GEN/ELASTICS/2

# THE MATRIX FORCE METHOD

# OF STRUCTURAL ANALYSIS

B. Hampton Feb. 1959

AVRO AIRCRAFT LIMITED

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THE MATRIX FORCE METHOD OF STRUCTURAL ANALYSIS

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SUMMARY

This report describes the method of structural analysis developed by J.H. Argyris. A scheme of analysis is presented with the object of making the analysis as automatic as possible. This can be achieved once the structure has been idealized and the redundancies selected.

## INTRODUCTION

This report is an introduction to and description of the method of structural analysis developed by J.H. Argyris.

The matrix force method employs the concept of redundancy systems in preferance to numerous individual redundant forces in order to facilitate matrix preparation and checking. The laws involved are simply the laws of statics and the Unit Load Theorem. This latter theorem expresses kinematic relations between strains and displacements by means of statically equivalent stresses, i.e. stresses satisfy the equilibrium conditions but not necessarily compatibility conditions. The theorem can be derived without recourse to strain energy or work and is valid for non-linear elastic structures provided static equilibrium is observed.

The results that can be achieved by this method include internal stresses, deflections in the direction of the applied loads, thermal stresses and deflections and modification (i.e. cut-out or geometry change) stresses and deflections. The procedure for analysing modifications is such that there is no need to prepare additional matrices embracing the whole of the modified structure. All that need be obtained are matrices pertaining to the elements involved in the modification and these can be used in conjunction with the matrices for the original unmodified structure.

INTRODUCTION (Cont'd)

This report discusses the assumptions involved and their significance with regard to analysing a delta wing structure. The steps to be taken in the analysis are also described. A simple problem is analysed by this method and the results compared with those obtained by Engineers Bending Theory.

ASSUMPTIONS

The structure to be analysed is assumed to consist of a network of members covered by flat sheet panels. In the case of a wing the members would be spars and ribs. Flanges may be assumed to exist at the intersection of webs with cover sheets. It is assumed that the cover sheets are flat between network lines and that they change slope only at the network lines.

All spars and ribs must be orthogonal to one another and must have little taper in plan view or elevation. If $\theta$ is the taper angle or lack of orthogonality of two intersecting members then it must be within a range such that $\cos 2\theta = 1$, $\sin 2\theta = 2\theta$. Although the large oblique angle condition is not allowed in the analysis a reference (App. Mech. Ref. July 1958 p2; Rech. Aero No. 23 1951 p.61) is given that might suggest a sutiable idealization to accommodate this condition.

The need for orthogonality in intersecting members is to validate the assumption that the shear flow in the panel is constant. This in itself is an approximation to a parabolic shear flow distribution which is implicit in a further assumption, namely, that direct stresses vary linearly between node points along the flanges. These assumptions are satisfactory provided the network spacing is sufficiently fine. If the basic structural network of spars and ribs is too coarse, fineness can be achieved by assuming further network lines drawn between the rib and spar lines. It does not

ASSUMPTIONS (Cont'd)

matter that there are no flanges along these lines.

Membrane conditions are assumed for the wing skins, i.e. bending of covers and flanges is excluded in a plane normal to the wing surface. The thick skins of wings with low thickness chord ratios will tend to invalidate this assumption. Poisson's Ratio effects are disregarded.

As the stresses are not known beforehand it is difficult to estimate what effective area should be used for the flanges. The theory was developed on the assumption that the flange areas were known and assumed constant between nodal points. Web stiffness being allowed for by adding 1/6 of its area to the flange area. However, a reference (Airc. Eng. March 55 P.87) is given to a method that allows for effective flange areas especially at discontinuities and at the root where stress distribution is difficult to estimate and Poisson's Ratio effects were pronounced. This refinement is known as the L-matrix technique.

It is assumed that the structure can be idealized to such an extent that the following redundancy systems are applicable. See Figs. 41, 42, 43, 44.

1. Rectangular stiffened panel. Type X = 1
2. Multi-web wing. Type X = 1
3. Multi-web wing. Longitudinal 4-boom tube. Type Y = 1
4. Multi-web wing. Transverse 4-boom tube. Type Z = 1

ASSUMPTIONS (Cont'd)

Diagrams are given of the arrangement of each type and of the load in each element due to unit applied loads to the systems.

NOTATION

b: Rectangular transformation matrix for stresses.

R: Column matrix of applied forces.

S: Column matrix of stressing loads on structural elements.

f: Flexibility matrix of unassembled structural elements.

F: Flexibility matrix of complete structure.

r: Column matrix of displacements.

X, Y, Z: Redundancy Systems.

$D = b_i' \cdot f \cdot b_i$

$D_o = b_i' \cdot f \cdot b_o$

$F_o = b_o' \cdot f \cdot b_o$

$b_o$: Rectangular transformation matrix for stresses in basic system.

$b_i$: Rectangular transformation matrix for stresses in redundancy system.

H: Column matrix of initial strains.

$\alpha$: Coefficient of thermal expansion.

$\theta$: Temperature.

$X_\theta = -D^{-1} \cdot b_{i_{La}} \cdot H$

Suffices

L, f, w, r and s: relate to elements.

m: Modification matrices.

c: Cut-out matrices.

Suffices (Cont'd)

$L_a$: Initial strain matrices.

e: Thermal or initial strain matrices.

g: Original elements of modified structure.

h: Removed or altered elements in modified structure.

