Model Rocket
ALTITUDE PREDICTION CHARTS
Including
AERODYNAMIC DRAG

by

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Estes Industries, Inc.
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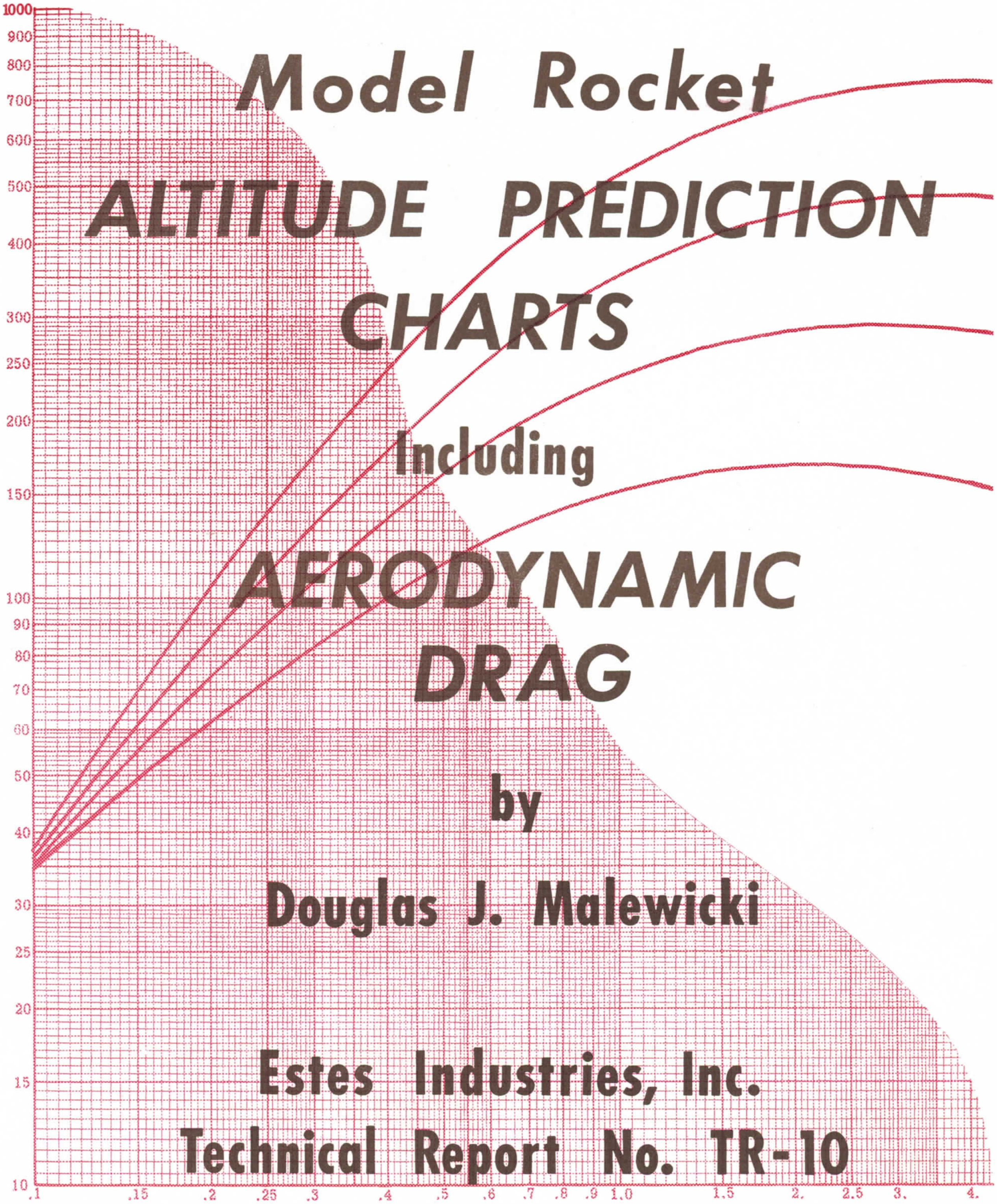

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PREFACE

This report presents a simple method by which aerodynamic drag effects can accurately be taken into account in the prediction of model rocket peak altitudes. With this data, altitudes can be determined for any rocket using any of the Estes motors (including 2 or 3 stage vehicles and cluster-powered birds). In addition, flight times can be easily found so that optimum engine delay times can be selected.

Using this simple method for including aerodynamic drag effects, many interesting experiments and research projects can be initiated which would have previously been too laborious to analyze. Also, a good basic understanding of the principles of aerodynamics and the aerospace sciences can be obtained by performing the simple drag experiments suggested at the end of this report.

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INTRODUCTION

DISCUSSION OF DRAG

The article on Rocket Math which appeared in the June, 1964 issue of Model Rocket News (Volume 4, Number 2) pointed out, among other things, the vast difference between the potential height a model rocket could reach and the much lower height to which it would actually climb due to the flight retarding forces of millions of air molecules that bombard the frontal surface of a model rocket during each second of forward flight.

These millions of impacts, of course, constitute the effect more commonly known as "aerodynamic drag" on a rocket or more simply as just "drag". Drag on any object has been found to follow this law:

$$D = C_D A \frac{1}{2} \rho V^2$$

where

- D is the drag force.
- C_D is a dimensionless "aerodynamic drag coefficient" that depends upon the shape and surface smoothness of the object.
- A is the reference area of the object (for model rockets we use the cross sectional area of the body tube as the reference).
- ρ is the density of the medium through which the object is moving (submarine designers use the density of water in their drag computations, whereas model rocketeers use the density of air). The density symbol "ρ" is the greek letter RHO pronounced ROW.

and

- V is the velocity of the object.

The full importance of drag has yet to be realized by most model rocketeers. It is hoped that through the use of the altitude prediction method presented herein that a more complete understanding of its effects will be acquired by all.

In the June, 1964 Rocket Math article (reference 1) it was pointed out that "typically drag will reduce the peak altitude of a shortened Astron Streak with a 1/2A.8-4S engine aboard from a theoretical drag free 1935 feet to an actual 710 feet". What this simply means is that merely due to the effect of air the altitude has been cut to less than half. The big question is, "Why is this effect so great and what is happening to the rocket to cause such a difference"?

To better understand the effects of the atmospheric forces that retard the forward motion of the rocket (we are talking about "drag" again) let us consider the following example. Suppose we have a one inch diameter rocket traveling at 100 ft/sec. We can determine the number of air molecules which collide with our rocket each second by using Loschmidts number, which is the number of molecules in a cubic centimeter of gas under standard conditions (temperature of 59° F and pressure of 14.7 PSI). Any basic physics or chemistry book gives this number; see reference 2, page 196 for example.

It turns out that during each second of flight time our 100 ft/sec rocket has to push its way through approximately 400,000,000,000,000,000,000,000 molecules that lie in its path. (Scientists and engineers usually write out such large numbers in the form 4×10^{23}, which means 4 followed by 23 zeros).

Even though molecules are very small, this extremely large number of them can add up to a drag force of about 1 ounce acting continuously at a speed of 100 ft/sec. Since drag is proportional to the square of the velocity* we would then have 4 ounces of drag at 200 ft/sec and 9 ounces of drag on the rocket at 300 ft/sec (approximately 200 miles per hour).

Fortunately, the full effect of these molecules sitting in the path of our rocket can be avoided by good streamlining. In this way most of the molecules are compressed in layers which tend to flow smoothly around the nose and body tube as the rocket passes them; relatively few molecules are really rammed head on.

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* The square of a number is obtained by multiplying it times itself.

The aerodynamic drag coefficient (C_D) is a measure of how easily a given shape goes through all those molecules. For example, a cube traveling flat side forward has a C_D of 1.05, a marble has a C_D of .47, a streamlined teardrop shape has a C_D of .05, and a teardrop with 1/3 of its length removed at the trailing end has a C_D of .1 (see reference 3). The product of the drag coefficient (C_D) and the frontal area (A) is commonly referred to as the Drag Form Factor (C_DA) of a body. Bodies with identical drag form factors will have identical drag values at any given speed. Thus, a cube having a one inch length has a frontal area of 1 inch^2 and a drag form factor of

$$C_DA_{Cube} = (1.05)(1 \text{ inch}^2) = 1.05 \text{ inch}^2$$

Our marble would require a diameter of 1.69 inch (its frontal area would then be 2.13 inch^2) in order to have a drag form factor identical to the cube.

$$C_DA_{Marble} = (.47)(2.13 \text{ inch}^2) = 1.05 \text{ inch}^2$$

A streamlined teardrop that would have the same amount of total drag as our 1 inch cube would require a diameter of 5.17 inches, corresponding to an area of 21 square inches, to match the cube's drag form factor.

$$C_DA_{Teardrop} = (.05)(21 \text{ inch}^2) = 1.05 \text{ inch}^2$$

Lastly, in order to have the same amount of drag as the cube our teardrop with 1/3 of its trailing end cut off would require a frontal area of 10.5 inch^2 resulting in a diameter of 3.66 inches.

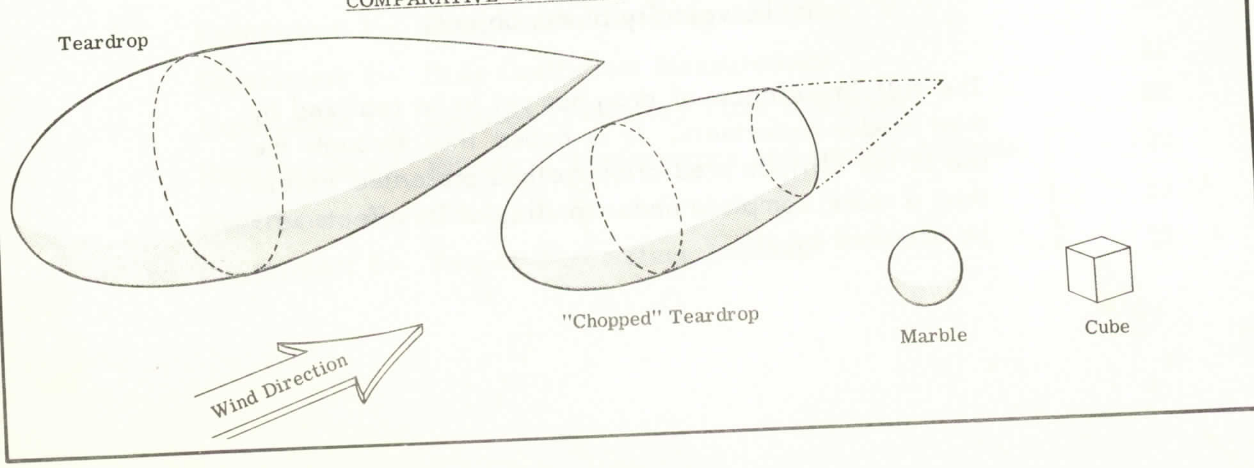
$$C_DA_{\substack{Chopped\\Teardrop}} = (.1)(10.5 \text{ inch}^2) = 1.05 \text{ inch}^2$$

Realizing that the deceivingly small cube has the same amount of drag as the large teardrop should make obvious the importance of streamlining on model rockets.

DISCUSSION OF THE BALLISTIC COEFFICIENT

There are several factors that affect the altitudes our rockets will reach. These are: the drag coefficient (C_D), the frontal area of the rocket (A), the rocket weight (W), the amount of propellant burned off during thrusting (W_P), the type of motor used (B3 down to 1/4A), and even the temperature of the air, and the altitude of the launch site.

COMPARATIVE SIZES OF "EQUAL DRAG" BODIES



The charts of this report were made possible by combining several of these variables into one new variable called the ballistic coefficient β (greek letter BETA) which is the ratio $\frac{W}{C_D A}$ (rocket weight divided by the drag form factor ($C_D A$), where the frontal area (A) is based on the body tube cross section area). The ballistic coefficient β is widely used by aerospace engineers as a trajectory parameter for space vehicles that are re-entering the atmosphere. It is also used by rifle designers who have to determine the best shape and weight of a bullet to use to obtain a given range and striking power.

A spacecraft with a high ballistic coefficient will come through the atmosphere faster than one with a low ballistic coefficient for a given set of initial re-entry conditions. A rifle bullet with a high ballistic coefficient will not slow down as fast as one with a lower ballistic coefficient. Using the charts of this report you will soon discover that for a given rocket weight and motor type that our models will reach higher altitudes with high ballistic coefficients as compared to lower ones.

SAMPLE PROBLEM

Assume we have a rocket that 1) weighs 1 ounce including motor, 2) is powered by an A.8-4 motor, and 3) uses a BT-20 body tube. For the present an aerodynamic drag coefficient of C_D = .75 shall be used. G. H. Stine's Handbook of Model Rocketry (Reference 4) presents this value on page 94. He also explains why it is considered accurate and valid. Experiment 2 on Drag Coefficient Measurement will enable you to make your own decisions as to how valid a C_D of .75 is.*

Using the C_D of .75 and a BT-20 body tube we look at Figure 1 to find that the drag form factor $C_D A$ = .32. The next step is to compute the ballistic coefficient to be used during powered flight. During the thrusting phase the average weight of the rocket should be used. The average weight, of course, equals the initial weight (W_I) minus one half the propellant weight (1/2 W_P). Figure 5 contains all the data for A type motors and states the value of 1/2 W_P as .0675 ounces. Thus, the ballistic coefficient to be used during thrusting is:

$$\beta = \frac{W_I - W_P}{C_D A} = \frac{1.0 - .0675}{.32}$$

$$\beta = \frac{.9325}{.32} = 2.92 \ \frac{\text{ounces}}{\text{inch}^2}$$

From Figure 5 we find that for a ballistic coefficient (β) of 2.92 and an initial weight (W_I) of 1.0 ounce that

Burnout Altitude (S_B) = 150 feet

and:

Burnout Velocity (V_B) = 310 Ft. per Sec.

During coasting we should use the empty weight of the rocket (initial weight (W_I) minus all the propellant (W_P)) to determine a more realistic ballistic coefficient (β). This will have a value

$$\beta = \frac{W_I - W_P}{C_D A} = \frac{1.0 - .135}{.32}$$

$$\beta = \frac{.865}{.32} = 2.71 \ \frac{\text{ounces}}{\text{inch}^2}$$

Figure 8 has all coasting data. Note that the coasting plots are independent of motor type, that is, Figure 8 is good for all coasting model rockets. Thus, for a coasting ballistic coefficient of β = 2.71 and a burnout velocity of V_B = 310 ft/sec we find that

Coast Altitude (S_C) = 550 feet

and:

Coast Time (t_c) = 5.0 Sec.
(From burnout to maximum height)

Note that ejection would have occurred at 4 seconds, however.

The total altitude that a rocket will reach, of course, is the sum of the altitude gained while thrusting (S_B) and the altitude gained during coasting (S_C), or

$$S_{Total} = S_B + S_C = 150 + 550 = 700 \text{ feet}$$

That is all that is involved in making a complete peak altitude calculation.

Does a question arise in your mind about accuracy of the method? Can something this simple be any good? Those of you who have that June, 1964 issue of the Model Rocket News (reference 1) can check that the altitude found for this same identical rocket without considering drag was 2600 feet (greater than a 300% error) and with drag was 750 feet.

You may also wonder why the 1964 calculations showed that an altitude of 750 feet was reached, whereas our new chart method shows it only reached 700 feet. This is due to neglecting in the 1964 calculations, the seemingly small amount of propellant being burned off. If you go through the sample problem again and don't include any weight loss you will find the rocket goes up to 750 feet just as the 1964 calculations said it would.

Now let's take a quick look at some of the effects "streamlining" can have for this same rocket. Suppose we have two people build this same rocket. Bill's is unpainted, unsanded, and the fins are square. Carl's bird, on the other hand, has a beautiful smooth paint job, is fully waxed, and the fins have a nice streamline airfoil shape. For comparison purposes let's assume Bill's aerodynamic drag coefficient has increased 20% over the C_D = .75 we previously used (Bill's C_D now equals .9) and Carl's drag coefficient has been reduced by 20% (Carl's C_D now equals .6).

Going through the same steps of the sample problem you will find Bill's rocket will reach 640 feet while Carl's will reach 790 feet.

As one becomes familiar with the method in this report he finds many more interesting aspects of aerodynamics to consider. It is very easy to investigate the effect on performance of any of the variables by just computing altitudes for various engine types, heavier

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* We also hope that since drag information can now be utilized in a simple way that its new usefulness creates more interest on the subject of drag and that this new interest, in turn, stimulates many more thorough aerodynamics research projects.

and lighter rocket weights, and different values of drag. With this kind of approach one soon gains a real understanding of these previously abstract principles.

"STREAMLINING" - ITS EFFECT ON DRAG		
	BILL	CARL
Drag Coefficient (C_D)	.9	.6
Thrusting Ballistic Coefficient (β)	2.43	3.65
Burnout Altitude (S_B)	148 feet	153 feet
Burnout Velocity (V_B)	304 Ft/Sec	319 Ft/Sec
Coasting Ballistic Coefficient (β)	2.26	3.39
Coast Altitude (S_C)	493 feet	640 feet
Coast Time (t_c)	4.72 Sec.	5.45 Sec.
Total Altitude ($S_B + S_C$)	640 feet	790 feet

CLUSTERED ROCKETS APPLICATION

The charts can also be used for any identical type motors clustered in a stage by making the following modifications:

During thrusting use the V_B and S_B graphs with

$$W_I = \frac{\text{Actual Weight}}{\text{Number of motors in cluster}} = \frac{W_{actual}}{N}$$

and:

$$\beta = \frac{W_{actual} - N(1/2\ W_P)}{C_D A}$$

During coasting use:

$$\beta = \frac{W_{actual} - N(W_P)}{C_D A}$$

Nothing else is required.

As an example, let us assume we have an Astron Cobra powered by a cluster of three B.8-6 motors. It weighs 6 ounces including motors and payload and is made with a BT-60 body tube. Again assume G. H. Stine's C_D of .75 is valid (reference 4).

