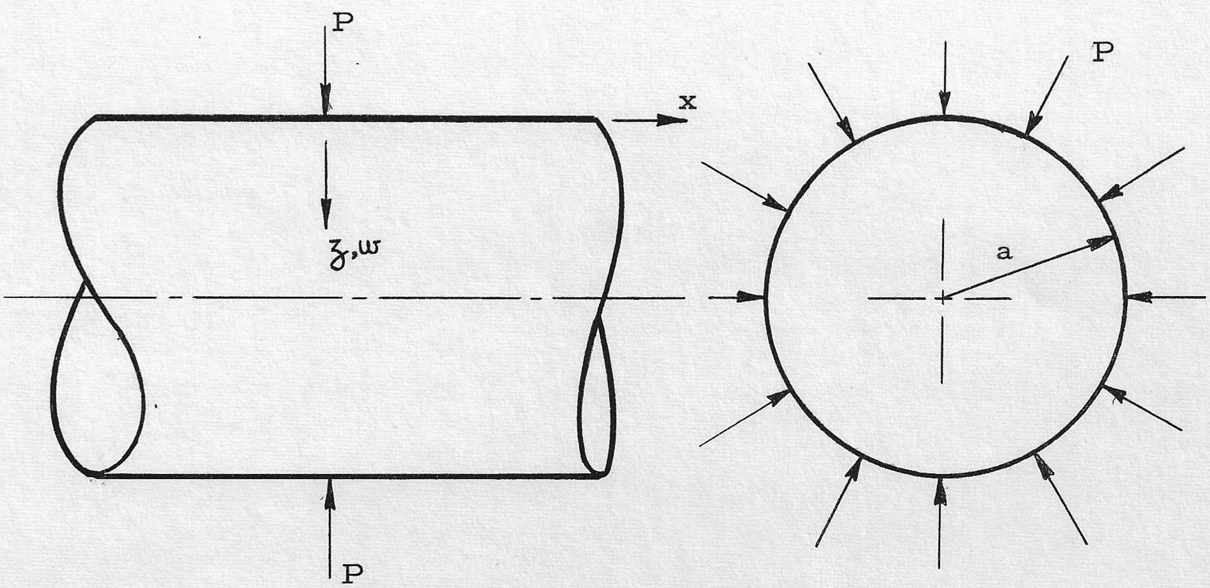

FIGURE 2 - EXTERNAL FORCES ON THE CYLINDER

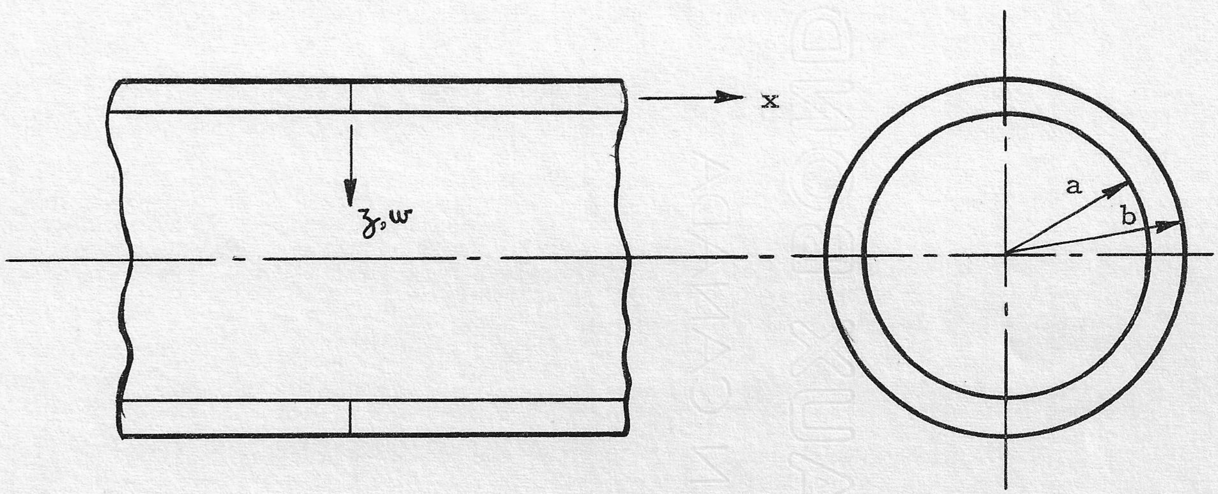

FIGURE 3 - GEOMETRY OF ENGINE BAY STRUCTURE

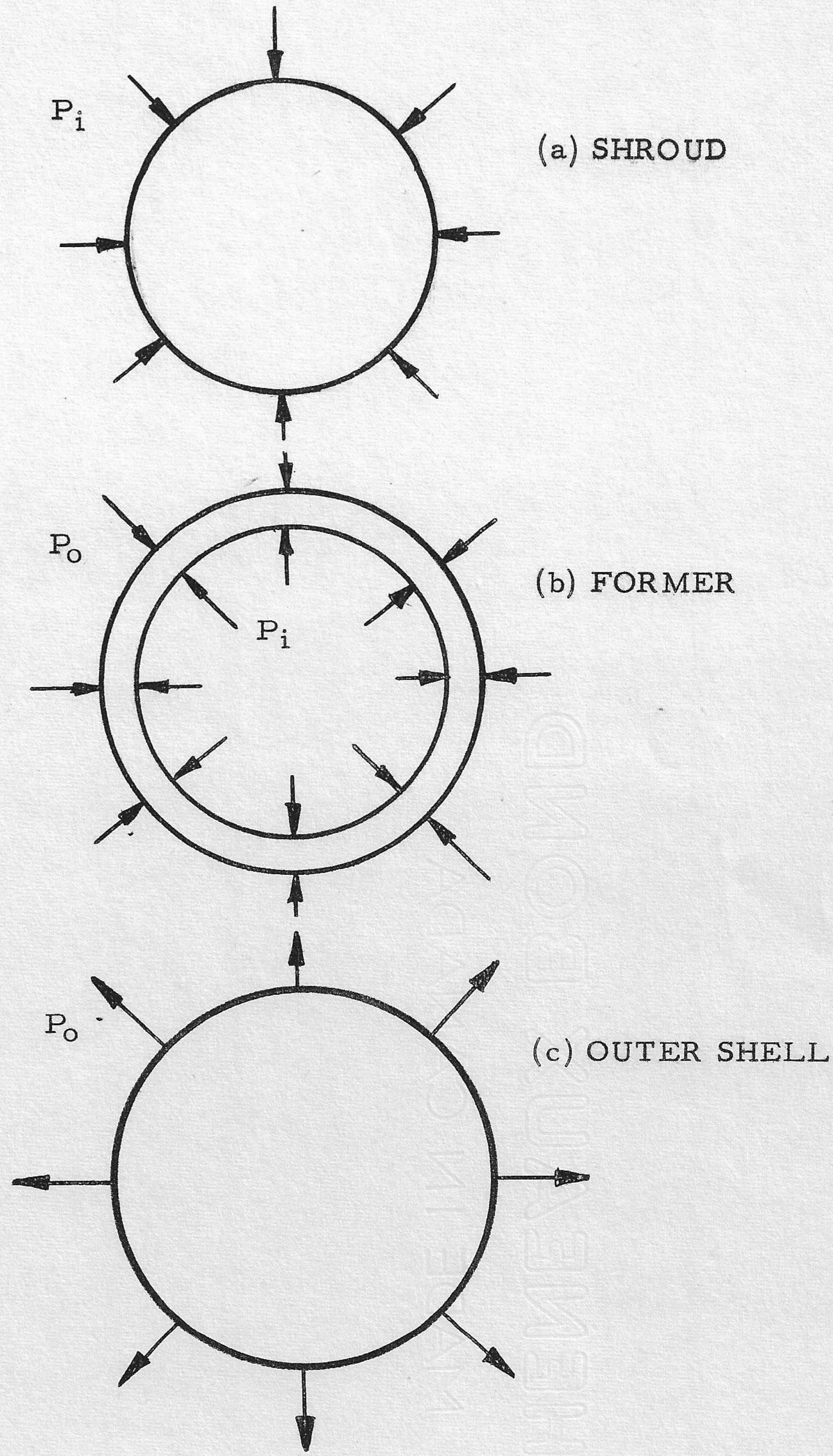

FIGURE 4 - EXTERNAL FORCES ON STRUCTURE COMPONENTS

AVRO AIRCRAFT

ERRATA

Please include the following corrections to the paper.

Page	Paragraph	Line	Original	Corrected
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11	1	5	antielastic	anticlastic
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28	2	2	the error, due to	the error due to
35				(Replace by new sheet)
C5				(Replace by new sheet)
D1	1	3	Reference	Reference 1