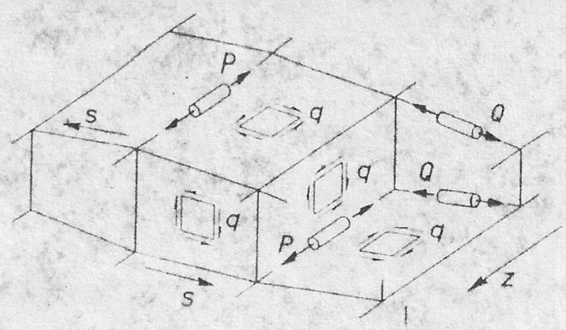
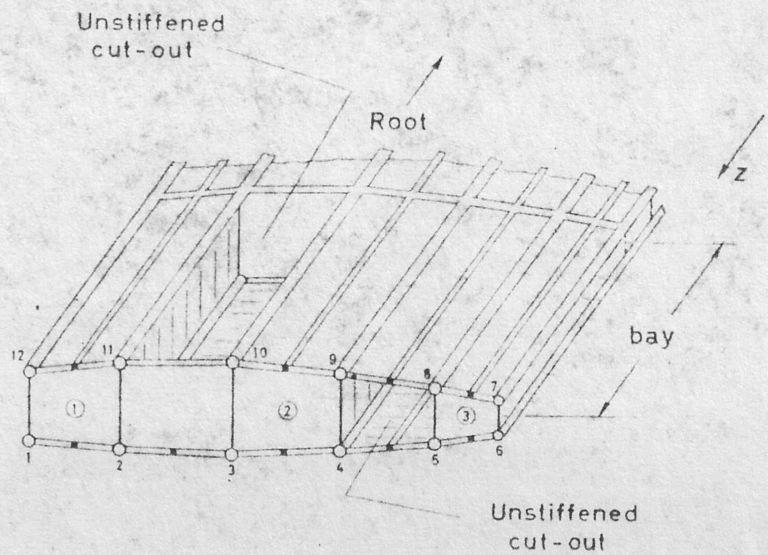