Now:

$$W_I = \frac{W_{actual}}{\text{Number of motors}} = \frac{6}{3} = 2 \text{ ounces}$$

and

$$\beta = \frac{W_{actual} - N(1/2 W_P)}{C_D A} = \frac{6 - 3\,(.1106)}{1.58}$$

$$\beta = \frac{6 - .3318}{1.58} = \frac{5.6682}{1.58} = 3.58$$

Figure 3 then gives:

$S_B = 175$ feet

$V_B = 233$ Ft/Sec

During coasting the rocket will have:

$$\beta = \frac{W_{actual} - N(W_P)}{C_D A} = \frac{6 - 3\,(.2212)}{1.58}$$

$$\beta = \frac{6 - .6636}{1.58} = \frac{5.3364}{1.58} = 3.37$$

From Figure 8 we find that for V_B = 233 ft/sec and β = 3.37:

$S_C = 450$ feet

$t_c = 4.8$ Sec.

and our total altitude is:

$$S_{total} = S_B + S_C = 175 + 450 = 625 \text{ feet}$$

Note that our choice of six second delay motors would allow the Cobra to be tracked to its peak altitude, whereas using four second delay motors we would have ejection occurring somewhat before the peak altitude was reached--thus spoiling our altitude measurement.

THE EFFECT OF LAUNCH ALTITUDE AND LAUNCH TEMPERATURE VARIATIONS

Our basic drag equation $D = C_D A\ 1/2 \rho\ V^2$ shows that the drag force (D) is among other things proportional to air density (ρ). Density of the air is both a function of altitude and temperature. Figure 9 presents a correction factor (K) for density that is based on the tabular data presented on page 92 of G. H. Stine's Handbook (reference 4). This factor (K) can fortunately be included in the ballistic coefficient (β) as follows:

$$\beta = \frac{W}{C_D A\ K}$$

Altitude calculations are performed just as described before with the exception of this single modification. The reason it can be included right in the ballistic coefficient will become apparent to those who follow the derivation of the motion equations.

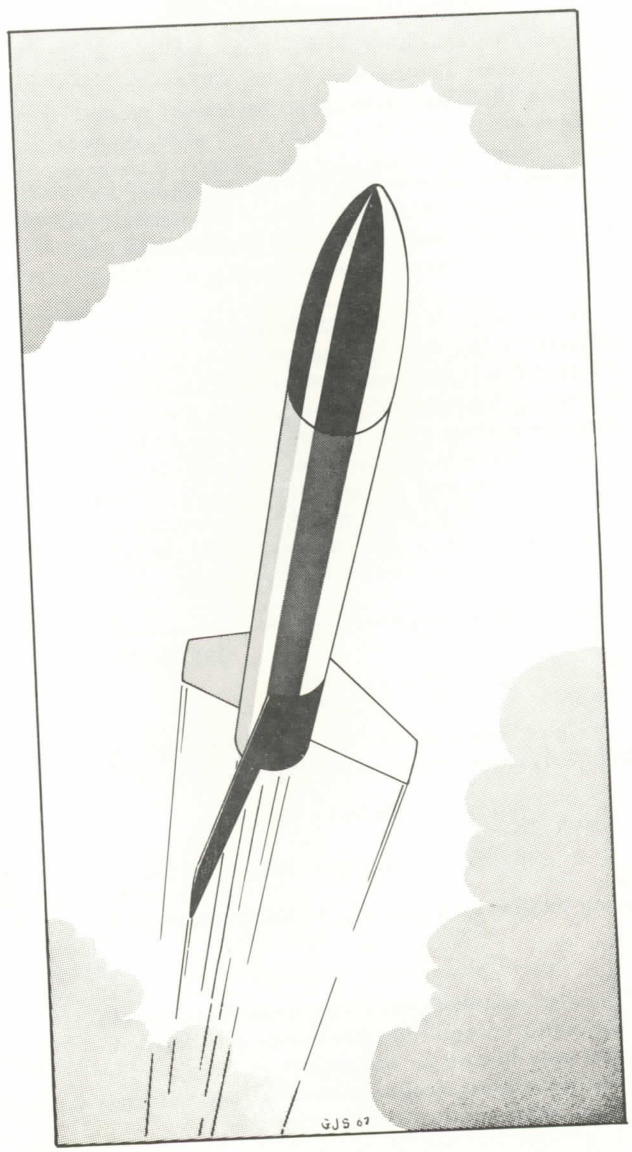


FIGURE 1

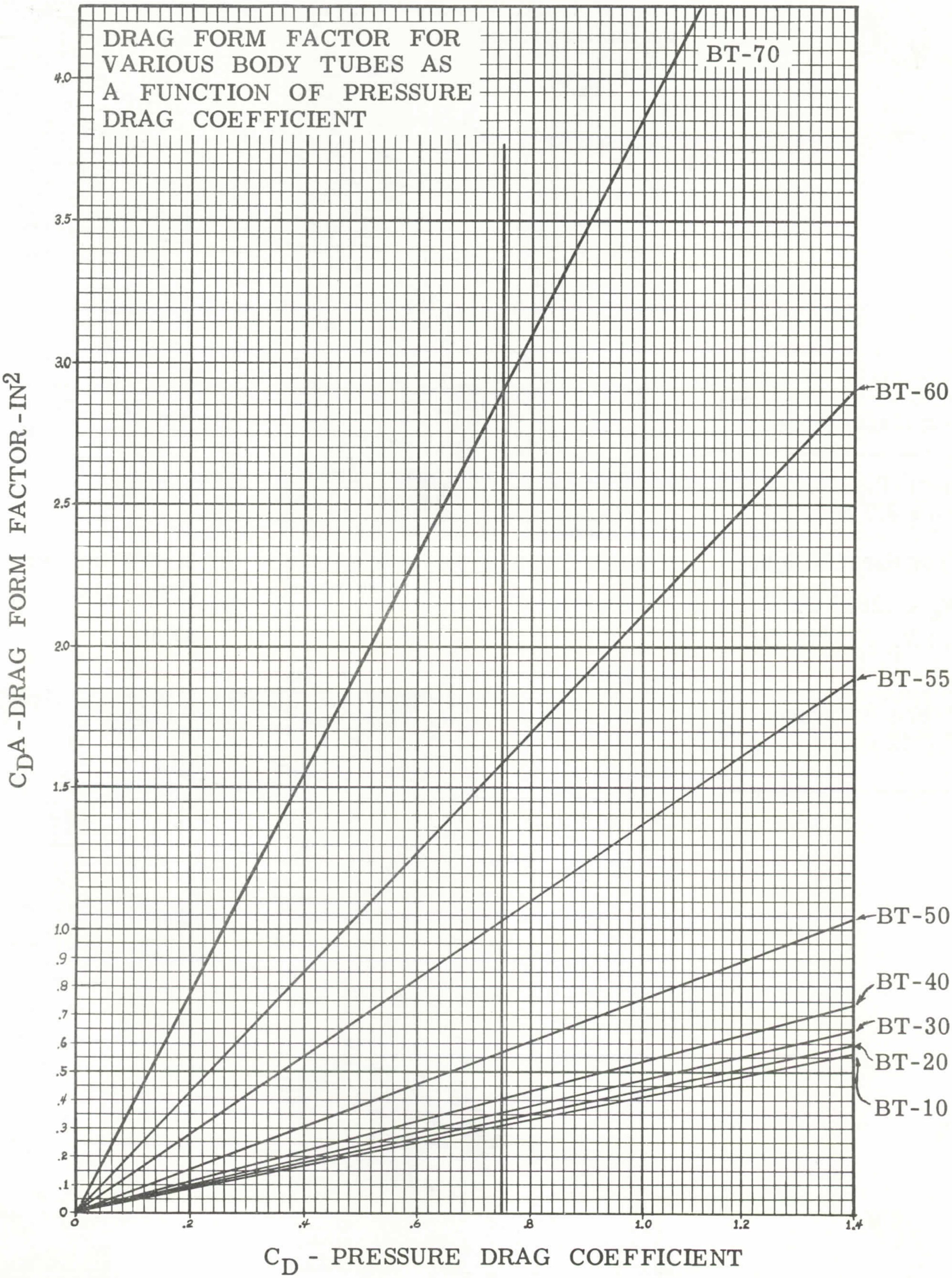
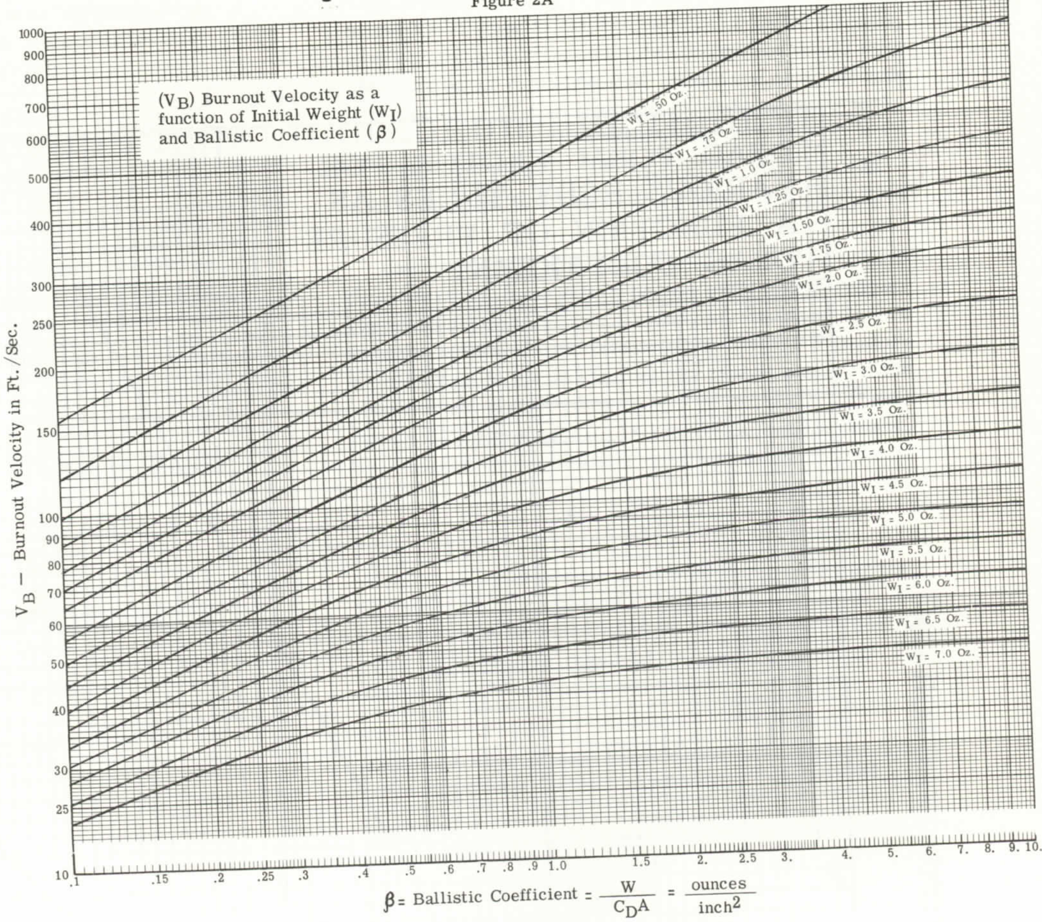

DRAG FORM FACTOR FOR VARIOUS BODY TUBES AS A FUNCTION OF PRESSURE DRAG COEFFICIENT
BT-70
BT-60
BT-55
BT-50
BT-40
BT-30
BT-20
BT-10
C_DA - DRAG FORM FACTOR - IN^2
C_D - PRESSURE DRAG COEFFICIENT
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.5
2.0
2.5
3.0
3.5
4.0
0
.2
.4
.6
.8
1.0
1.2
1.4

Figure 2A



C MOTOR

Burn Time
t_b = 2.0 Sec.

Propellent Weight

W_p = .2895 Oz.
$1/2W_p$ = .1448 Oz.

Average Thrust
T = 12 Oz.

Figure 2B

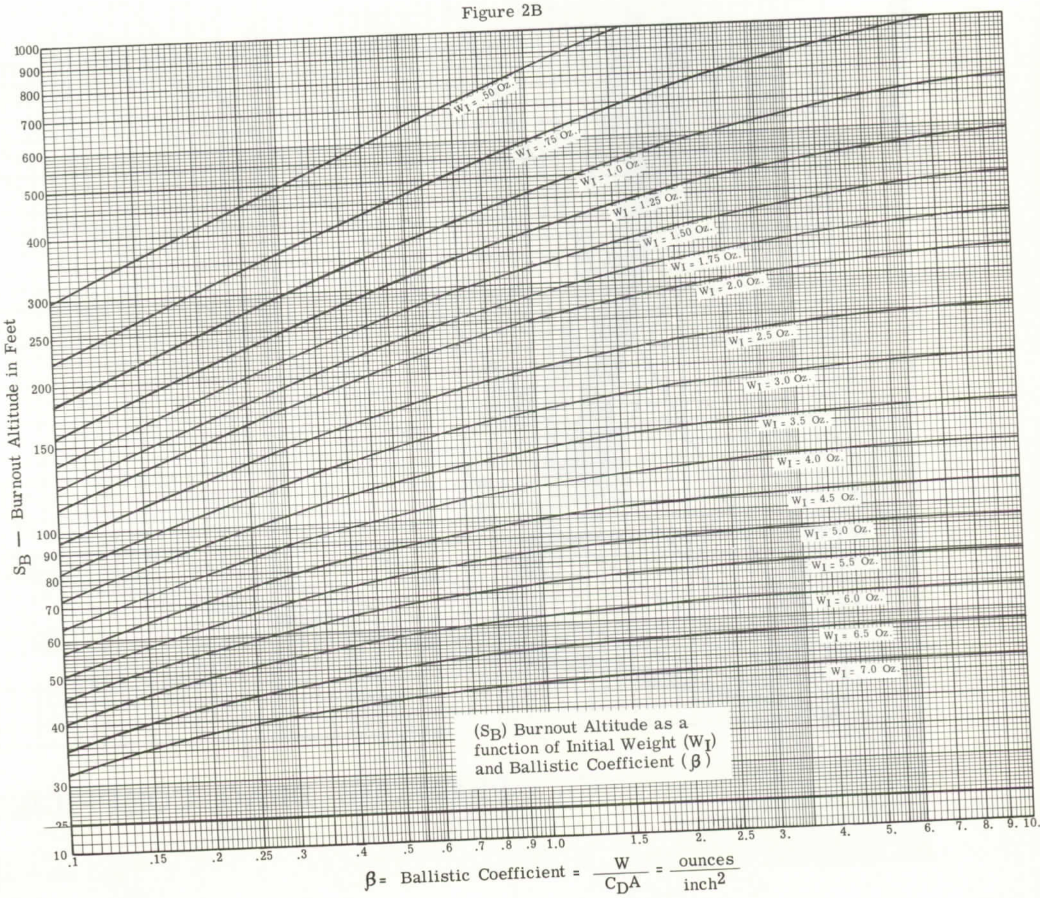
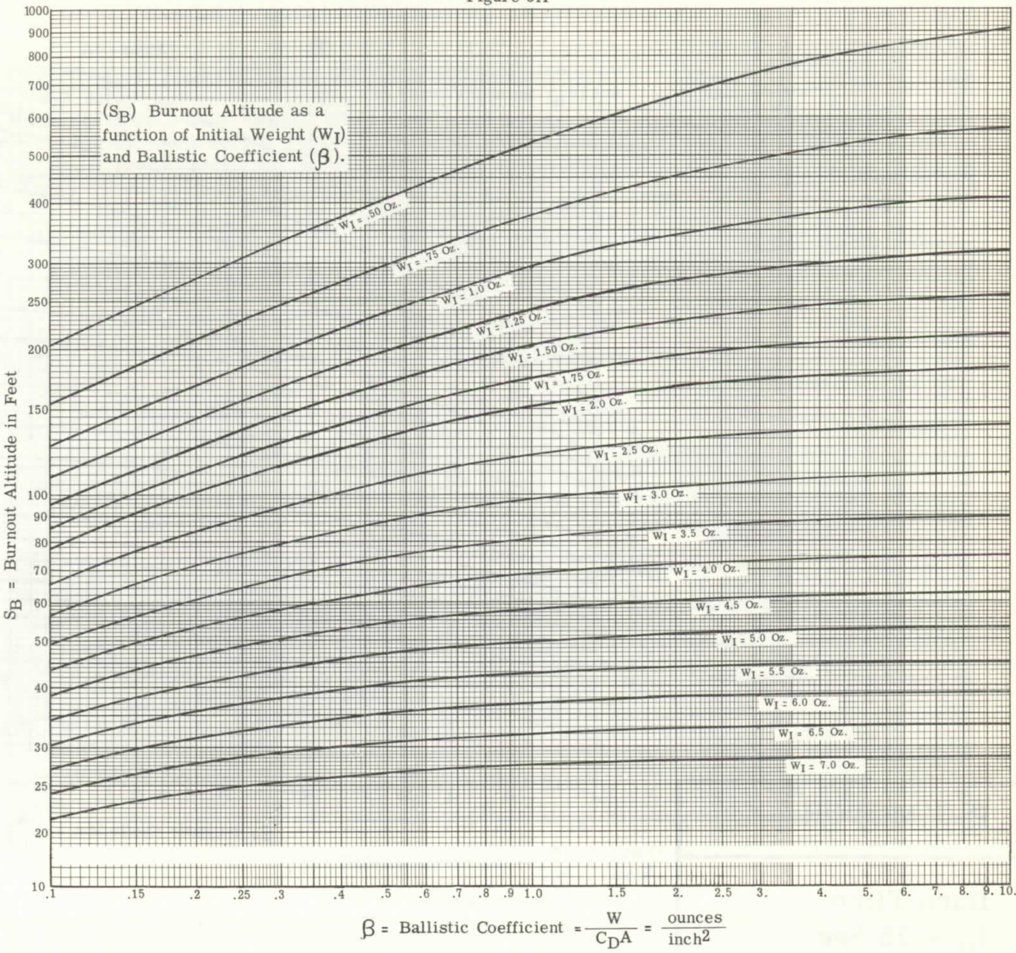


Figure 3A



B MOTOR

Burn Time
$t_b = 1.40$ Sec.

Propellent Weight
$W_P = .2212$ Oz.

$1/2\ W_P = .1106$ Oz.

Average Thrust
$T = 13.13$ Oz.