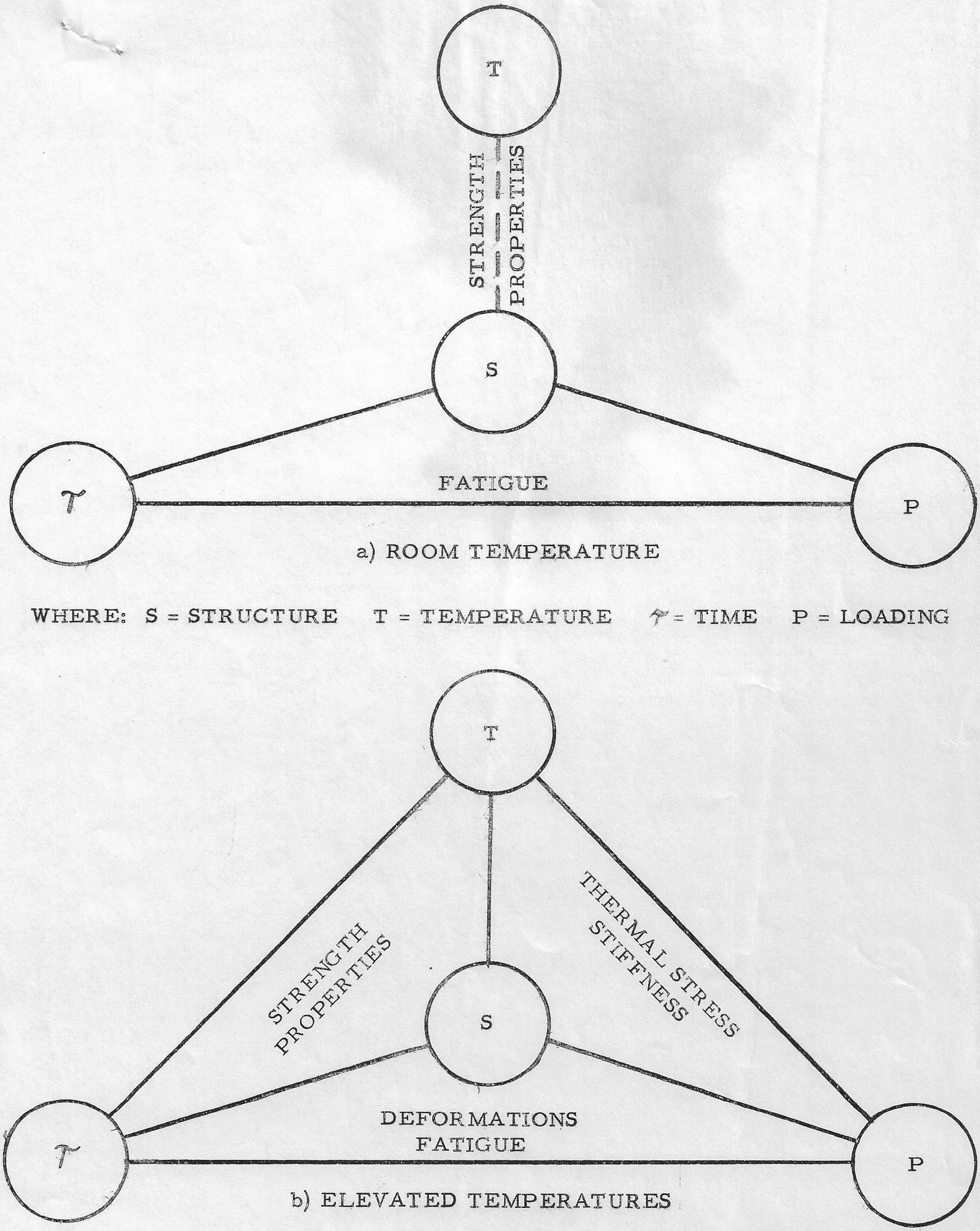

FIGURE 3 - INTERACTIONS ON THE STRUCTURE

Another equation of the form of (9) can be written expressing the relationship between θ_3 , θ_β and (a - r)

$$(\theta_\beta - \theta_3) = \frac{(a-r)}{224K}\left\{\frac{2(a-r)}{3}(136\dot{\theta}_3) - \dot{r}(136\theta_3 + 39\theta_\beta)\right\} \tag{13}$$

Hence the three equations (9), (12) and (13) can be solved simultaneously to give the time variations of θ_1 , θ_3 and r starting the solution with

$$\left.\begin{aligned} \theta_3 &= 0 \\ r &= a/2 \\ \theta_1 &= \theta_\infty\left(1 - e^{-t/(1+\nu)\tau}\right) \end{aligned}\right\} \quad @\ t = t_1$$

For times greater than t_2 the value of θ_2 will no longer be given by (10)

Let us define θ_4 as the value shroud temperature would assume in the absence of formers.

Thus

$$\theta_4 = \theta_2 \qquad t \leq t_2$$

$$\theta_4(t) = \theta_\infty\left(1 - e^{-t/\tau}\right) \qquad t > t_2$$

Further we can apply the result of (9) to the shroud temperature decrease below θ_4 hence.

$$(\theta_4 - \theta_1) - (\theta_4 - \theta_2) = \frac{l^2}{336K}\left\{136\frac{d}{dt}(\theta_4 - \theta_2) + 32\frac{d}{dt}(\theta_4 - \theta_1)\right\}$$

and expressing l^2/K in terms of the transit time this becomes

$$(\theta_2 - \theta_1) = \frac{7t_2}{26}\left(21\dot{\theta}_4 - 17\dot{\theta}_2 - 4\dot{\theta}_1\right) \tag{14}$$

The heat balance equation now becomes

$$\int_0^l H\,dy = k\rho c\int_0^l \theta_s\,dy + k_1\rho c\int_0^r \theta_F\,dy$$

and following a similar approach to that taken in deriving equation (12) gives

$$(\theta_\infty - \theta_1) = \tau\left(1 + \frac{\gamma r}{l}\right)\dot{\theta}_1 + \frac{2\tau\gamma r}{l}\dot{\theta}_3 + \frac{\tau\gamma\dot{r}}{l}(2\theta_3 + \theta_1) + 2\left\{\tau\dot{\theta}_2 - (\theta_\infty - \theta_2)\right\} \tag{15}$$