Fig. 38.—Sign convention for flange loads and shear flows




○ Longitudinal flanges continuous at junction } $\beta = 12$

• Longitudinal flanges interrupted at junction

N = 3

Fig. 39.—Geometry of typical bay for determination of number of redundancies

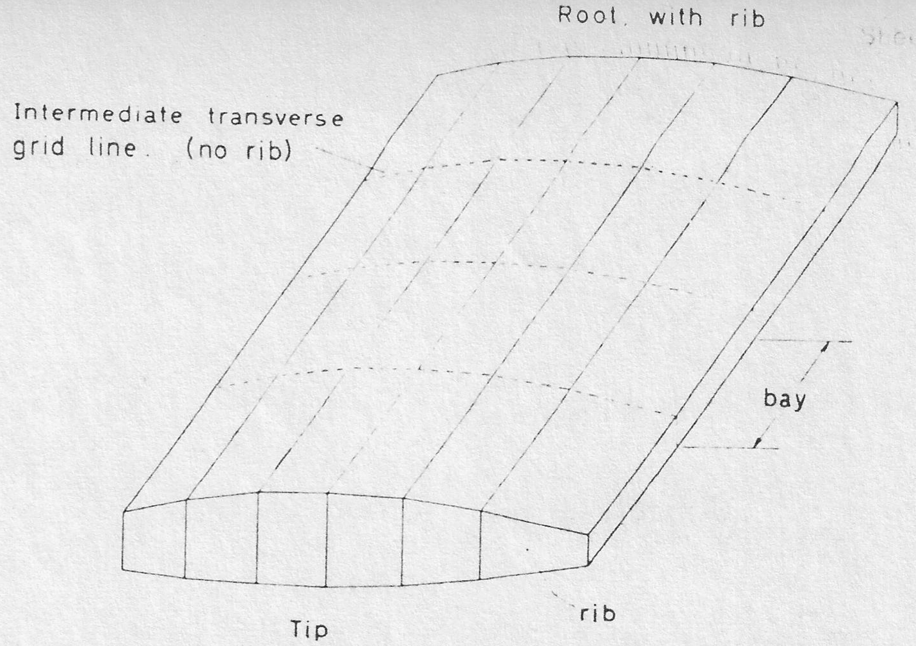