Figure 3B

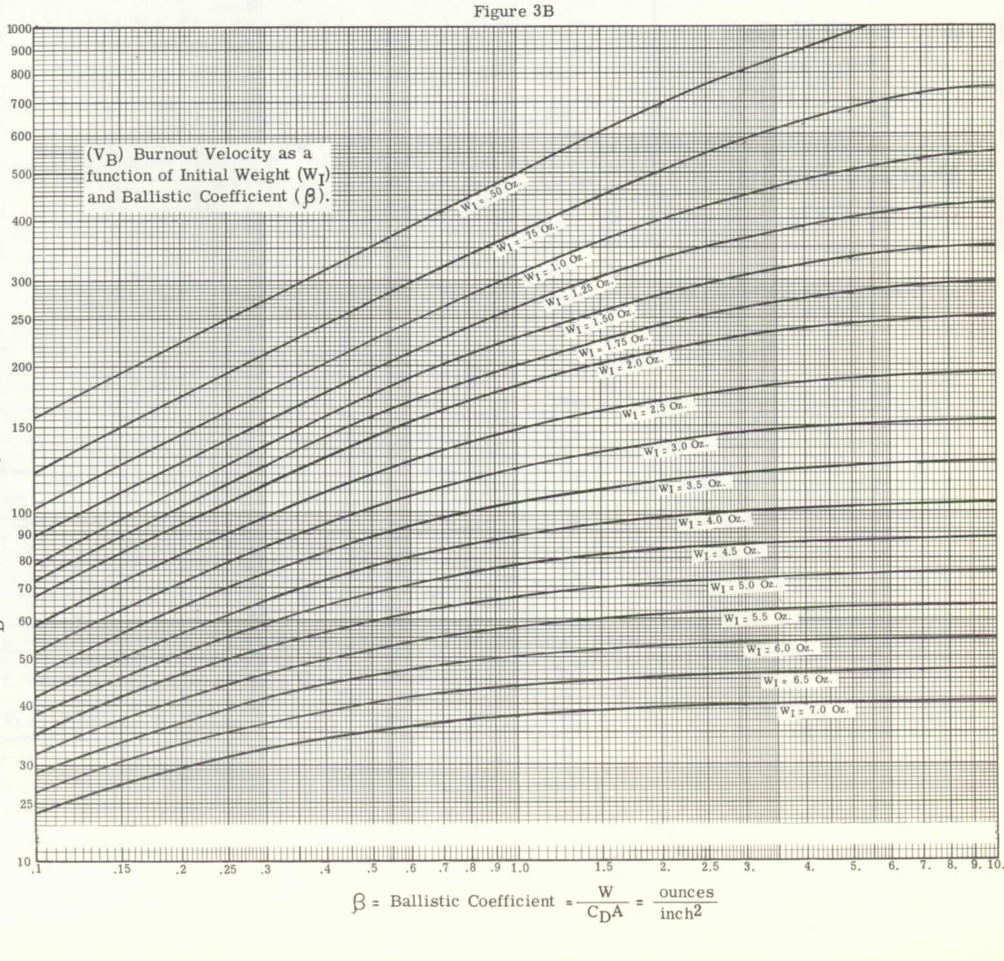
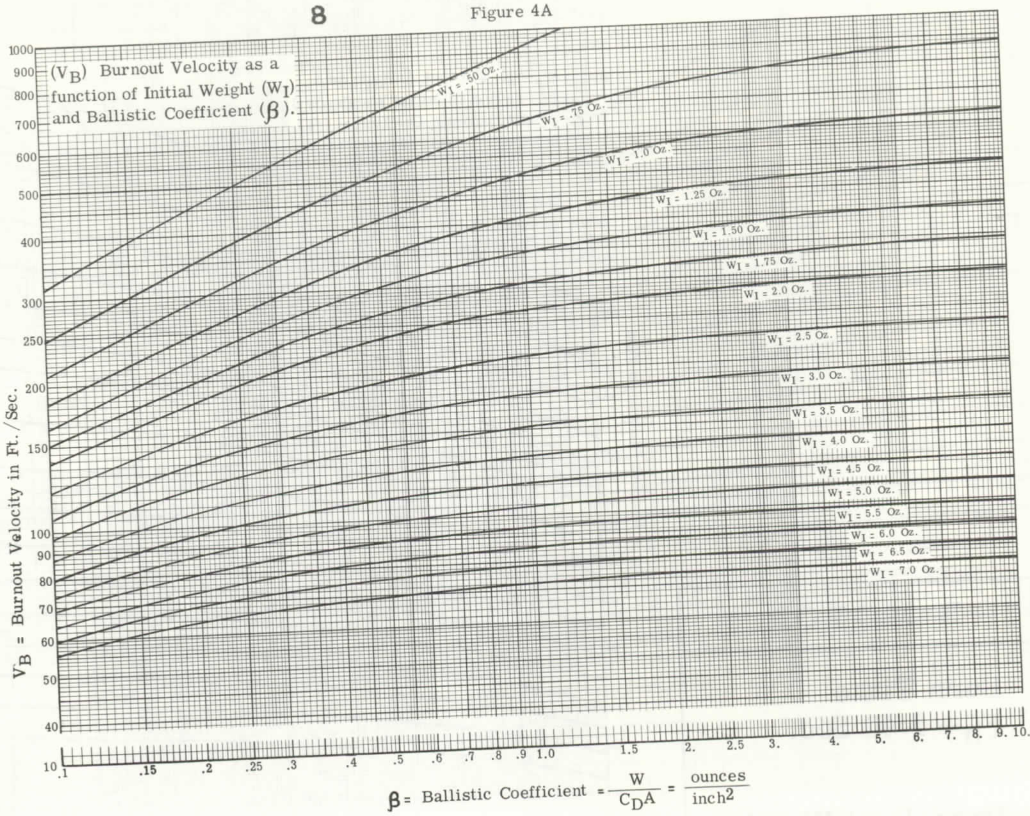


Figure 4A



B-3 MOTOR

Burn Time
t_b = .35 Sec.

Propellent Weight
W_P = .2212 Oz.

$1/2W_P$ = .1106 Oz.

Average Thrust
T = 52.7 Oz.

Figure 4B

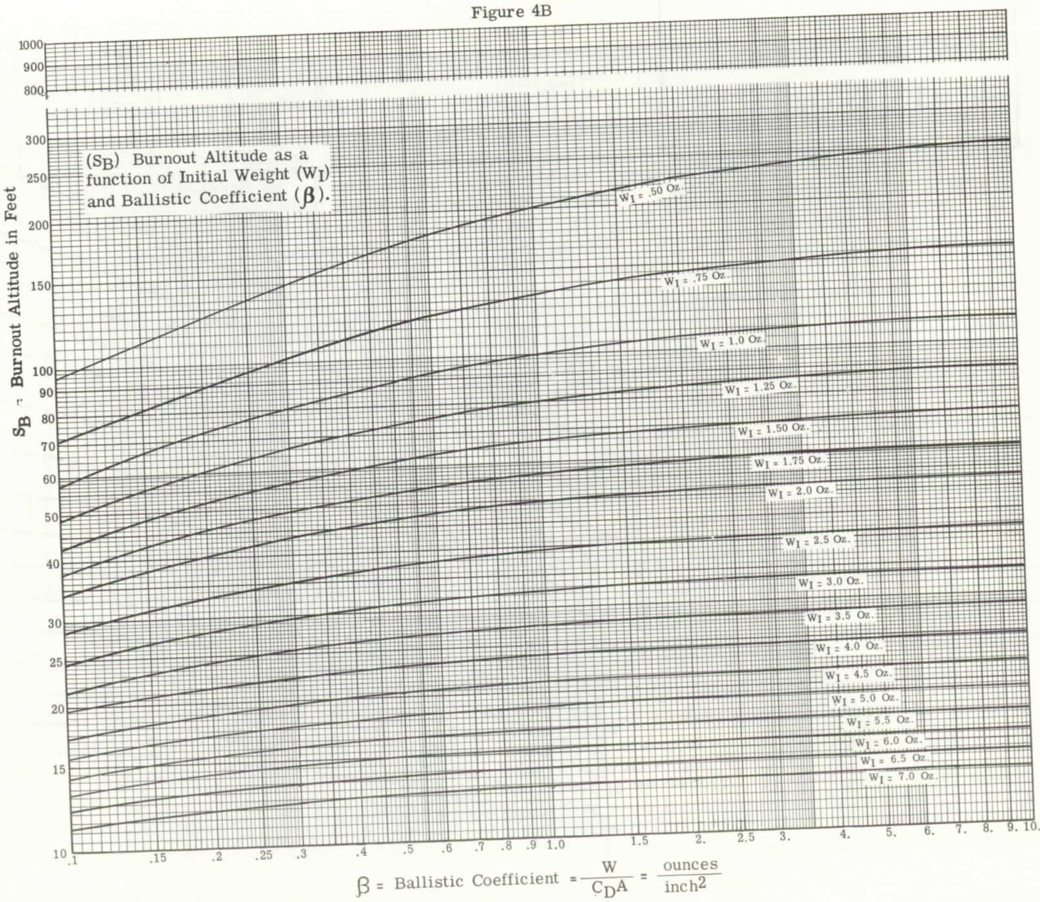


Figure 5A

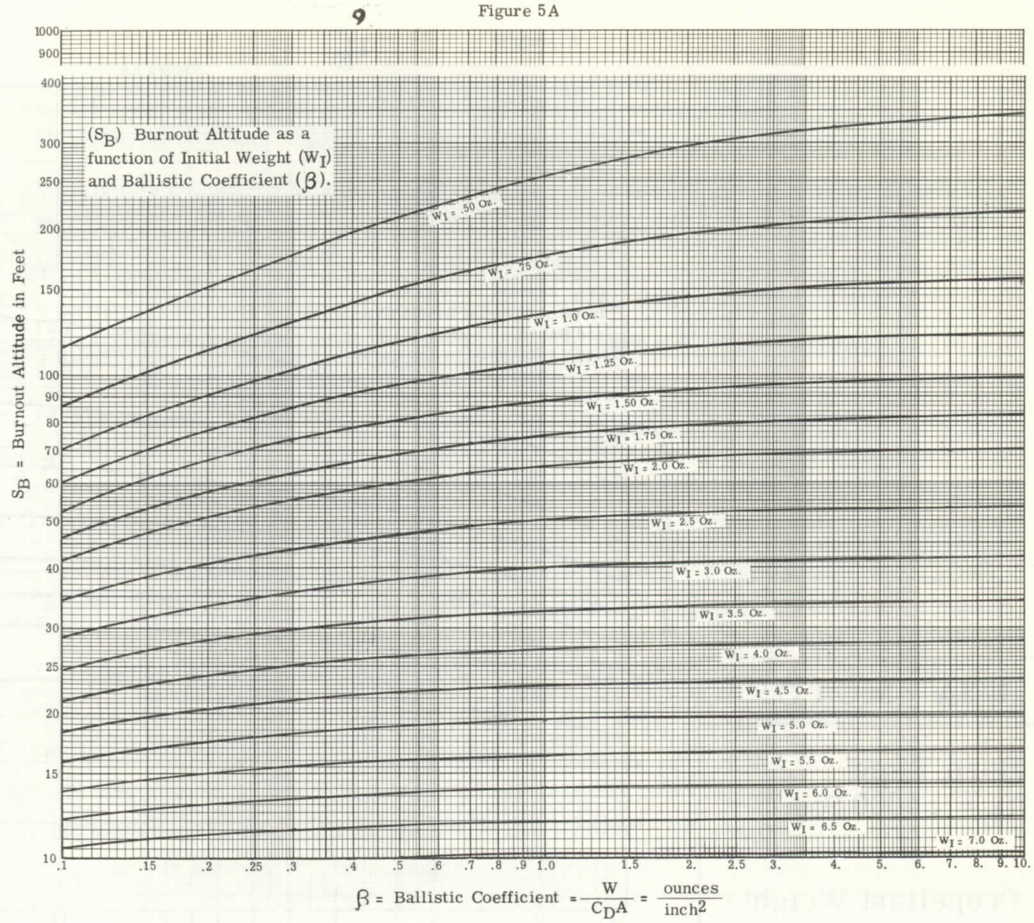
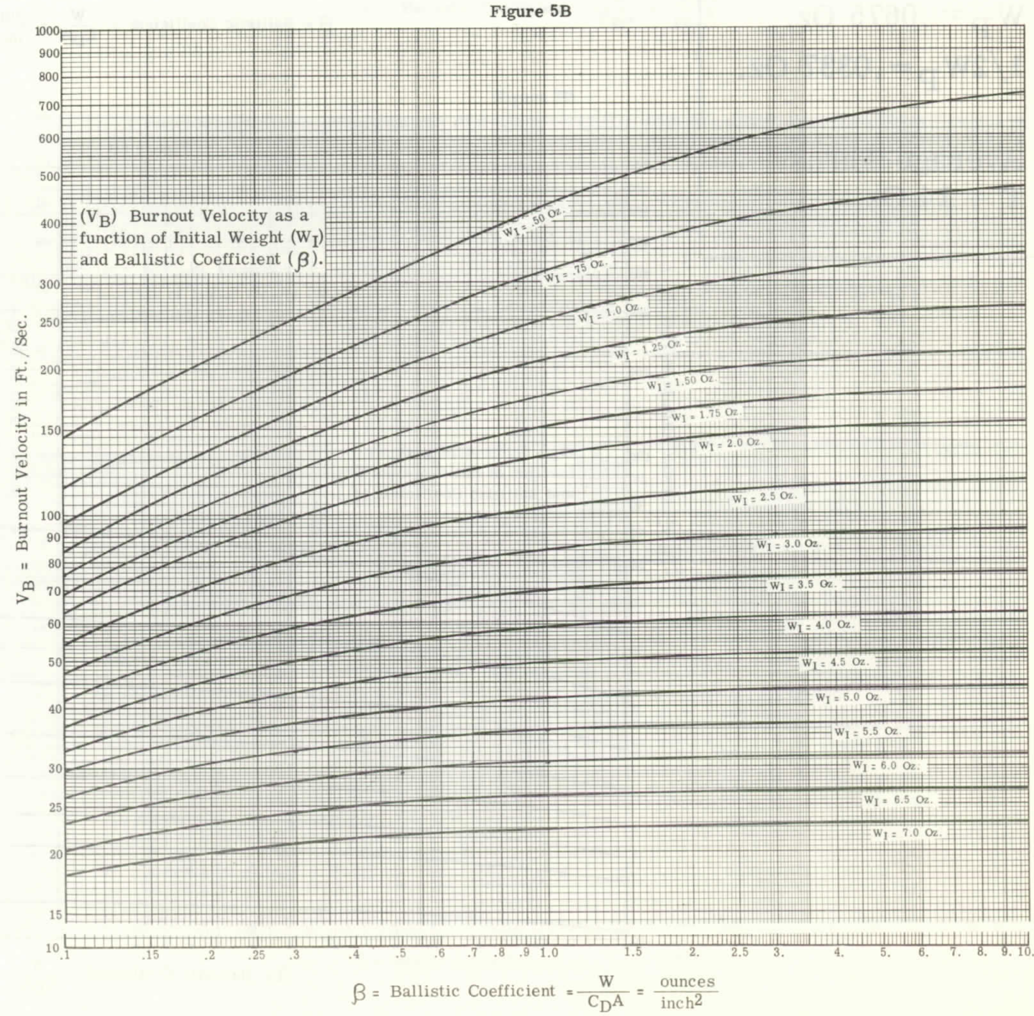


Figure 5B



A MOTOR

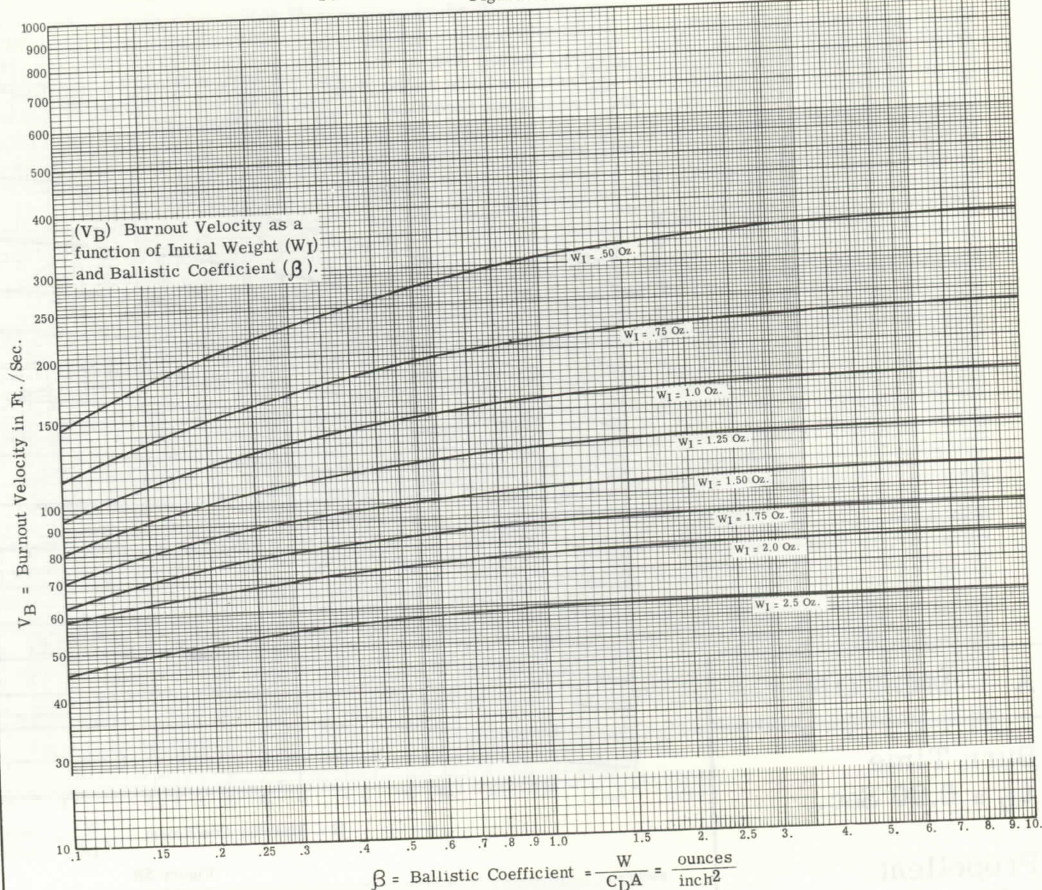
Burn Time
t_b = 0.90 Sec.