**Fig. 40.—Multi-web wing without intermediate ribs**

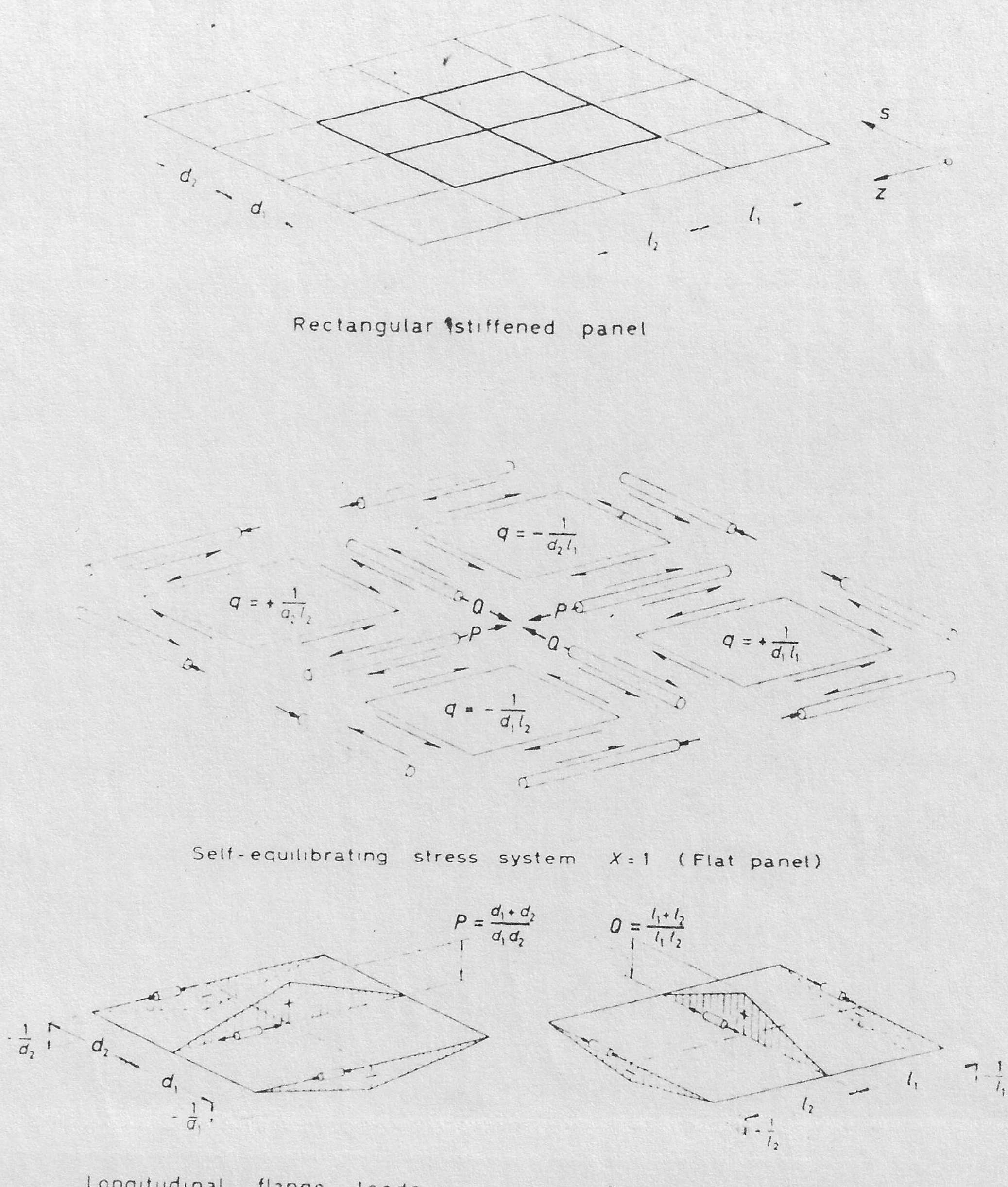



**Fig. 41.—Rectangular stiffened panel. Unit self-equilibrating stress system of type $X$ 1**

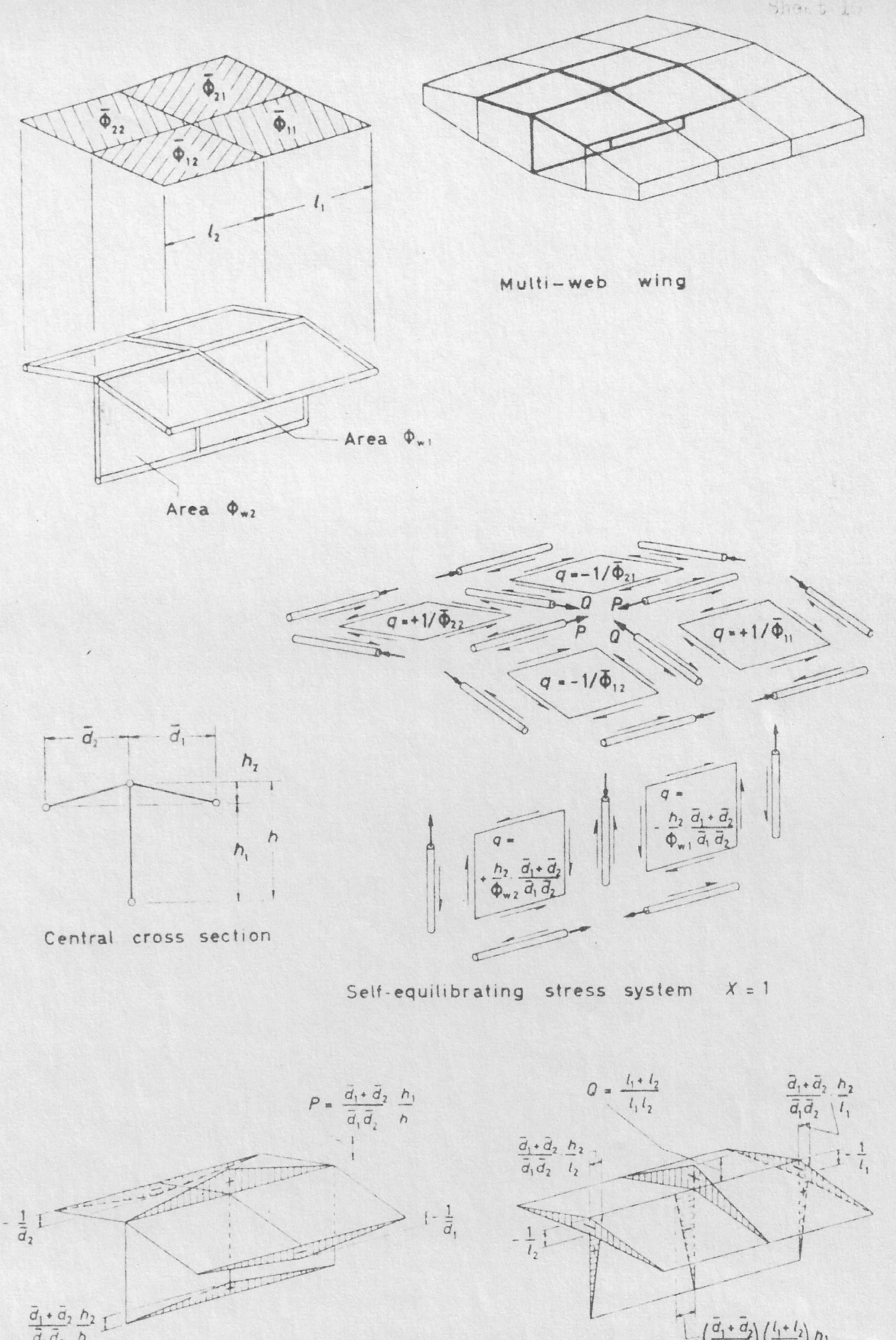
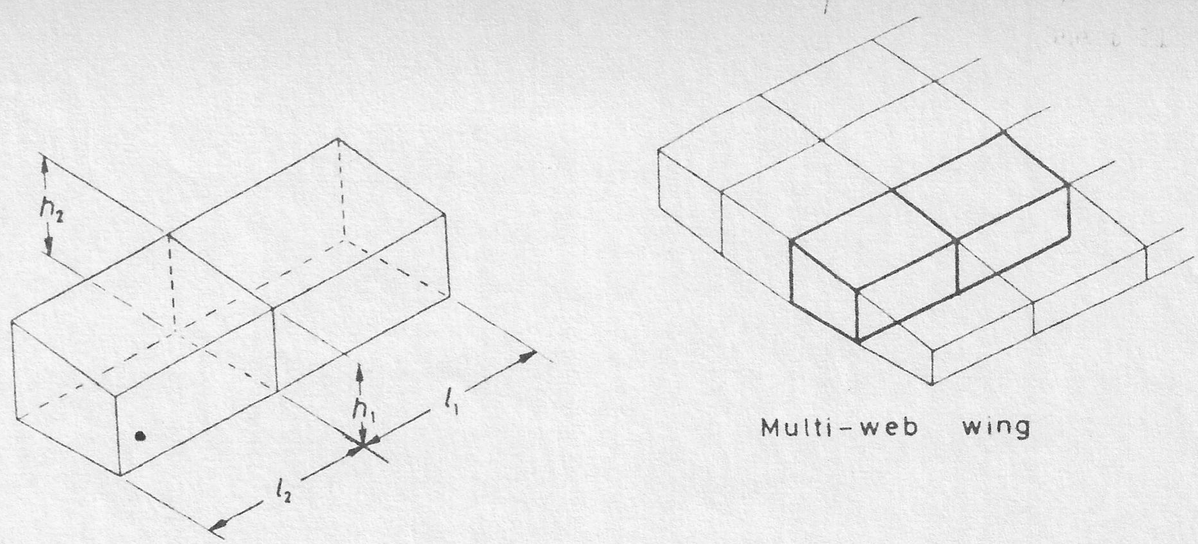
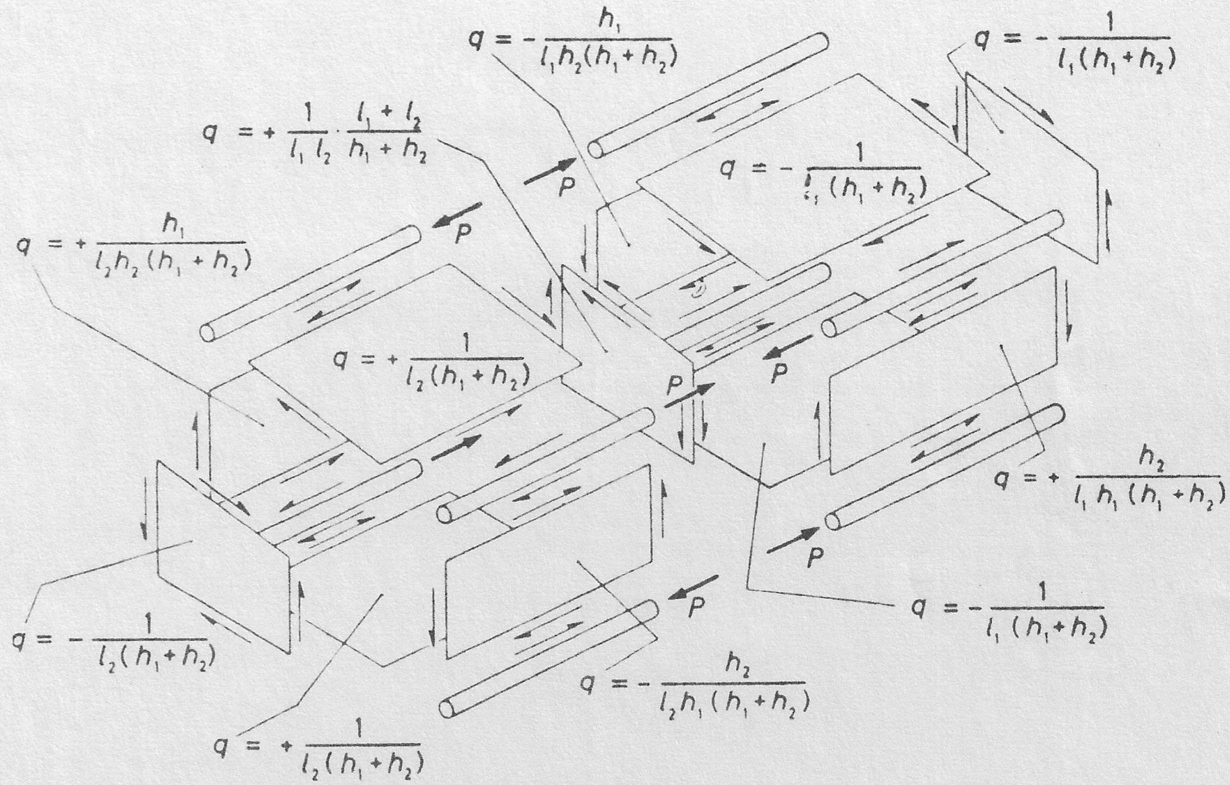


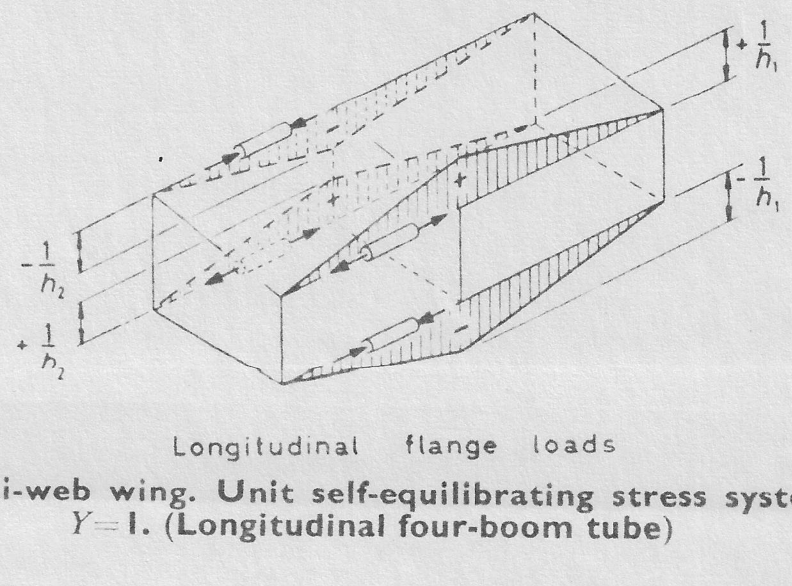

**Fig. 42.—Multi-web wing. Unit self-equilibrating stress system of type $X = 1$**




Multi-web wing

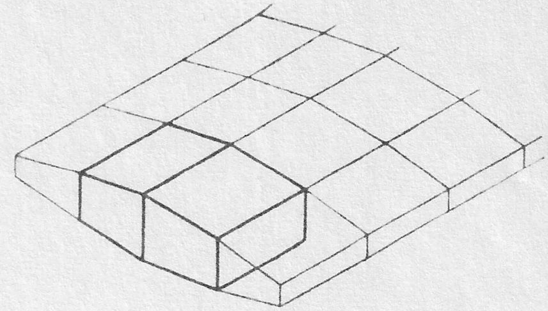



Self-equilibrating stress system $Y = 1$

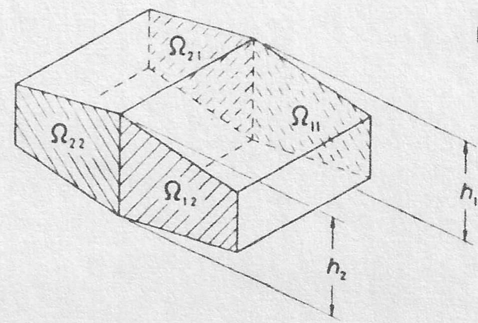


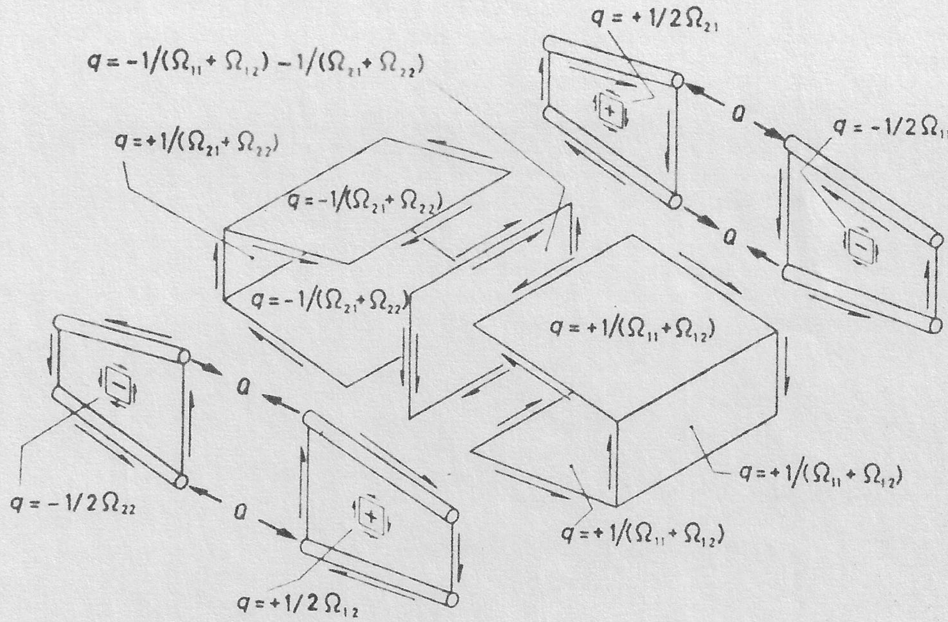

Longitudinal flange loads

**Fig. 43.—Multi-web wing. Unit self-equilibrating stress system of type $Y = 1$. (Longitudinal four-boom tube)**

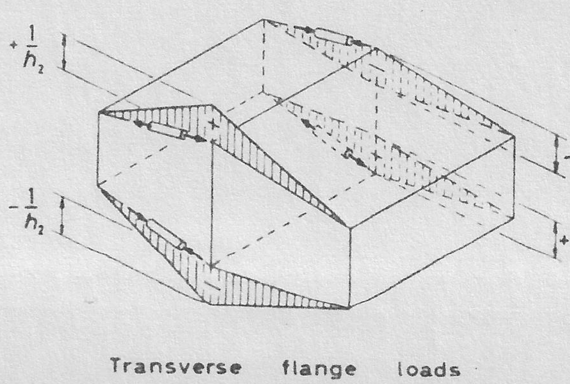


Multi-web wing




Self-equilibrating stress system $Z = 1$




Transverse flange loads

**Fig. 44.—Multi-web wing. Unit self-equilibrating stress system of type $Z = 1$. (Transverse four-boom tube)**

METHOD OF ANALYSIS

The first step in the anlysis is the idealization of the structure. The actual structure should be approximated to a network of orthogonal lines with cover plates and webs occupying the spaces between the lines. The network can be made sufficiently fine by assuming the existence of additional lines. These additional lines need not correspond to flange areas, which may be zero, but will increase the number of shear fields and axial elements by geometrical subdivision.

Out of the idealized structure two systems are obtained; the basic system and the system comprising the redundancies. The basic system must be statically determinate and should be selected in a form that permits ease of calculation. It must also be capable of supporting unit loads at the points of application. For example, in a multi-spar wing where the loads are applied at the intersection of spars and ribs, the spars above would form a satisfactory basic system. They would be a system of cantilevers supporting the unit loads, each cantilever being independent of the remainder. From shear and bending considerations the loads in each element comprising the cantilevers can be determined easily andthe first matrix $b_o$ formed.

Before the redundancy system can be derived the number of redundancies $\eta$ must be evaluated. This is achieved by use of the formula.

METHOD OF ANALYSIS (Cont'd)

$$\eta = \varphi.(B + N - 4)$$

where $\varphi$ = number of bays

B = Number of longitudinal effective flanges which are continuous across the junction i.e. are not interrupted there.

N = Number of closed cells that are stiffened by ribs at the tip end of the bay. See Fig. 39.

Having obtained the degree of redundancy the number and types of redundancy systems can be chosen. The three types of system are referred to in the assumptions as X-, Y- and Z- systems. If there is any choice available Y- systems are preferable because of their better conditioned equations. Whether or not there is a choice of redundancies that can be used will depend on the idealized form of the structure. The systems chosen will have to represent the structure and its form may eliminate some of the redundancy systems. For example, in a single torque tube X and Z systems are inapplicable. In a 2-cell tube of trapezoidal section Z- systems would not apply but X and Y systems would apply. Overlapping may be necessary, i.e. more than one system, or type of system, relating to a portion of the structure. In general it is best to take as many as possible of each type of system. This reduces overlapping to a minimum.