Propellent Weight
W_P = 0.1350 Oz.

$1/2 W_P$ = 0.0675 Oz.

Average Thrust
T = 12. 45 Oz.

Figure 6A



1/2A MOTOR

Burn Time
$t_b = .4000$ Sec.

Propellent Weight
$W_P = .0675$ Oz.
$1/2 W_P = .0337$ Oz.

Average Thrust
T = 14 Oz.

Figure 6B

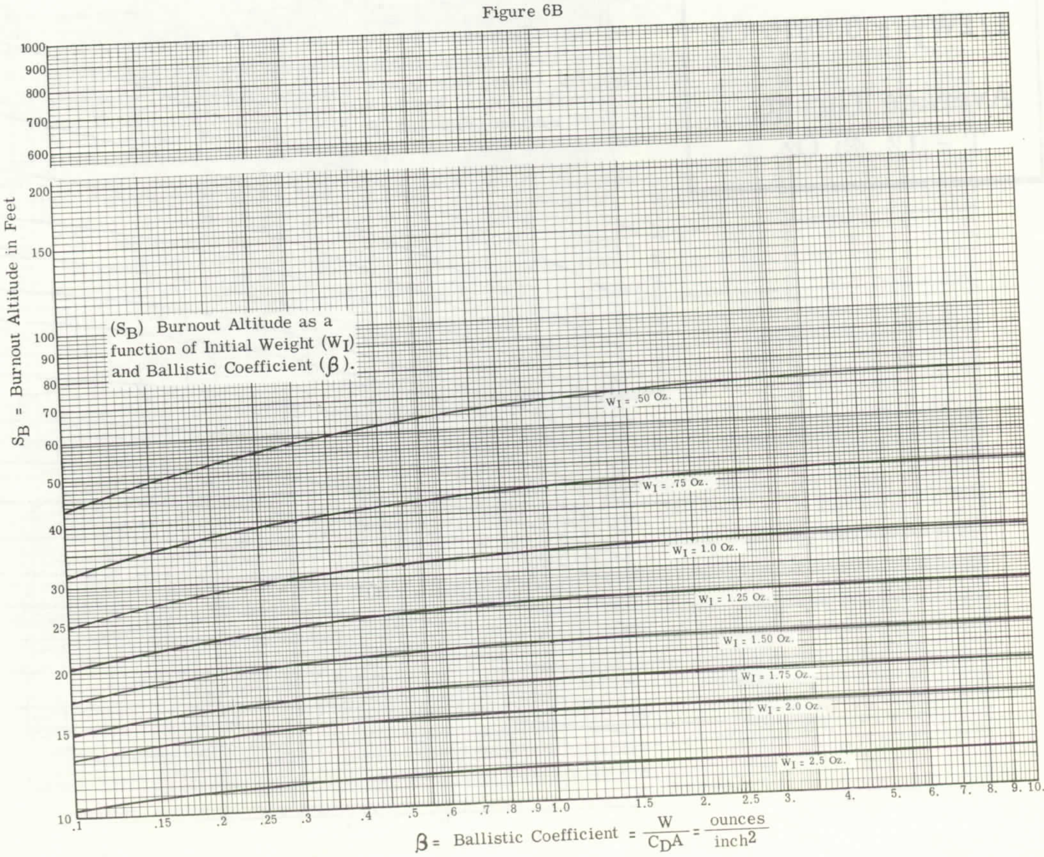
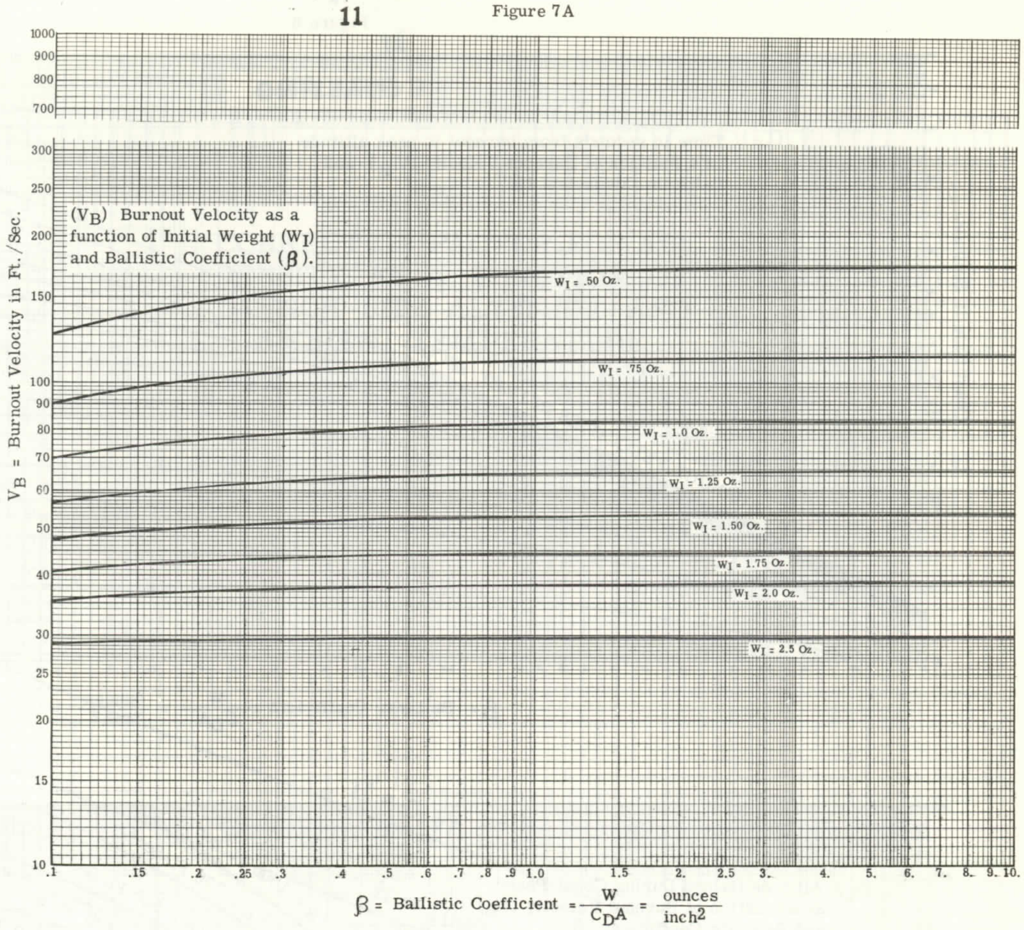


Figure 7A



1/4A MOTOR

Burn Time
$t_b = .1700$ Sec.

Propellent Weight
$W_P = .0338$ Oz.
$1/2 W_P = .0169$ Oz.

Average Thrust
T = 16 Oz.

Figure 7B

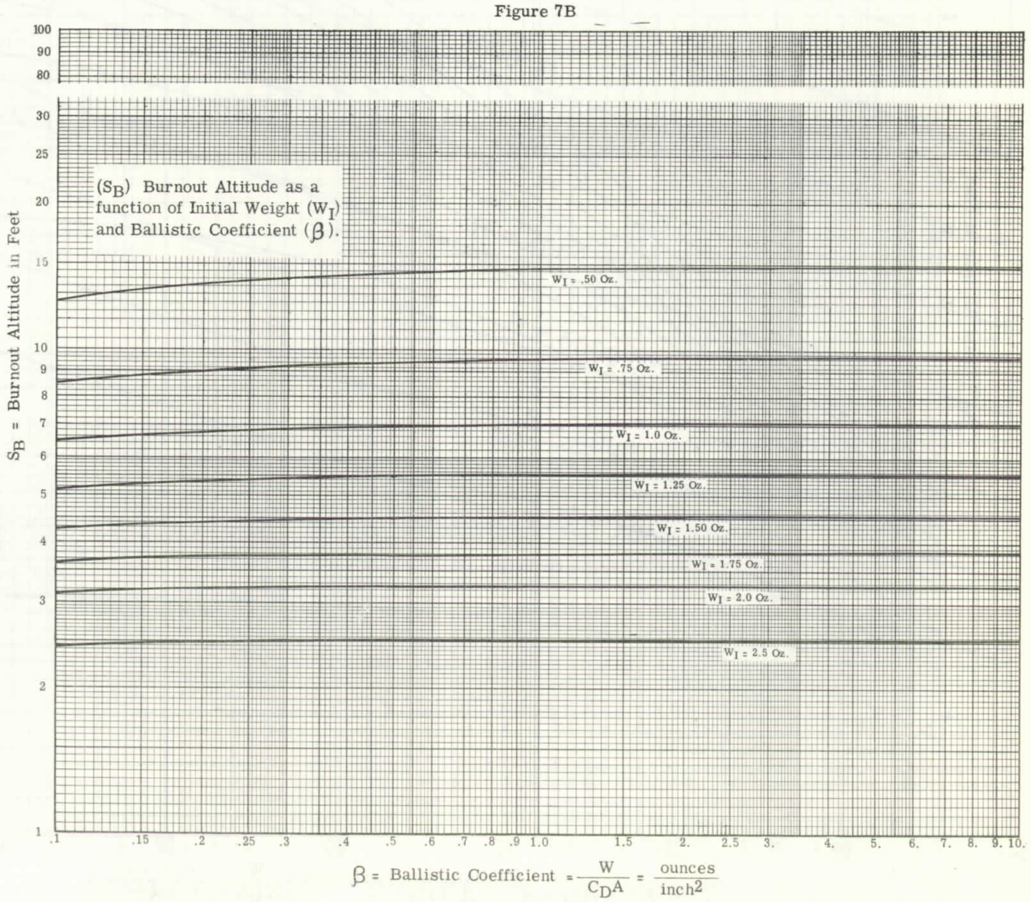
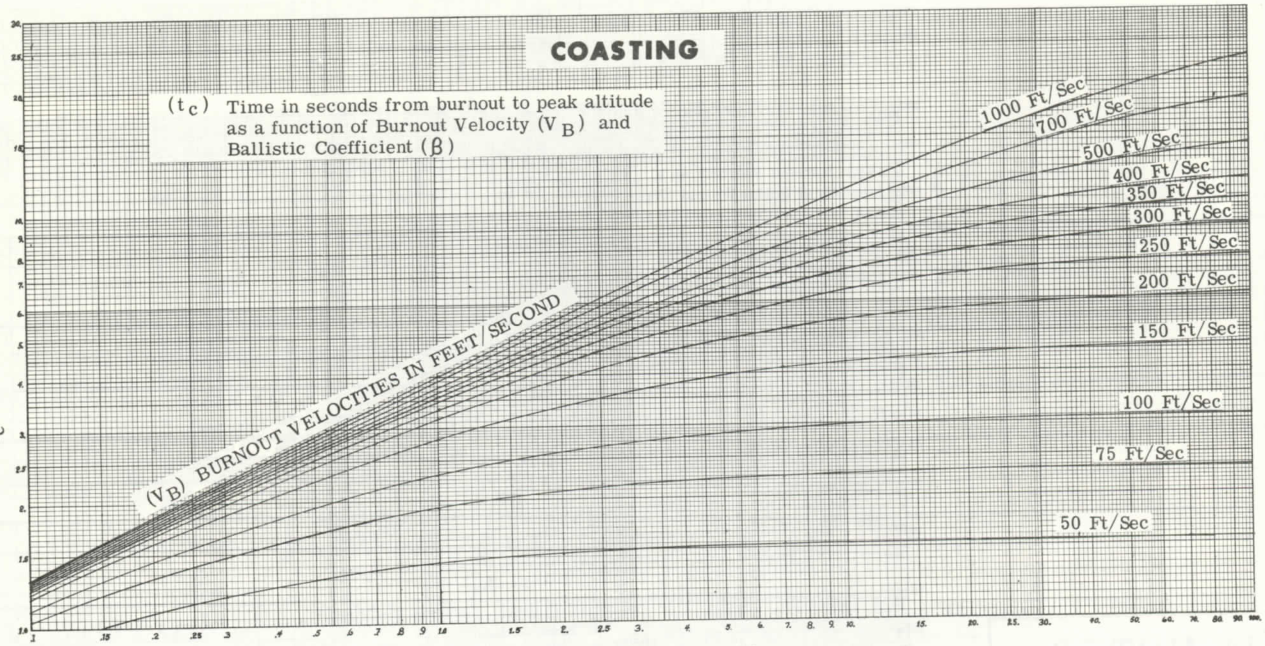
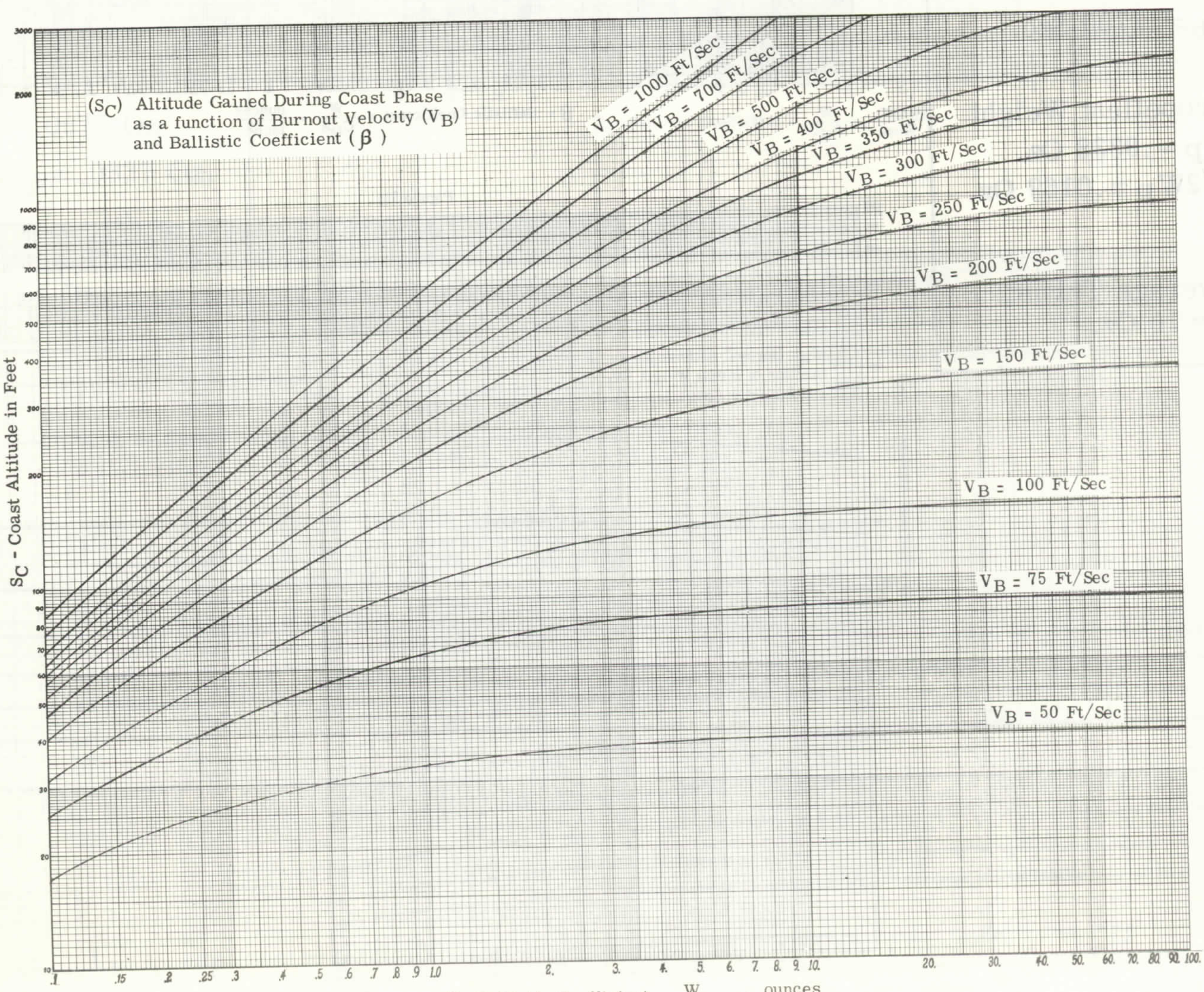


Figure 8

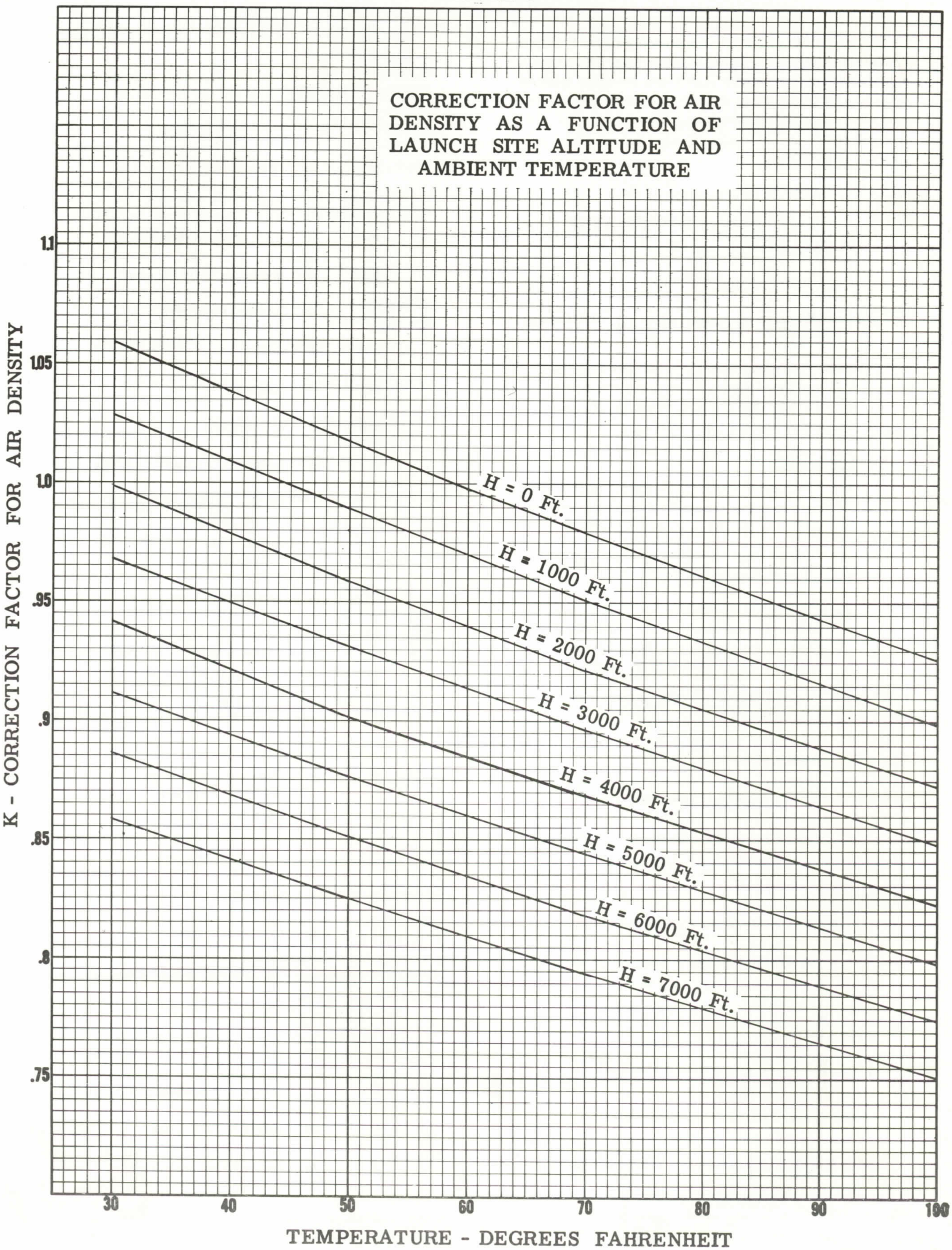


β = Ballistic Coefficient = $\frac{W}{C_D A}$ = $\frac{\text{ounces}}{\text{Inch}^2}$



β = Ballistic Coefficient = $\frac{W}{C_D A}$ = $\frac{\text{ounces}}{\text{inch}^2}$

FIGURE 9


CORRECTION FACTOR FOR AIR DENSITY AS A FUNCTION OF LAUNCH SITE ALTITUDE AND AMBIENT TEMPERATURE
K - CORRECTION FACTOR FOR AIR DENSITY
1.1
1.05
1.0
.95
.9
.85
.8
.75
H = 0 Ft.
H = 1000 Ft.
H = 2000 Ft.
H = 3000 Ft.
H = 4000 Ft.
H = 5000 Ft.
H = 6000 Ft.
H = 7000 Ft.
30
40
50
60
70
80
90
100
TEMPERATURE - DEGREES FAHRENHEIT

GENERAL EQUATIONS OF MODEL ROCKET MOTION, TERMINOLOGY, AND ASSUMPTIONS

The equations of motion are presented here for those rocketeers who are interested in obtaining the complete time history of their rocket's motion, for those with an interest in mathematics, and for those who are curious as to the origin of the charts.

A. THRUSTING

1. Altitude as a function of time

$$S = \frac{1}{g}\,\frac{W}{C_D A\,1/2\,\rho}\,\ln\cosh\left\{g\sqrt{\left(\frac{T}{W}-1\right)\frac{C_D A\,1/2\,\rho}{W}}\;t\right\}$$

2. Velocity as a function of time

$$V = \sqrt{\frac{W}{C_D A\,1/2\,\rho}\left(\frac{T}{W}-1\right)}\;\tanh\left\{g\sqrt{\left(\frac{T}{W}-1\right)\frac{C_D A\,1/2\,\rho}{W}}\;t\right\}$$

Note that when time (t) is the burnout time (t_B) that the altitude (S) becomes the burnout altitude (S_B) and the velocity (V) becomes the burnout velocity (V_B).

B. COASTING

1. The coast altitude (S_C) is the distance gained between the time of burnout and the time when the rocket reaches its peak.

$$S_C = \frac{1}{2g}\,\frac{W}{C_D A\,1/2\,\rho}\,\ln\left\{1+\frac{C_D A\,1/2\,\rho\,V_B^2}{W}\right\}$$

2. The coast time (t_C) is the time it takes the rocket to slow down to zero velocity ($V=0$). That is, the peak is reached and the rocket will now begin to fall back to the ground.

$$t_C = \frac{1}{g}\sqrt{\frac{W}{C_D A\,1/2\,\rho}}\;\tan^{-1}\sqrt{\frac{C_D A\,1/2\,\rho\,V_B^2}{W}}$$

3. Additional altitude gained as a function of time from burnout.

$$S = S_C + \frac{1}{g}\,\frac{W}{C_D A\,1/2\,\rho}\;\ln\cos\left\{g\sqrt{\frac{C_D A\,1/2\,\rho}{W}}\,(t_C - t)\right\}$$

4. Velocity as a function of time from burnout

$$V = \sqrt{\frac{W}{C_D A\,1/2\,\rho}}\;\tan\left\{g\sqrt{\frac{C_D A\,1/2\,\rho}{W}}\;(t_C - t)\right\}$$

C. SECOND AND SUBSEQUENT STAGES

1. Altitude gained as a function of time from the previous stage's burnout.

$$S_2 = \frac{1}{g}\,\frac{W}{C_D A\,1/2\,\rho}\;\ln\left\{\cosh\left[g\sqrt{\frac{C_D A\,1/2\,\rho}{W}\left(\frac{T}{W}-1\right)}\;t\right] + V_{B_1}\sqrt{\frac{C_D A\,1/2\,\rho}{W}\,\frac{1}{\left(\frac{T}{W}-1\right)}}\;\sinh\left[g\sqrt{\frac{C_D A\,1/2\,\rho}{W}\left(\frac{T}{W}-1\right)}\;t\right]\right\}$$

2. Velocity as a function of time

$$V_2 = \sqrt{\frac{W}{C_D A\,1/2\,\rho}\left(\frac{T}{W}-1\right)}\;\tanh\left\{g\sqrt{\frac{C_D A\,1/2\,\rho}{W}\left(\frac{T}{W}-1\right)}\;t + \tanh^{-1}\sqrt{\frac{C_D A\,1/2\,\rho\,V_{B_1}^2}{W}\,\frac{1}{\left(\frac{T}{W}-1\right)}}\right\}$$

Again note that when time (t) is the burnout time (t_B) that the altitude (S_2) becomes the second (or third) stage altitude increment (S_{B2}) and the velocity (V_2) becomes the second (or third) stage burnout velocity (V_{B2}).

TERMINOLOGY

The terminology used is as follows:

sinh = hyperbolic sine

cosh = hyperbolic cosine

tanh = hyperbolic tangent

$\tanh^{-1}$ = arc hyperbolic tangent

ln = natural logarithm

cos = cosine

tan = tangent

$\tan^{-1}$ = arc tangent

W = weight of rocket in pounds

- (a) use average weight during thrusting
- (b) use final weight during coasting

C_D = aerodynamic drag coefficient

A = reference area in ft.^{2} for professional rocketry or in.^{2} for model rocketry

ρ = sea level density of air

$= .002378\ \frac{\text{lb-sec}^2}{\text{ft}^4} = .0765\ \frac{\text{lb}}{\text{ft}^3} = 1.22\ \frac{\text{ounces}}{\text{ft}^3}$

T = average thrust in pounds for professional rocketry or ounces for model rocketry

g = earth's gravitational constant

= 32.174 ft/sec^2

t = time in seconds

S = altitude in feet

V = velocity in feet per second

Subscript terminology:

B = refers to burnout

C = refers to coasting

1 = refers to first stage

2 = refers to second stage

ASSUMPTIONS

The assumptions used in deriving these equations are essentially the same as used in the June, 1964 Model Rocket News (reference 1) article. In review:

1. Weight is assumed to be constant during thrusting (an average value is used).

2. Drag is proportional to V^2

   $D = C_D A \, 1/2 \, \rho V^2$ (which is quite reasonable up to 700 ft/sec.)

3. Thrust is constant during burning (again an average value is used):

   $$T = \frac{\text{Total Impulse}}{\text{Burn Time}}$$

4. The rocket is launched straight up and no cross-wind or turbulence exists, so the flight continues straight up at zero angle-of-attack.

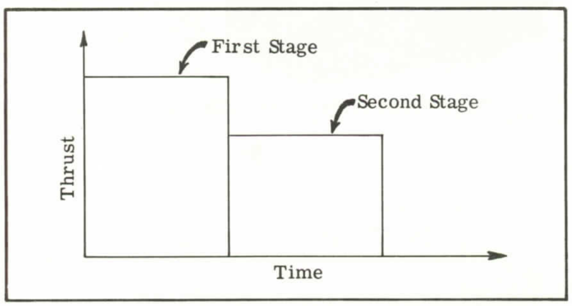
5. Thrust due to smoke delay burnoff is negligible.

6. The overall drag coefficient does not increase during the coast phase due to base drag (i.e. the smoke delay burnoff cancels it).

7. Additional weight losses due to smoke delay burnoff are neglected.

8. Atmospheric density is uniform and does not vary from the launch pad value.

9. Second and subsequent stages have no time delay for ignition and buildup to the average thrust value of the next stage. An example of the assumed time history is shown:



GENERAL EQUATIONS CONVERTED FOR MODEL ROCKETRY APPLICATIONS

The basic equations are converted for model rocketry calculation use by first substituting the known values for the constants ρ and g. Also since we deal primarily with weights in ounces instead of pounds and frontal areas in square inches instead of square feet we use the additional conversion factors:

1 pound = 16 ounces

1 square foot = 144 square inches

We can also shorten the repetitive calculations considerably by precomputing the ballistic coefficient (β)

$$\beta = \frac{W}{C_D A} \text{ in } \frac{\text{ounces}}{\text{Square inches of frontal area}}$$

and the drag-free acceleration (a)

$$a = \left(\frac{T}{W} - 1\right) \text{ in g's of acceleration}$$

(Note that the drag-free acceleration (a) is non-dimensional so both thrust (T) and weight (W) must be in ounces.

With these modifications the A, B, and C equations for altitude and velocity reduce to the following:

A. THRUSTING

1. Altitude as a function of time

   $$S = 235.26\,\beta \ln \cosh\left[.36981 \frac{\sqrt{a}}{\sqrt{\beta}} t\right]$$

2. Velocity as a function of time

   $$V = 87.0 \sqrt{\beta} \sqrt{a} \tanh\left[.36981 \frac{\sqrt{a}}{\sqrt{\beta}} t\right]$$

   Again note that when time (t) is equal to the motor burnout time (t_B) that the altitude (S) equals the burnout altitude (S_B) and the velocity (V) equals the burnout velocity (V_B).

B. COASTING

1. The coast altitude (S_C) is the distance gained between the time of burnout and the time at which the rocket reaches its peak.

   $$S_C = 117.63\,\beta \ln\left[1 + \frac{V_B^2}{7659.3\beta}\right]$$

2. The coast time (t_C) is the time it takes the rocket to slow down to zero velocity (V = 0). That is, the peak is reached on the rocket and will now begin to fall back down.

   $$t_c = 2.7041 \sqrt{\beta} \tan^{-1}\left[\frac{V_B}{87.0\sqrt{\beta}}\right]$$

3. Additional altitude gained as a general function of time from burnout.

   $$S = S_C + 235.26\,\beta \ln \cos\left\{.36981 \frac{(t_c - t)}{\sqrt{\beta}}\right\}$$

4. Velocity as a general function of time from burnout.

   $$V = 87.0\sqrt{\beta} \tan\left\{.36981 \frac{(t_c - t)}{\sqrt{\beta}}\right\}$$

C. SECOND AND SUBSEQUENT STAGES

1. Altitude gained as a function of time from burnout of the previous stage.

   $$S_2 = 235.26\,\beta \ln\left\{\cosh\left[.36981 \frac{\sqrt{a}}{\sqrt{\beta}} t\right] + \frac{V_{B1}}{87.0\sqrt{\beta}\sqrt{a}} \sinh\left[.36981 \frac{\sqrt{a}}{\sqrt{\beta}} t\right]\right\}$$

2. Velocity of the second (or third) stage as a

function of time from burnout of the previous stage.

$$V_{B_2} = 87.0\sqrt{\beta}\sqrt{a}\ \tanh\left\{.36981\ \frac{\sqrt{a}}{\sqrt{\beta}}\,t + \tanh^{-1}\frac{V_{B1}}{87.0\sqrt{\beta}\sqrt{a}}\right\}$$

Again note that when time (t) is equal to the burnout time (t_B) for this stage that the altitude (S_2) becomes the second (or third) stage altitude increment ($^{S}B_2$) and the velocity (V_2) becomes the second (or third) stage burnout velocity ($^{V}B_2$).

HOW TO ANALYZE TWO OR THREE STAGE ROCKETS

In order to predict altitudes for multiple stage rockets one must resort to using equations (C1) and (C2). A book of mathematical tables (such as reference 5) must be obtained in order to evaluate the hyperbolic sines, cosines, tangents and arc tangents (perhaps one of your teachers can help you the first time through).

The first stage burnout velocity and altitude can be found using the charts in the usual way. The ballistic coefficient (β) and drag-free acceleration (a) are then calculated for the second stage. Equations (C1) and (C2) use these values in conjunction with the burnout velocity of the first stage ($^{V}B_1$) and the motor burn time of second stage (t_B) to obtain the altitude gained during second stage thrusting ($^{S}B_2$) and the new velocity at second stage burnout ($^{V}B_2$).

If it is just a two-stage vehicle it will now coast to the peak; and so the coasting chart data can be utilized as usual.

A three-stage vehicle on the other hand has to make use of equations (C1) and (C2) again to find the third stage burnout velocity and the increment of altitude gained during third-stage thrusting. The coast altitude and time can then be found as a function of the ballistic coefficient (as based on the empty weight of the third stage) and the third stage burnout velocity.

Needless to say, it is important to use weight values in your calculations that properly reflect the effect of the booster stages falling away after they burn out.

The reason that the second and third stage data had to be calculated, instead of just simply read from a chart, is that there are too many variable factors.

You will notice that four variables affect the velocity and altitude gained during second or third stage thrusting: 1) type of rocket motor, 2) weight of rocket, 3) drag of rocket, and 4) initial velocity due to previous stage. Single-stage rockets only have three variables that effect velocity and altitude: 1) type of rocket motor, 2) weight of rocket, and 3) drag of rocket. Plotting each motor type as a separate graph for single-stage rockets essentially reduces the number of variables from 3 to 2, thus lending itself to plotting on two-dimensional graph paper. Using this same procedure for multiple-stage rockets still leaves us with trying to plot mathematical functions of three variables on two-dimensional graph paper.

REFERENCES

1) "Rocket Math", Model Rocket News, Volume 4, Number 2, June, 1964 and Volume 4, Number 3, November, 1964.

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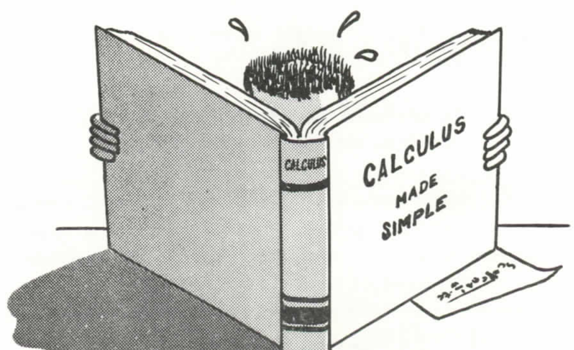
3) Dr. S. F. Hoerner, Fluid-Dynamic Drag, Published by the Author, Midland Park, New Jersey, 1965.

4) G. Harry Stine, Handbook of Model Rocketry, Follett Publishing Company, 1965.

5) H. B. Dwight, Tables of Integrals and Other Mathematical Data, The MacMillian Company, 1957.

6) D. Bergamini, Mathematics - Life Science Library, Time Incorporated, 1963.



APPENDIX I

DISCUSSION OF CALCULUS

Calculus is the branch of mathematics which allows one to analyze movement and change. Complex motions can essentially be broken into small increments, and the changes occurring at each instant can be investigated.

Prior to this report, model rocket altitudes with aerodynamic drag effects were frequently determined by using small time interval step-by-step computation methods. At a given velocity the drag can be computed and subtracted from the thrust to find the net force acting on the rocket. This, in turn, gives the net acceleration for that time period. The assumption is then made that this acceleration will be constant for the next short time period. Next the increase in velocity due to the assumed constant acceleration is computed and the entire process is ready to be repeated. We do know, however, that the velocity and drag do not take small jumps after each time period but in reality are smoothly increasing during thrusting, and are smoothly decreasing during coasting.

The step-by-step method thus introduces slight errors in the predicted altitude. Taking smaller time periods increases the accuracy. With calculus the time periods are infinitely small and as a result the velocity and drag become the perfectly smooth variables with time that we know them to be. A very well illustrated, interesting, and simple to understand layman's description of calculus is given in Chapter 5 of the "Mathematics" volume of the Life Science Library (reference 6). This chapter helps very much to convey the importance of calculus as a fundamental tool in today's modern world of spacecraft and electronics.

The following two sections of the report can be bypassed with no loss in continuity. These detailed derivations of the thrusting and coasting equations for model rockets are presented primarily for those persons with the prerequisite background in the calculus operations of "differentiation" and "integration", who desire a more complete understanding of where the equations for the altitude prediction charts originate.

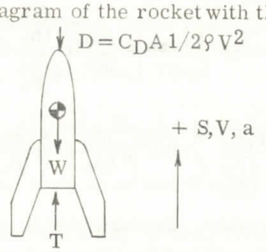
DERIVATION OF THE THRUSTING EQUATIONS

The equation of motion for the rocket during thrusting is obtained by applying Sir Isaac Newton's first law, which can be written as:

$$F = ma \qquad (1)$$

Where: F = The summation of all external forces applied to the rocket.
m = The mass of the rocket.
and: a = The acceleration of the rocket.

A free body diagram of the rocket with the proper forces is shown.



Let us, for convenience, choose the displacement S, the velocity V, and the acceleration a, as positive in the upward direction. Since we are interested only in the one-dimensional upward trajectory of the rocket we eliminate all possible forces acting in any other direction than S.