METHOD OF ANALYSIS (Cont'd)

Each of the described systems has been analysed for the equilibrium loads on its elements due to the system as a whole being a unit self-equilibrating system. This enables the redundancy matrix $b_i$ to be calculated.

The flexibility matrix, f, is formed as a result of application of the Unit Load Theorem to the basic elements i.e. the field under constant shear and the flange under linearly varying strain (see Appendix I). Care has to be taken when forming flexibility matrices that where a redundancy system overlaps an axis of symmetry, e.g. the wing centre line, only half the relevant areas are used i.e. half the centre rib flange and web cross-sectional area.

Each of the matrices $b_o$, $b_i$ and f contain groups of submatrices. Each submatrix relates to a group of elements. Using Argyris notation the groups of elements are denoted by the following suffixes: Spar flanges L; Rib flanges t ; Spar webs w; rib webs r and cover panels s.

Having obtained these three basic matrices the deflections and stresses follow from the equations ; (for derivation see Appendix II)

$$D = D_L + D_w + \text{etc.}, = (b'_{iL}.f_L.b_{iL}) + (b'_{iw}.f_w.b_{iw}) + \ldots\ldots \quad (1)$$

$$D_o = D_{oL} + D_{ow} + \text{etc.}, = (b'_{iL}.f_L.\ b_{oL}) + (b'_{iw}.f_w.b_{ow}) + \ldots\ldots \quad (2)$$

$$F_o = F_{oL} + F_{ow} + \text{etc.}, = (b'_{ol}.f_L.b_{oL}) + (b'_{ow}.f_w.b_{ow}) + \ldots\ldots \quad (3)$$

$$b_L = b_{oL} - b_{iL}.D^{-1}.D_o \qquad b_W = b_{ow} - b_{iw}.D^{-1}.D_o \ \text{etc.}, \quad (4)$$

$$S_L = b_L.R \qquad S_W = b_W.R \qquad \text{etc.}, \quad (5)$$

S = Column matrix of the loads in the elements.

R = Column matrix of the applied loads.

$$F = F_o - D'_o.D^{-1}.D_o \quad (6)$$

$$r = F.R \quad (7)$$

= Column matrix of the deflection at the points of application of the applied loads and in the direction of these loads.

MODIFICATIONS AND THERMAL DISTORTIONS

a) Cut-out or geometry changes.

Two matrices are required in addition to the matrices produced for the basic unmodified structure. The first one $b_{i_h}$ is obtained directly from $b_i$ and is simply the extraction of the relevant elements. Hence if $b_i = b_{i_g} + b_{i_h}$ then $b_{i_g}$ are the remaining elements of the original $b_i$ matrix after matrix $b_{i_h}$ has been removed. $b_h$ is a corresponding extraction for the same elements from the original b matrix and $f_h$ is the flexibility of the elements. Whereas f used to be $\begin{pmatrix} f_g & \\ & f_h \end{pmatrix}$ it will become $\begin{pmatrix} f_g & \\ & f_h + \Delta f_h \end{pmatrix}$ under modification. $\Delta f_h$ is then the difference in the flexibilities of the affected elements calculated under the modified and unmodified dimensions. For cut-out analysis $\Delta f_h = o$.

The following equations yield the desired results;

i. Modifications.

$$b_m = b - b_i . D^{-1} . b'_{i_h} . (b_{i_h} . D^{-1} . b'_{i_h} + \Delta f_h{}^{-1})^{-1} . b_h$$

$$F_m = F + b'_h . (b_{i_h} . D^{-1} . b'_{i_h} + \Delta f_h{}^{-1})^{-1} . b_h$$

$$r_m = F_m . R \qquad\qquad S_m = b_m . R$$

ii. Cut-outs.

$$b_c = b - b_i . D^{-1} . b'_{i_h} (b_{i_h} . D^{-1} . b'_{i_h})^{-1} . b_h$$

$$F_c = F + b'_h . (b_{i_h} . D^{-1} . b'_{i_h})^{-1} . b_h$$

$$r_c = F_c . R \qquad S_c = b_c . R$$

When additional modifications are to be made to a structure already modified the procedure is to form matrices $b_{i_h}$ and $b_h$ that pertain to the sum of the modifications for the current structure. These are then used in conjunction with the matrices for the original unmodified structure in the procedure outlined above.

Care has to be taken in forming matrices $b_{i_h}$ and $b_h$ that superfluous terms are not included. For instance, the addition of a cut-out to the original structure will mean the reduction to zero stress of a certain group of elements. It may be, however, that zero stress in one element automatically causes zero stress in another element. e.g. the ends of two flange elements meeting at one point have the same end load at that point. Removal of one flange will automatically imply zero loading at that point in the other flange and there will be no need to include the row for this element in matrices $b_{i_h}$ and $b_h$.

b) Thermal strains or other initial strains.

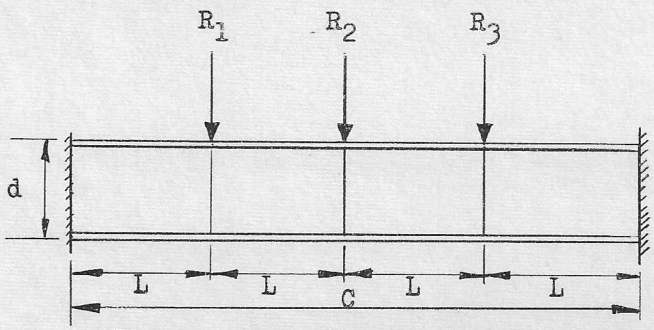
The generalized initial strains are given by $H = L.\alpha\theta$ where the matrix L is comprised of submatrices, $L_1$, $L_2$ etc., (see Appendix I equation (4) ) and $\theta$ is the matrix of temperature. The matrices $L_1$ $L_2$, etc. correspond to each element affected by temperature and are given by application of the unit load theorem.

From H is derived $X_\theta = -D^{-1}.b'_{i_{La}}.H$ where $b_{i_{La}}$ is the submatrix from $b_i$ due to the elements concerned.