The following forces are acting on the rocket:

1. THRUST (T)
   The thrust is considered to be constant from ignition to burnout. It is easily seen that the thrust is acting in the positive direction and thus is a positive force.
2. WEIGHT (W)
   This is the earth's gravitational field acting on the mass of the rocket. We assume that the weight loss from ignition to burnout is small enough so that taking the average weight during this period gives sufficient accuracy. Thus, the weight will be a constant and, of course, can be seen to be acting in the negative direction.
3. DRAG (D) $D = C_D A\, 1/2\, \rho\, V^2$
   The aerodynamic drag is assumed to hold to the V^2 law for all flight speeds. It too is a negative force.

Thus, equation (1) becomes:

$$T - D - W = ma \qquad (2)$$

From physics we know the mass (m) is equal to the weight (W) divided by (g) the acceleration due to gravity.

$$m = \frac{W}{g}$$

Substituting the values of m and D into equation (2) we obtain:

$$T - C_D A\, 1/2\, \rho\, V^2 - W = \frac{W}{g} a$$

For a given rocket the term $C_D A\, 1/2\, \rho$ is a constant. Instead of carrying all these terms through the derivation we call it the constant K.

$$K = C_D A\, 1/2\, \rho$$

Thus, our equation becomes:

$$T - KV^2 - W = \frac{W}{g} a$$

Now the acceleration (a) is the rate of change of velocity with respect to time. In calculus this is written as:

$$a = \frac{dV}{dt}$$ and is called the first derivative of velocity.

our equation now becomes:

$$T - KV^2 - W = \frac{W}{g}\frac{dV}{dt} \qquad (3)$$

This is called a first order differential equation. Luckily, with a bit of rearranging it can be integrated.

Multiplying both sides by $\dfrac{dt}{T - W - KV^2}$

We obtain:

$$dt = \frac{W}{g}\,\frac{dV}{(T-W) - KV^2} \qquad (4)$$

Next we integrate both sides:
dt from an initial time (t_o) to some time (t), and dV from an initial velocity (V_o) to velocity (V) at time (t).

thus:

$$\int_{t_o}^{t} dt = \frac{W}{g}\int_{V_o}^{V}\frac{dV}{(T-W) - KV^2}$$

The more difficult right-hand side can be reduced to the standard integral form given on page 29 of reference 5. This is accomplished by making the following substitutions:

$$a^2 = T - W$$
$$b^2 = K$$
$$U = V$$

Reference 5 gives the solution as:

$$\int \frac{dU}{a^2 - b^2U^2} = \frac{1}{ab}\tanh^{-1}\frac{bU}{a}$$

When applied to our equation we find that:

$$t\Big]_{t_o}^{t} = \frac{W}{g}\,\frac{1}{\sqrt{T-W}\sqrt{K}}\,\tanh^{-1}\left(\sqrt{\frac{K}{T-W}}\,V\right)\Bigg]_{V_o}^{V}$$

or:

$$t - t_o = \frac{W}{g}\,\frac{1}{\sqrt{(T-W)K}}\tanh^{-1}\left(\sqrt{\frac{K}{T-W}}\,V\right) - \frac{W}{g}\,\frac{1}{\sqrt{(T-W)K}}\tanh^{-1}\left(\sqrt{\frac{K}{T-W}}\,V_o\right)$$

Now at ignition time $t_o = 0$. We will have the rocket's velocity $V_o = 0$. Upon substitution of these values we get:

$$t = \frac{W}{g}\frac{1}{\sqrt{(T-W)K}}\tanh^{-1}\left(\sqrt{\frac{K}{T-W}}\,V\right) \qquad (5)$$

since the value of $\tanh^{-1}(0)$ is zero.

We now find it convenient to define a constant β_o called the density ballistic coefficient

$$\beta_o = \frac{W}{C_D A\, 1/2\rho} = \frac{W}{K}$$

Where K is the constant previously defined. Note that the "density ballistic coefficient" must be distinguished from the "ballistic coefficient" defined in the introduction and subsequently used throughout the report.

Seeing the "density ballistic coefficient" terms appearing in each of the final equations summarized earlier should give one a clue as to why the density correction factor due to temperature and launch altitude variations can be included right in the "ballistic coefficient". This correction factor is simply the ratio of density at any altitude and temperature ($\rho_{H,T}$) to the sea level density (ρ).

Thus, the actual density is simply: $\rho_{H,T} = \left(\frac{\rho_{H,T}}{\rho}\right)\rho$

where $\left(\frac{\rho_{H,T}}{\rho}\right)$ is the correction factor for air density presented in Figure 9.

For values of density other than sea level we will have

$$\beta_o = \frac{W}{C_D A\, 1/2\rho_{H,T}} = \frac{W}{C_D A\, 1/2\left(\frac{\rho_{H,T}}{\rho}\right)\rho} = \frac{\beta}{1/2\rho}$$

Where our regular β includes the density correction term in a mathematically acceptable manner.

$$\beta = \frac{W}{C_D A\left(\frac{\rho_{H,T}}{\rho}\right)}$$

We also find it convenient to use the drag free acceleration in g's

$$a_o = \frac{T}{W} - 1$$

Note that we now use the subscript "o" for the drag free acceleration as a means for distinguishing it from the true rocket acceleration term (a) as used in Newton's basic motion equation F = ma.

a_o and β_o can be utilized by working with the constants of equation (5). These constants are re-arranged in the following manner:

$$\frac{W}{g}\frac{1}{\sqrt{(T-W)K}} = \frac{1}{g}\sqrt{\frac{W^2}{(T-W)K}} = \frac{1}{g}\sqrt{\frac{W}{K}\frac{1}{\left(\frac{T-W}{W}\right)}} = \frac{1}{g}\sqrt{\frac{W}{K}\frac{1}{\left(\frac{T}{W}-1\right)}} = \frac{1}{g}\sqrt{\frac{\beta_o}{a_o}}$$

also:

$$\sqrt{\frac{K}{T-W}} = \sqrt{\frac{K/W}{(T-W)/W}} = \sqrt{\frac{1/\beta_o}{\left(\frac{T}{W}-1\right)}} = \sqrt{\frac{1}{\beta_o a_o}} = \frac{1}{\sqrt{\beta_o a_o}}$$

thus, equation (5) becomes:

$$t = \frac{1}{g}\sqrt{\frac{\beta_o}{a_o}}\tanh^{-1}\left(\frac{V}{\sqrt{\beta_o a_o}}\right) \qquad (6)$$

Noting that the identity form $M = \tanh^{-1}(N)$ means that $N = \tanh(M)$ gives us a method by which we can obtain velocity (V) as a function of time (t) from equation (6) so

$$\boxed{V = \sqrt{\beta_o a_o}\tanh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right]} \qquad (7)$$

which is equation (A2) as can be verified by substituting in the terms represented by β_o and a_o.

This equation gives us the velocity of the rocket at any time (t) as long as the motor is burning. At the time (t_B) when the motor burns out, we then have the burnout velocity (V_B) of the rocket.

To find the height that the rocket will achieve as a function of time during thrusting, we note that velocity (V) is the rate of change of displacement (or distance if you like) with respect to time. In calculus this is written as

$V = \frac{dS}{dt}$ called the first derivative of displacement.

Thus for equation (7) we have:

$$\frac{dS}{dt} = \sqrt{\beta_o a_o}\tanh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right]$$

This is another first order differential equation and can be solved as before;

$$dS = \sqrt{\beta_o a_o}\tanh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right]dt$$

integrating both sides for "dS" from an initial displacement (S_o) to some displacement (S), and "dt" from the initial time (t_o) to the corresponding time (t) we obtain:

$$\int_{S_o}^{S} dS = \sqrt{\beta_o a_o}\int_{t_o}^{t}\tanh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right]dt$$

A standard integral form on Page 155 of reference 5 is:

$$\int \tanh X\, dX = \ln\cosh X$$

In our case

$$X = g\sqrt{\frac{a_o}{\beta_o}}\,t$$

and the differential "dX" must have the value

$$dX = g\sqrt{\frac{a_o}{\beta_o}}\,dt$$

This value is inserted as follows:

$$\int_{S_o}^{S} dS = \sqrt{\beta_o a_o}\int_{t_o}^{t}\tanh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right]\frac{g\sqrt{\frac{a_o}{\beta_o}}}{g\sqrt{\frac{a_o}{\beta_o}}}dt$$

$$\int_{S_o}^{S} dS = \frac{\sqrt{\beta_o a_o}}{g\sqrt{\frac{a_o}{\beta_o}}}\int_{t_o}^{t}\tanh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right]\left(g\sqrt{\frac{a_o}{\beta_o}}\,dt\right)$$

and is now integrable. Performing the integration we obtain:

$$S - S_o = \frac{\beta_o}{g}\ln\cosh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right] - \frac{\beta_o}{g}\ln\cosh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t_o\right]$$

Note that at ignition time $t_o = 0$, we have zero distance $S_o = 0$ and zero velocity. Thus we obtain:

$$\boxed{S = \frac{\beta_o}{g}\ln\cosh\left(g\sqrt{\frac{a_o}{\beta_o}}\,t\right)}$$

Since the value of

$$\cosh(0) = 1$$

and

$$\ln(1) = 0$$

or

$$\ln\cosh(0) = 0$$

The above equation is identical to (A1) and again can be verified by substituting in all the terms represented by β_o and a_o. This equation gives the burnout altitude (S_B) by substituting the burnout time (t_B) for t.

DERIVATION OF THE COASTING EQUATIONS

We now return to our initial equation.

$$T - D - W = m a = \frac{W}{g}\frac{dV}{dt}$$

It is easily seen that we have a coasting situation when thrust (T) = 0, and weight (W) equals the final weight of the rocket at burnout. Therefore, our coasting equation becomes:

$$-KV^2 - W = \frac{W}{g}\frac{dV}{dt} \qquad (8)$$

Multiplying both sides by $\frac{dt}{-KV^2-W}$ we get:

$$dt = -\frac{W}{g}\frac{dV}{(W+KV^2)} \qquad (9)$$

which will be integrated from the burnout time (t_B) to some time (t), and from the burnout velocity (V_B) to the velocity (V) which corresponds to time (t).

$$\int_{t_B}^{t} dt = -\frac{W}{g}\int_{V_B}^{V}\frac{dV}{W+KV^2} \qquad (10)$$

The right-hand side can be reduced to the standard integral form given on page 25 of reference 5 by the following substitutions:

$$a^2 = W$$
$$b^2 = K$$
$$U = V$$

The standard form is:

$$\frac{dU}{a^2+b^2U^2} = \frac{1}{ab}\tan^{-1}\frac{bU}{a}$$

Applying the standard form to equation (10) we obtain:

$$t\Big]_{t_B}^{t} = -\frac{W}{g}\frac{1}{\sqrt{WK}}\tan^{-1}\left(\sqrt{\frac{K}{W}}V\right)\Big]_{V_B}^{V}$$

or:

$$t - t_B = -\frac{W}{g}\frac{1}{\sqrt{WK}}\tan^{-1}\left(\sqrt{\frac{K}{W}}V\right) + \frac{W}{g}\frac{1}{\sqrt{WK}}\tan^{-1}\left(\sqrt{\frac{K}{W}}V_B\right)$$

In order to predict delay times easily we start counting time from the time of motor burnout (t_B). This is accomplished by setting t_B to zero. Thus:

$$t = -\frac{W}{g}\frac{1}{\sqrt{WK}}\tan^{-1}\left(\sqrt{\frac{K}{W}}V\right) + \frac{W}{g}\frac{1}{\sqrt{WK}}\tan^{-1}\left(\sqrt{\frac{K}{W}}V_B\right) \qquad (11)$$

where t is the time from motor burnout.

Now we again introduce the "density ballistic coefficient" in order to reduce the complexity of the equation.

$$\beta_o = \frac{W}{C_D A\, 1/2\rho} = \frac{W}{K}$$

now:

$$\frac{W}{g}\frac{1}{\sqrt{WK}} = \frac{1}{g}\sqrt{\frac{W^2}{WK}} = \frac{1}{g}\sqrt{\frac{W}{K}} = \frac{1}{g}\sqrt{\beta_o}$$

also:

$$\sqrt{\frac{K}{W}} = \sqrt{\frac{1}{\beta_o}} = \frac{1}{\sqrt{\beta_o}}$$

Therefore, equation (11) becomes:

$$t = -\frac{1}{g}\sqrt{\beta_o}\tan^{-1}\left(\frac{V}{\sqrt{\beta_o}}\right) + \frac{1}{g}\sqrt{\beta_o}\tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right) \qquad (12)$$

which can be rearranged as follows to solve for velocity (V) as a function of time (t).

$$+\frac{gt}{\sqrt{\beta_o}} = -\tan^{-1}\left(\frac{V}{\sqrt{\beta_o}}\right) + \tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right)$$

$$+\tan^{-1}\left(\frac{V}{\sqrt{\beta_o}}\right) = \tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right) - \frac{gt}{\sqrt{\beta_o}}$$

Noting that the following is true;

If $M = \tan^{-1}(N)$

then $N = \tan(M)$

we can proceed to solve for V. Thus:

$$\boxed{V = \sqrt{\beta_o}\tan\left\{\tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right) - \frac{gt}{\sqrt{\beta_o}}\right\}} \qquad (13)$$

which is essentially equation (B4), a general equation for rocket velocity after burnout as a function of time after burnout. (Further manipulations will get (13) into the exact form as presented in the report.) We can also use (13) to determine the coast time (t_c) of the rocket. This will be the time at which the rocket slows down to zero vertical velocity.

Substituting V=0 on the left side of (13) and noting that tan(0)=0 we obtain:

$$0 = \tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right) - \frac{gt_c}{\sqrt{\beta_o}}$$

Therefore:

$$\boxed{t_c = \frac{1}{g}\sqrt{\beta_o}\tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right)} \qquad (14)$$

which is equation (B2) with the proper β_o substitution.

Next we want to find the distance the rocket coasts as a function of time from motor burnout. We can make a first order differential equation relating distance (S) and time (t) by again using the substitution:

$$V = \frac{dS}{dt}$$

thus, for equation (13) we get:

$$\frac{dS}{dt} = \sqrt{\beta_o}\tan\left\{\tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right) - \frac{gt}{\sqrt{\beta_o}}\right\} \qquad (15)$$

A reduction in complexity can be accomplished by multiplying both sides of equation (14) by $\frac{g}{\sqrt{\beta_o}}$.

$$\frac{g[t_c]}{\sqrt{\beta_o}} = \frac{g}{\sqrt{\beta_o}}\left[\frac{1}{g}\sqrt{\beta_o}\tan^{-1}\frac{V_B}{\sqrt{\beta_o}}\right] = \tan^{-1}\frac{V_B}{\sqrt{\beta_o}} \qquad (16)$$

thus, (15) becomes:

$$\frac{dS}{dt} = \sqrt{\beta_o}\tan\left\{\frac{gt_c}{\sqrt{\beta_o}} - \frac{gt}{\sqrt{\beta_o}}\right\}$$

$$\frac{dS}{dt} = \boxed{V = \sqrt{\beta_o}\tan\left[\frac{g}{\sqrt{\beta_o}}(t_c - t)\right]} \qquad (17)$$

Note that the velocity (equation 17) is now in the form as presented in equation (B4). Now rearranging equation (17) and integrating dS from the burnout displacement (S_B) to some displacement (S) and dt from the burnout time (t_B) to the time (t) corresponding to the displacement (S), we obtain:

$$\int_{S_B}^{S} dS = \sqrt{\beta_o}\int_{t_B}^{t}\tan\left[\frac{g}{\sqrt{\beta_o}}(t_c - t)\right]dt$$

and can use the standard integral form on page 110 of reference 5 to solve the right-hand side;

$$\int \tan X \, dX = -\ln \cos X$$

in this situation our $X = \frac{g}{\sqrt{\beta_o}}(t_c - t)$

and so the differential dX must have the value

$$dX = -\frac{g}{\sqrt{\beta_o}}$$

The factor is included as before and the result is:

$$\int_{S_B}^{S} dS = \frac{\sqrt{\beta_o}}{\frac{-g}{\sqrt{\beta_o}}} \int_{t_B}^{t} \tan\left[\frac{g}{\sqrt{\beta_o}}(t_c - t)\right]\left(\frac{-g}{\sqrt{\beta_o}}\, dt\right)$$

which is now integrable. Thus:

$$S - S_B = -\frac{\beta_o}{-g} \ln \cos\left[\frac{g}{\sqrt{\beta_o}}(t_c - t)\right]$$

$$-(-)\frac{\beta_o}{-g} \ln \cos\left[\frac{g}{\sqrt{\beta_o}}(t_c - t_B)\right]$$

In order to have the burnout conditions as a reference point to measure coast altitude and delay time, we set $S_B = 0$ and $t_B = 0$, our equation becomes:

$$S = +\frac{\beta_o}{g} \ln \cos\left[\frac{g}{\sqrt{\beta_o}}(t_c - t)\right] - \frac{\beta_o}{g} \ln \cos\left(\frac{g}{\sqrt{\beta_o}} t_c\right) \qquad (18)$$

it can readily be seen that when (t) the flight time equals the coast time to peak (t_c) the rocket will have reached its maximum altitude from burnout (S_c).

$$\boxed{S_c = \frac{-\beta_o}{g} \ln \cos\left(\frac{g}{\sqrt{\beta_o}} t_c\right)} \qquad (19)$$

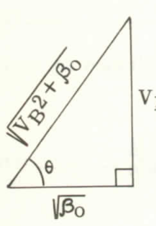
Since $\cos(0) = 1$ and $\ln(1) = 0$ therefore $\ln \cos(0) = 0$. Note that equation (19) is an alternate form of equation (B1) and is useful when coast time (t_c) is also computed.

Substitution for the value of t_c from equation (14) into this latest equation yields:

$$S_c = \frac{-\beta_o}{g} \ln \cos\left[\frac{g}{\sqrt{\beta_o}}\left(\frac{\sqrt{\beta_o}}{g} \tan^{-1} \frac{V_B}{\sqrt{\beta_o}}\right)\right]$$

$$S_c = \frac{-\beta_o}{g} \ln \cos \tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right) \qquad (20)$$

The dual trigonometric functions suggests some simplifying can be done. Let us look at the right triangle:



(The value for the hypotenuse is found by using the Pythagorean Theorem.)

Now: $\tan\theta = \frac{V_B}{\sqrt{\beta_o}}$

or: $\theta = \tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right)$

also: $\cos(\theta) = \frac{\sqrt{\beta_o}}{\sqrt{V_B^2 + \beta_o}} = \cos\left(\tan^{-1} \frac{V_B}{\sqrt{\beta_o}}\right)$

Thus our coasting altitude formula (20) can be simplified to:

$$S_c = \frac{-\beta_o}{g} \ln \frac{\sqrt{\beta_o}}{\sqrt{V_B^2 + \beta_o}}$$

$$= \frac{-\beta_o}{g} \ln \frac{1}{\sqrt{\frac{V_B^2}{\beta_o} + 1}}$$

$$= \frac{-\beta_o}{g} \ln\left(\frac{V_B^2}{\beta_o} + 1\right)^{-1/2}$$

$$= -(-1/2)\frac{\beta_o}{g} \ln\left(\frac{V_B^2}{\beta_o} + 1\right)$$

$$\boxed{S_c = \frac{\beta_o}{2g} \ln\left[1 + \frac{V_B^2}{\beta_o}\right]}$$

This is equation (B1) for the coast altitude as can be verified by substitution of the terms comprising (β_o).

The general equation for distance at any time between burnout and peak altitudes, which is presented in the report as equation (B3), is found by substituting the value of (S_c) found in equation (19), directly into equation (18).

Thus we obtain equation (B3) as:

$$\boxed{S = S_c + \frac{\beta_o}{g} \ln \cos\left\{\frac{g}{\sqrt{\beta_o}}(t_c - t)\right\}}$$

Some of you may be interested in verifying that the equations degenerate to the drag free projectile equations when drag equals zero. This can easily be done by applying L'hopital's limit rule for $C_D \rightarrow 0$.

NOTE: An easy method for finding natural logarithms of numbers less than 1.0 that should prove useful when doing hand computations involves noting:

that $\ln(AB) = \ln A + \ln B$

and $\ln(A/B) = \ln A - \ln B$

Next, assume we want to compute ln(.572). This can easily be accomplished as follows:

$$\ln(.572) = \ln\left(\frac{5.72}{10}\right) = \ln(.572) - \ln(10)$$

$$= 1.7440 - 2.3026$$

$$\ln(.572) = -.5586$$

A remark or two concerning the use of the equations and graphs are in order at this point. No calculated value or burnout altitude, burnout velocity, coast altitude, and coast time can be regarded as perfectly exact. The formulas used are based on certain assumptions as to the magnitude and repeatability of the average thrust durations for each type engine. Also, the slight decrease in air density as the rocket climbs, the minor perturbations to the flight path that will surely occur, and the true propellant weight burnoff time history are not taken into consideration.

The mathematical analysis, on the other hand, is exact and perhaps elaborate and impressive to the uninitiated. The disadvantage in the using of these allegedly "precise" formulas is the possibility of being misled into thinking that the results they yield correspond exactly to the real condition. It must be kept in mind that the results in reality are just close approximations and are limited by the basic assumptions made.

In actuality for this work, as in altitude tracking, great precision in numerical work is not justified and slide rule calculations giving results to three significant figures are sufficient. Admittedly, though, the crudest results obtained using the methods of this report will be much more realistic than the grossly erroneous altitudes calculated without any consideration of aerodynamic drag effects.

Perhaps the main value of this paper lies in the fact that it is simple to use for all rocketeers and at the same time contains some scientific aspects which will keep the more advanced rocketeers busy investigating and eventually understanding the more sophisticated principles involved.

APPENDIX II

SUGGESTIONS FOR EXPERIMENTS

The following experiments will require precise altitude measurement. Setting up a two station tracking system as outlined in Estes Industries technical report TR-3, "Altitude Tracking", should be adequate. In addition, you will need a scale to weigh your rockets. A stop watch to time the ascent will also be useful.

Experiment I

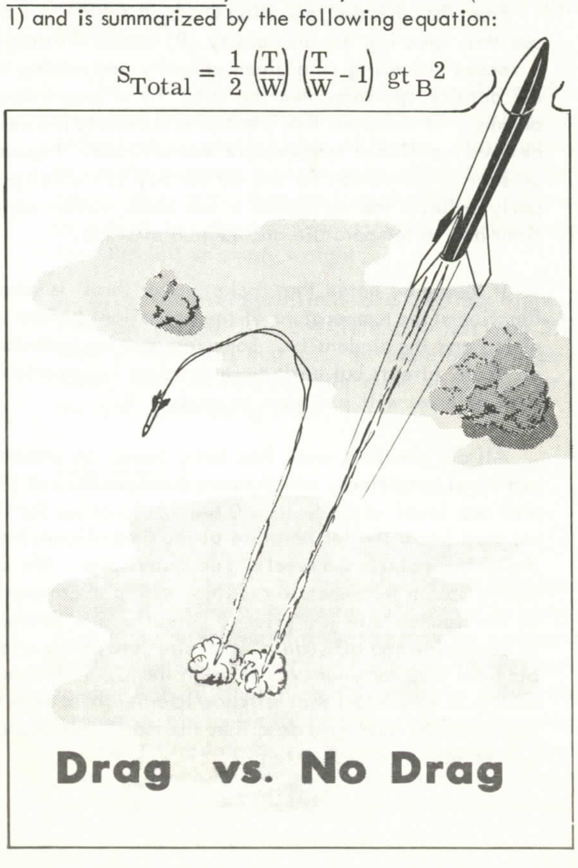
DRAG VERSUS NO-DRAG ALTITUDES

In this experiment you will perform some flight tests to compare the actual altitudes reached by one of your rockets to altitudes computed a) with drag effects, and b) without drag effects. Use an aerodynamic drag coefficient of $C_D = .75$ to start with and be sure to use a motor that has a delay time slightly greater than the computed coast time (t_C). This will insure that your model will reach its peak altitude prior to nose-cone ejection. Calculate your model's altitude in the same manner as outlined in the sample problem.

By performing this experiment with both large and small diameter rockets for both large and medium total impulse motors you will have gained a real understanding of the importance of aerodynamic drag. It will also help you realize that neglecting the effects of drag gives completely ridiculous results.

For convenience a chart giving the single stage no-drag altitudes for any weight rocket that uses any of the Estes motor types is included. Figure 10 is based on the no-drag altitude computation method presented in Model Rocket News, Volume 4, Number 2 (reference 1) and is summarized by the following equation:

$$S_{Total} = \frac{1}{2}\left(\frac{T}{W}\right)\left(\frac{T}{W} - 1\right) g t_B^{\,2}$$



Experiment 2

DRAG COEFFICIENT MEASUREMENT

Up until now all we talked about was using Mr. G. H. Stine's aerodynamic drag coefficient of $C_D = .75$ as a standard for every rocket, inasmuch as that value was determined using an accurate wind tunnel. This experiment uses a method whereby a good value for the drag coefficient of a rocket can be determined without having to build an expensive wind tunnel. All you will have to do is fly your birds a few times and measure the peak altitudes. That's your favorite pastime anyway--we hope.

First calculate total altitudes for various assumed drag coefficient values for your rocket as shown in table I. Plot the data as shown in Figure 11. Note that for ballistic coefficient (β) values greater than 10 we just use $\beta = 10$ during motor burning, and use the actual value of the coasting ballistic coefficient to obtain our coasting altitudes and times. The reason for this can be seen by looking at the thrusting graphs. The curves are almost flat at $\beta = 10$ and higher values of β will give no significant increases in either burnout velocity (V_B) or burnout altitude (S_B). The .5 ounce and .75 ounce curves for the bigger motors seems to disagree with what has just been said, but one must keep in mind that these larger motors weigh almost .75 ounce each, exclusive of the rocket. The .5 ounce and .75 ounce curves were included in these graphs for theoretical comparisons only.

Once the actual altitude reached by this rocket is measured you work backwards with this graph to determine the drag coefficient C_D. Find the point on the curve which corresponds with the measured altitude and mark it. The altitude reached will vary slightly from flight to flight so it is best to make at least three good vertical flights and then use the average drag coefficient value obtained. You can also measure the flight time with a stop watch and compare that to the plotted values of total flight time versus drag coefficient. The total flight time (t_{Total}) is simply the sum of the motor burn time (t_B) plus the coast time (t_C)

$$t_{Total} = t_B + t_C$$

Measuring both flight time and altitude for each flight gives you two data points per flight to use instead of one per flight for your drag coefficient measurement experiment. As a result you save both time and money.

You might find it very interesting to repeat the experiment using different total impulse motors. By ballasting the rocket with say an NAR standard 1 ounce payload you can plot even more curves by which the coefficient value can be verified. The results of such experimentation will be surprisingly good as long as you don't change the external shape or paint finish between flights. The results of such a test should come out looking something like the graph of figure 12 and figure 13.

LEARN TO USE A SLIDE RULE...

it will help you here...

and will be useful in later years too!

Perhaps a few words about the advantages of learning to use a slide rule are in order here. Completing the above calculations using a slide rule will take just a couple of minutes, without a slide rule probably 10 minutes. By the time you have done a couple of other motor and ballast conditions you can save quite a bit of time and effort. Admittedly, it takes at least a month of using a slide rule to become proficient at it, but by then you have acquired knowledge of another powerful tool which you will have at your disposal for the rest of your life. Any engineer or scientist will readily tell you he would be lost without his.

Experiment 3

DETAILED TRAJECTORY TIME HISTORIES

With natural logarithm, cosine, tangent, hyperbolic sine, hyperbolic cosine, and hyperbolic tangent tables (check the math section of your library for books similar to reference 5) a complete time history of the motion for the rocket whose drag coefficient was previously determined can be calculated. Using equations (A1) and (A2) for a few time increments from lift off to burnout we can calculate the corresponding velocity and distance. Similarly, after burnout we can use equations (B3) and (B4) in conjunction with (B1) and (B2) to calculate the velocity and altitude at various times. To get the actual altitude time history we must add the altitude gained during coasting to the altitude at burnout and the corresponding coast flight times to the burnout time. Using the formula

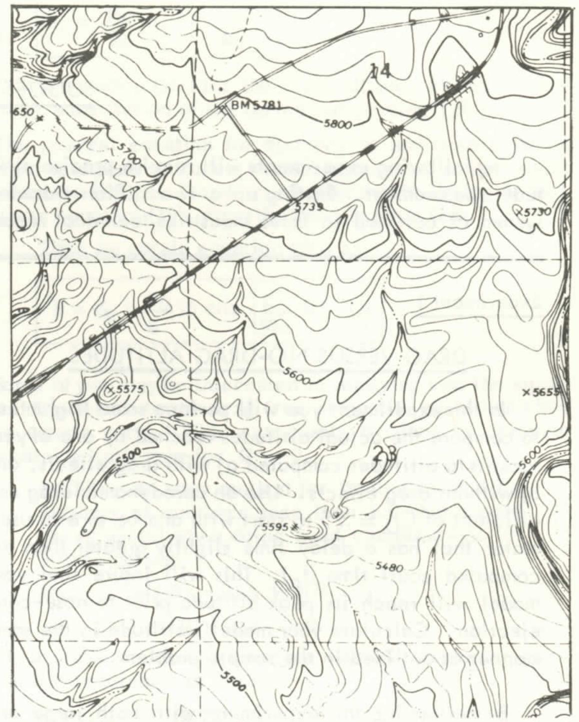
$$D = C_D A\, 1/2 \rho\, V^2 = .0001321\, C_D A\, V^2$$

we can calculate the drag in ounces at any time. With the formula

$$a = \frac{T - D}{W} - 1$$

we obtain the net acceleration in g's on our rocket at any time during the upward flight.

We can verify the accuracy of our time history by using engines which will cause ejection to occur before the peak altitude is reached. The altitude at the instant of ejection should be close to the indicated time history value at that time. For other time points a stop watch will be required. A verbal signal to the trackers can be given for any desired time after liftoff. It should be noted, however, that the recorded altitudes are more susceptible to errors during the portions of the flight when the rocket is traveling at high velocity. A time history plot of the typical performance of an A.8-4 powered 1 ounce rocket is given in Model Rocket News, Volume 4, Number 2.



Each intermediate line means a rise or fall of 20' elevation of land surface in the sample of a topological map shown above. Primary lines mark each 100', and exact figures are given for the high and low points in the topography.

Experiment 4

EFFECTS OF TEMPERATURE AND LAUNCH ALTITUDE ON ROCKET PERFORMANCE

From the basic drag equation $D = C_D A\, 1/2 \rho\, V^2$ we can see that lowering the air density (ρ) lowers the drag on a rocket traveling at a given velocity and raising the air density (ρ) increases the drag by a proportional amount. It turns out that atmospheric density is a well-defined function of temperature and altitude. Figure 9 presents a correction for the air density (ρ) which properly reflects the variations in our basic motion equations due to temperature and launch altitude.

It should be noted that rocket motor thrust is also a function of the temperature of the propellant before ignition and the air density. Some research has been done on these effects but until more detailed information is available we will just have to neglect it.

All our previous work has been based on standard sea level conditions, which means a temperature of 59°F and sea level altitude (H = 0 feet). To allow for this we must know the temperature at the time of launch and the altitude above sea level of our launch site. We can easily obtain temperature readings with a thermometer at the launch site just prior to lift-off. The best way to determine the altitude of your site is to obtain a topological map for your area through the U. S. Government. A free folder that tells how to order these maps for any location (and also describes the map symbols used) can be obtained by writing:

U. S. Geological Survey
Denver 25, Colorado, or
Washington 25, D. C.

The topological maps are only 25¢ each and are very interesting in themselves. Some libraries carry sets of them and some of the larger cities have Geological Survey offices at which you can browse through these maps at your leisure.

Once you have this data you can try to verify the effects on peak altitudes. Since your launch altitude is fixed, it would probably be easiest to make flights for which only temperature is a variable. (Flights in early morning when it is cool and also during the warmer afternoon temperatures should prove to be adequate). By calculating total altitudes for our rocket at different temperatures we can generate a theoretical graph as shown in figure 14. This graph should contain all information relative to the rocket as included on this sample. Note that points with appropriate comments have been included to represent recorded flight test data.

If nothing else, it should be enlightening to consider such facts as that the clubs in Denver, Colorado (altitude above sea level H= 5000 feet) who try for altitude records on hot days have a definite advantage over the rest of us in the U. S. A. If a Denver club is out in 90 degree weather the peak theoretical altitude reached by our 1 ounce A.8-4 sample problem rocket becomes 780 feet. This is a very good improvement over the 700 foot altitude obtained under the standard conditions of sea level altitude and 59°F.

We anxiously await news of new altitude records obtained by those who drive up Pike's Peak in Colorado (elevation 14,110 feet) to fly their birds. Launched from the top of the mountain at a mid-afternoon temperature of 40°F our 1 ounce A.8-4 sample problem rocket would reach 855 feet.

The drag coefficient measurement experiment can be refined by including temperature and launch altitude effects.

The assumption that air density is uniform through the entire flight of a model is not particularly valid for rockets which reach higher altitudes. It might be wiser to use the average altitude upon which to base density corrections, rather than the launch altitude. This would be similar to using the average weight during thrusting than the full or empty weight.

Using the average altitude of a 2000 foot flight would increase the ballistic coefficient by less than 3%. In view of the errors that arise in tracking at such altitudes, such calculations for most of us may not merit the time spent doing them.

Experiment 5

CLUSTERS

Clustered rockets are primarily used for heavier payloads. It would probably be very useful to know how high a cluster-powered rocket can go for various payload weights and what ejection delay times should be used. Actually, plotting this data as a function of weight will be quite informative. Super-imposing the corresponding data for a single motor booster of the same shape and weight will give a clear understanding of the performance improvements obtained by clustering.

Experiment 6

MULTIPLE STAGE ROCKETS

Perhaps one of the most interesting comparisons one can make involves comparing the altitude reached by a three-stage rocket using identical "type" motors (such as 1/2A.8-0, 1/2A.8-0, and 1/2A.8-4) to a two-stage rocket of identical initial total weight, size and shape which has a cluster of two motors for the first stage (such as two 1/2A.8-0 motors in the first stage and a 1/2A.8-4 in the second stage). Both rockets have the same total impulse input, but which will go higher? Using the methods of this report you can predict the results with confidence before the firing button is pushed and actual tracking measurements are made.

As one becomes more familiar with the effects of drag on different shape and size rockets through the use of this report, one eventually will be able to follow the above procedures even before one starts building more complex original rockets in order to decide, in fact, what is the best way to accomplish a desired mission. This same type of pre-flight performance analysis would carry the name "optimization study" if done by the aerospace engineers and scientists who design our country's big birds.