Hence stresses are given by $S_\theta = b_i.X_\theta$

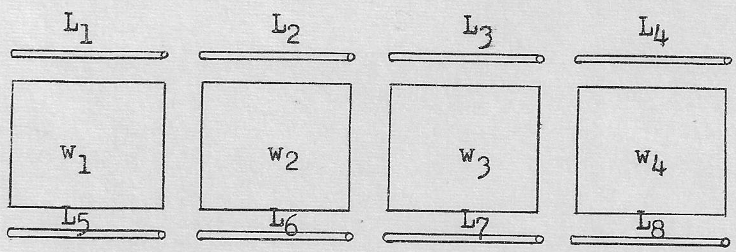
For deflections a sub-matrix $b_{La}$ is found from b using the relative rows and then $r_H = b'_{La}.H$

## ANALYSIS OF SIMPLE BEAM PROBLEM




A = Flange c.s.a.

It is required to determine the deflections at the load points of application and the stresses in each of the elements. Consider the above idealized structure as a combination of a statically determinate system and a redundancy system. Let the statically determinate system consist of a cantilever and then the redundancies will be represented by a fixing moment M and a support reaction R at the right hand side, say.




Locality of elements

Elements L are flanges with linearly varying end load.

Elements W are webs with constant shear fields.

Let span C = 40 ft. L = 10 ft. d= 2 ft. A = 0.025 ft.$^2$

Let t = web thickness = 0.01 ft.

$E = 10^7$ p.s.i. $= 14.4 \times 10^8$ LB/FT$^2$ $G = 4 \times 10^6$ p.s.i. $= 5.76 \times 10^8$ LB/FT$^2$

Formation of Basic Matrices

For the L elements a row will refer to the reaction at one end of the element. In this case the upper row will relate to the left hand end of the element and the lower row to the right hand end. Positive signs indicate tension loads and negative signs compression loads.

$$bo = \begin{bmatrix} b_{o_L} \\ b_{o_W} \end{bmatrix}$$

From $S = bo.R + b_i.X$ -----(Appendix II)

Structural Element

$$b_{o_L} = \frac{L}{d} \times \begin{array}{c} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 2 \\ \hline -1 & -2 & -3 \\ 0 & -1 & -2 \\ \hline 0 & 1 & 2 \\ 0 & 0 & 1 \\ \hline 0 & -1 & -2 \\ 0 & 0 & -1 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \text{---- } L1 \\ \\ L_5 \\ \\ L_2 \\ \\ L_6 \\ \\ L_3 \\ \\ L_7 \\ \\ L_4 \\ \\ L_8 \\ \\ \end{array}$$

The shear matrix $b_{o_W}$ is the change in end load along an element divided by the length of the element and is easily obtained by subtracting one row of figures in $b_{o_L}$ from the other row for each element.

$$b_{o_W} = \frac{1}{d} \times \begin{array}{cc} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} & \underline{\text{Structural Element}} \\ \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] & \begin{array}{c} w_1 \\ w_2 \\ w_3 \\ w_4 \end{array} \end{array}$$

$$b_i = \left[\begin{array}{c} b_{i_L} \\ b_{i_W} \end{array}\right] \qquad \text{From: } S = b_o.R + b_iX \text{ ------(Appendix II)}$$

$$b_{i_L} = \frac{1}{d} \times \begin{array}{cc} \begin{array}{cc} M & R \end{array} & \underline{\text{Structural Element}} \\ \left[\begin{array}{cc} 1 & 4L \\ 1 & 3L \\ \hline -1 & -4L \\ -1 & -3L \\ \hline 1 & 3L \\ 1 & 2L \\ \hline -1 & -3L \\ -1 & -2L \\ \hline 1 & 2L \\ 1 & L \\ \hline -1 & -2L \\ -1 & -L \\ \hline 1 & L \\ 1 & 0 \\ \hline -1 & -L \\ -1 & 0 \end{array}\right] & \begin{array}{c} L_1 \\ \\ L_5 \\ \\ L_2 \\ \\ L_6 \\ \\ L_3 \\ \\ L_7 \\ \\ L_4 \\ \\ L_8 \\ \\ \end{array} \end{array}$$

$$b_{i_w} = \frac{1}{d} \times \begin{array}{cc} M & R \\ \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right. & \left.\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right] \end{array} \quad \begin{array}{c} \underline{\text{Structural Element}} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{array}$$

The flexibility matrix is obtained by using equations (2) and (3) in Appendix 1.

$$f = \begin{bmatrix} f_L & o \\ o & f_W \end{bmatrix}$$

From: $v = f.s.$----- (Appendix II )

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Structural Element

$$f_L = \frac{L}{EA} \times \left[\begin{array}{cccccccccccccccc}
1/3 & 1/6 &&&&&&&&&&&&&& \\
1/6 & 1/3 &&&&&&&&&&&&&& \\ \hline
&& 1/3 & 1/6 &&&&&&&&&&&& \\
&& 1/6 & 1/3 &&&&&&&&&&&& \\ \hline
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&&&&&&&&&&&&&& 1/3 & 1/6 \\
&&&&&&&&&&&&&& 1/6 & 1/3 \\ \hline
\end{array}\right]
\begin{array}{l}
L_1 \\ \\ L_5 \\ \\ L_2 \\ \\ L_6 \\ \\ L_3 \\ \\ L_7 \\ \\ L_4 \\ \\ L_8 \\ \\
\end{array}$$

Structural Element

$$f_W = \frac{L.d}{G.t} \times \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{bmatrix}
\begin{array}{l} w_1 \\ w_2 \\ w_3 \\ w_4 \end{array}$$

The remainder of the work is computational. Values have to be substituted for the constants pre-multiplying each sub-matrix and then these are ready for use in the equations (1) to (7) given in the method of analysis.

This yields the following matrices.

b =

| | | | Structural Element |
|---|---|---|---|
| 2.77 | 2.50 | 0.98 | $L_1$ |
| -1.43 | 0.00 | 0.177 | |
| -2.77 | -2.50 | -0.98 | $L_5$ |
| 1.43 | 0.00 | -0.177 | |
| -1.43 | 0.00 | 0.177 | $L_2$ |
| -0.625 | -2.5 | -0.625 | |
| 1.43 | 0.00 | -0.177 | $L_6$ |
| 0.625 | 2.50 | 0.625 | |
| -0.625 | -2.50 | -0.625 | $L_3$ |
| 0.177 | 0.00 | -1.43 | |
| 0.625 | 2.50 | 0.625 | $L_7$ |
| -0.177 | 0.00 | 1.43 | |
| 0.177 | 0.00 | -1.43 | $L_4$ |
| 0.98 | 2.50 | 2.77 | |
| -0.177 | 0.00 | 1.43 | $L_8$ |
| -0.98 | -2.50 | -2.77 | |
| 0.42 | 0.25 | 0.08 | $w_1$ |
| -0.08 | 0.25 | 0.08 | $w_2$ |
| -0.08 | -0.25 | 0.08 | $w_3$ |
| -0.08 | -0.25 | -0.42 | $w_4$ |

Using this matrix to pre-multiply the load column matrix gives the load in each member.

$$F = 10^{-8} \times \begin{bmatrix} 263 & 274 & 113 \\ 274 & 548 & 274 \\ 113 & 274 & 263 \end{bmatrix}$$

From Engineers Bending Theory with allowance for shear deflection

$$F = 10^{-8} \times \begin{bmatrix} 260 & 275 & 116 \\ 275 & 549 & 275 \\ 116 & 275 & 260 \end{bmatrix}$$

## CONCLUSIONS

The method has been used to analyse a simple beam problem and although it does not illustrate the method's potentiatities completely it serves as an introduction. In the main the procedure is automatic but engineering judgement has to be used in at least two instances. These are:

i. Idealizing the structure to a form that is considered to be a sufficiently fine network of spars and ribs;

ii. Making the most suitable selection of redundancies.

With regard to the former it is inferred that the network of spars and ribs for a thin skinned wing is usually of adequate fineness to represent the wing structure. For decisions of the latter sort there are several indications mentioned in the methods of analysis.

It is expected that on a Ferranti - Pegasus computer the computer time for the analysis of a delta wing with 30 load points and with cut-outs will be less than 50 hours.

## REFERENCES

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APPENDIX I

The Unit Load Theorem and the Element Flexibilities obtained from it.

Unit Load Theorem: $1 . r_j = \int_V \bar{\sigma}_j' . \xi . dv$ - (1)

where $\xi, \bar{\sigma}_j$ are the column matrices,

$\xi = \{\xi xx, \xi yy, \xi zz, \xi xy, \xi yz, \xi zx\}, \bar{\sigma}_j = \{\bar{\sigma} xx, \bar{\sigma} yy, \bar{\sigma} zz, \bar{\sigma} xy, \bar{\sigma} yz, \bar{\sigma} zx\}$

$\xi xx$, etc are the true direct and shear strains in a structure due to a given set of loads, prescribed displacements, thermal strains or any other initial strains.

$\bar{\sigma} xx$, etc, are the direct and shear stresses statically equivalent to a unit load applied at the point and direction j.

$r_j$ is the deformation at the point j due to the above causes.

Element 1: Field with constant shear.

$$f = \frac{ab}{Gt} \qquad - (2)$$

where a.b is the field area and t the thickness; G is the shear modulus.

Element 2: Flange under linearly varying end load.

Let the end loads at the extremities of the element be $S_1$ and $S_2$, then, for the Unit Load Theorem, the generalized displacement is $v = f.S$ where $S = \{S_1 \; S_2\}$

$$f = \begin{bmatrix} \frac{L}{3EA} & \frac{L}{6EA} \\ \frac{L}{6EA} & \frac{L}{3EA} \end{bmatrix} \qquad - (3)$$

APPENDIX I (Cont'd)

Element 3: Flange under linearly varying strain. Let the strains be $\eta_1$ and $\eta_2$ at the extremities of the element, then from the Unit Load Theorem the generalized displacement is

$v = L\eta$ where $\eta = \{\eta_1 \ \eta_2\}$

$$L = \begin{bmatrix} \frac{L}{3} & \frac{L}{6} \\ \frac{L}{6} & \frac{L}{3} \end{bmatrix} \qquad - (4)$$

APPENDIX II

Derivation of Basic Formula.

(1) $v = f.S$ (by definition) Strains v due to a linearly elastic structure under a load system R producing loads S internally

$= f.b.R$ (by definition)

f = Flexibility matrix of the elements

b = Rectangular matrix for stress transformation

(2) $r = \bar{b}'.v$ Unit Load Theorem (matrix form)

$\bar{b}$ indicates compliance of stresses with static equilibrium only i.e. $\bar{b}$ can = $b_o$ or $b_i$

$v_j$ is the strain at the point j

$r_j$ is the deformation at the point j.

Substituting (1) into (2) gives.

APPENDIX II (Cont'd)

(3) $r = \bar{b}'.f.b.R$ $\quad b_i = \bar{b}$ will be used in determining the redundancies X in terms of R;

For the compatibility of the original basic structure the deformation at each redundancy cut must be zero. i.e. $r = o$

and by definition,

(4) $S = b.R = b_oR + b_iX$ $\quad b_o$ = Rectangular matrix for stress transformation of basic structure

$b_i$ = Rectangular matrix for stress transformation of redundancy systems.

Hence $b_i'.f.b_oR + b_i'.f.b_iX = o$

$$\therefore X = -\frac{1}{b_i'.f.b_i}.b_i'.f.b_o.R$$

$$= -D^{-1}.D_o.R$$

Substituting in $S = b_oR + b_iX$

$$= b_oR - b_i.D^{-1}.D_oR$$

(6) i.e. $b = b_o - b_i.D^{-1}.D_o$

As in (3) previously $r = \bar{b}'.f.b.R$

i.e. the flexibility of the system is

APPENDIX II (Cont'd)

(7) $$F = \bar{b}'.f.b.$$

Using $b_o$ for $\bar{b}$, $F = b_o'.f.b.$

But from (6) previously $b = b_o - b_i.D^{-1}.D_o$

(8) Therefore $F = b_o'.f.b_o - b_o'.f.b_i.D^{-1}.D_o$

$$= F_o - D_o'.D^{-1}.D_o \qquad \text{where } F_o = b_o'.f.b_o$$