Figure 10

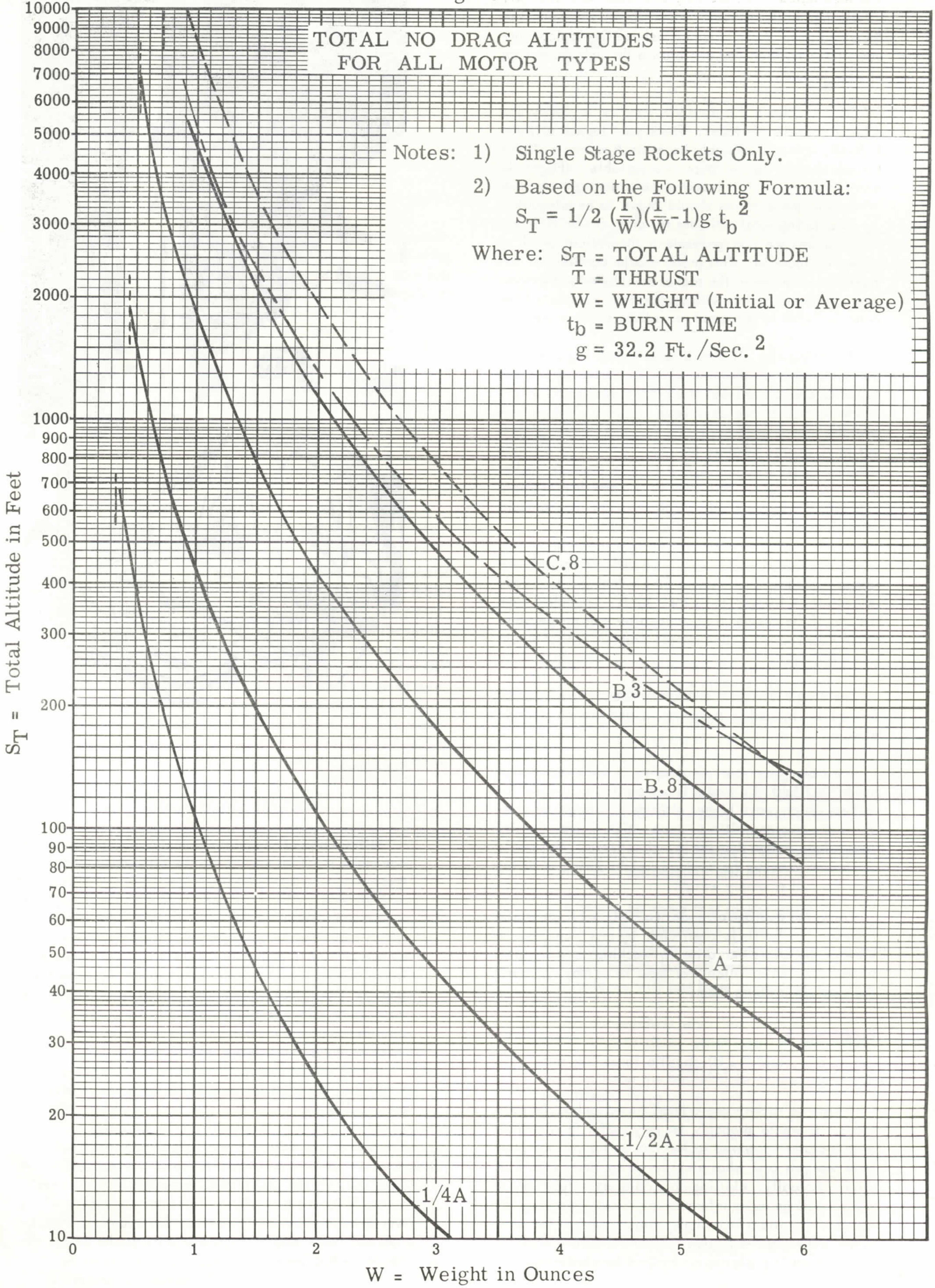

TOTAL NO DRAG ALTITUDES
FOR ALL MOTOR TYPES
Notes: 1) Single Stage Rockets Only.
2) Based on the Following Formula:
$S_T = 1/2\ (\frac{T}{W})(\frac{T}{W}-1)g\ t_b^{\ 2}$
Where: S_T = TOTAL ALTITUDE
T = THRUST
W = WEIGHT (Initial or Average)
t_b = BURN TIME
g = 32.2 Ft./Sec. ^{2}
S_T = Total Altitude in Feet
W = Weight in Ounces
C.8
B 3
B.8
A
1/2A
1/4A
10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
900
800
700
600
500
400
300
200
100
90
80
70
60
50
40
30
20
10
0
1
2
3
4
5
6

TABLE I

CALCULATIONS FOR THE DRAG COEFFICIENT MEASUREMENT EXPERIMENT						
ROCKET: SPURTNIK II MOTOR: B 3-7 AVERAGE THRUST (T): 52.7 Oz. BURN TIME (t_B): .35 Sec. BODY TUBE: BT-20	INITIAL WEIGHT WITH MOTOR $W_I =$ 2 Oz. 1/2 PROPELLENT WEIGHT $\frac{1}{2}W_P =$ .1106 Oz. PROPELLENT WEIGHT $W_P =$ .2212 Oz. $W_I - \frac{1}{2}W_P =$ 1.8894 Ounces $W_I - W_P =$ 1.7788 Ounces					
ASSUMED DRAG COEFFICIENT C_D	.1	.2	.4	.6	.8	1.0
FORM FACTOR $C_D A$	.0425	.0850	.170	.255	.34	.425
BALLISTIC COEFFICIENT DURING THRUSTING $\beta = \frac{W_I - \frac{1}{2}W_P}{C_D A} = \frac{\text{Ounces}}{\text{Inch}^2}$	44.3	22.2	11.1	7.41	5.55	4.43
BURNOUT ALTITUDE S_B in FEET	52.5 ft.	52.5 ft.	52.5 ft.	52.5 ft.	52 ft.	52 ft.
BURNOUT VELOCITY V_B in Feet/Second	300	300	300	297	295	292
BALLISTIC COEFFICIENT DURING COASTING $\beta = \frac{W_I - W_P}{C_D A} = \frac{\text{Ounce}}{\text{Inch}^2}$	41.8	20.9	10.45	6.97	5.22	4.18
COAST ALTITUDE S_C in FEET	1230 ft.	1110 ft.	940 ft.	815 ft.	715 ft.	645 ft.
COAST TIME t_C in SECONDS	8.55	8.0	7.1	6.4	5.9	5.5
TOTAL ALTITUDE $S_{total} = S_B + S_C$	1283	1163	993	868	767	697
TOTAL FLIGHT TIME $t_{total} = t_B + t_C$	8.9	8.35	7.45	6.75	6.25	5.85

$C_D = 0$ or ZERO DRAG COMPUTATIONS

DRAG FREE ACCELERATION IN Ft/Sec.^{2}; $a = \left(\frac{T}{W} - 1\right) 32.2 = \left(\frac{52.7}{1.8894} - 1\right) 32.2$

$a = (27.9 - 1)\,32.2 = 26.9\,(32.2) = 866$ ft/sec^2

$V_B = a t_B = (866)(.35) = 303$ ft/sec

$S_B = \frac{1}{2} V_B t_B = 1/2\,(303).35 = 53.0$ ft.

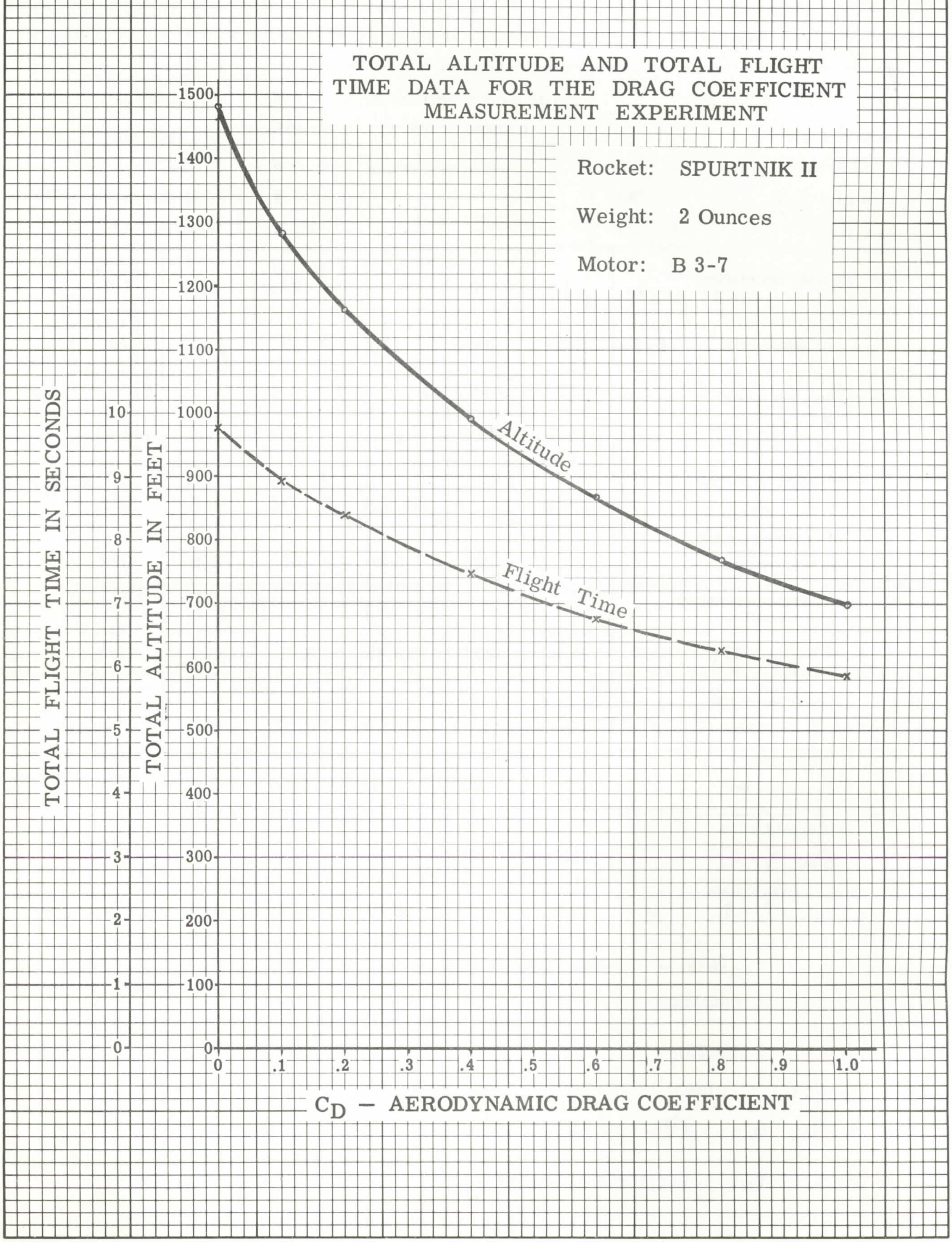
$t_C = V_B/g = V_B/32.2 = \frac{303}{32.2} = 9.41$ sec.

$S_C = 1/2\, V_B t_C = 1/2\,(303)\,9.41 = 1425$ ft.

$S_{total} = S_B + S_C = 53 + 1425 = 1478$ ft.

$t_{total} = t_B + t_C$
$= .35 + 9.41 = 9.76$ sec.

Figure 11


TOTAL ALTITUDE AND TOTAL FLIGHT TIME DATA FOR THE DRAG COEFFICIENT MEASUREMENT EXPERIMENT
Rocket: SPURTNIK II
Weight: 2 Ounces
Motor: B 3-7
Altitude
Flight Time
TOTAL FLIGHT TIME IN SECONDS
TOTAL ALTITUDE IN FEET
0
1
2
3
4
5
6
7
8
9
10
0
100
200
300
400
500
600
700
800
900
1000
1100
1200
1300
1400
1500
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
C_D — AERODYNAMIC DRAG COEFFICIENT

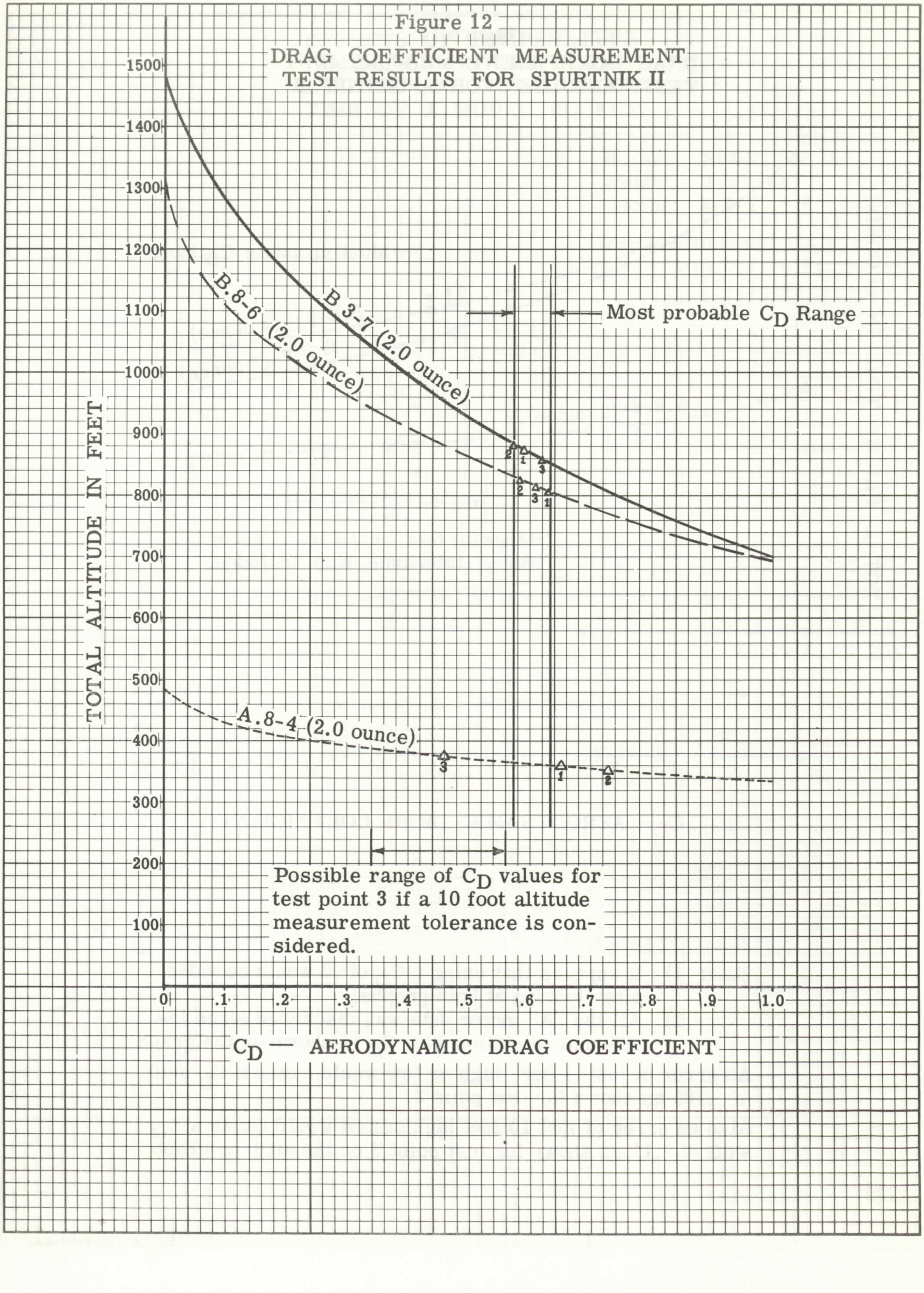
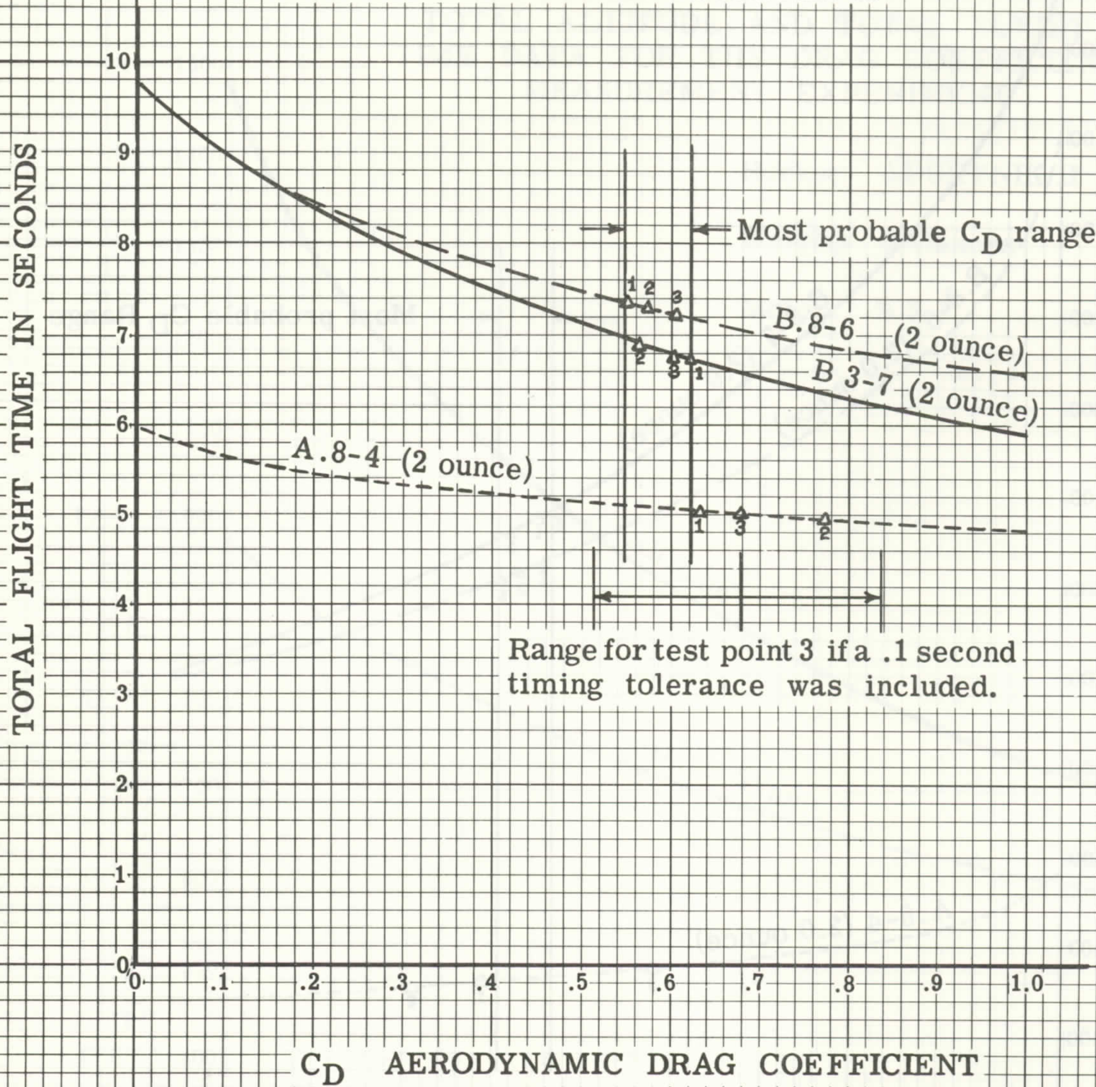


Figure 12
DRAG COEFFICIENT MEASUREMENT
TEST RESULTS FOR SPURTNIK II
Most probable C_D Range
B 3-7 (2.0 ounce)
B.8-6 (2.0 ounce)
A.8-4 (2.0 ounce)
Possible range of C_D values for test point 3 if a 10 foot altitude measurement tolerance is considered.
TOTAL ALTITUDE IN FEET
1500
1400
1300
1200
1100
1000
900
800
700
600
500
400
300
200
100
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
C_D — AERODYNAMIC DRAG COEFFICIENT

Figure 13
DRAG COEFFICIENT MEASUREMENT
TEST RESULTS FOR SPURTNIK II



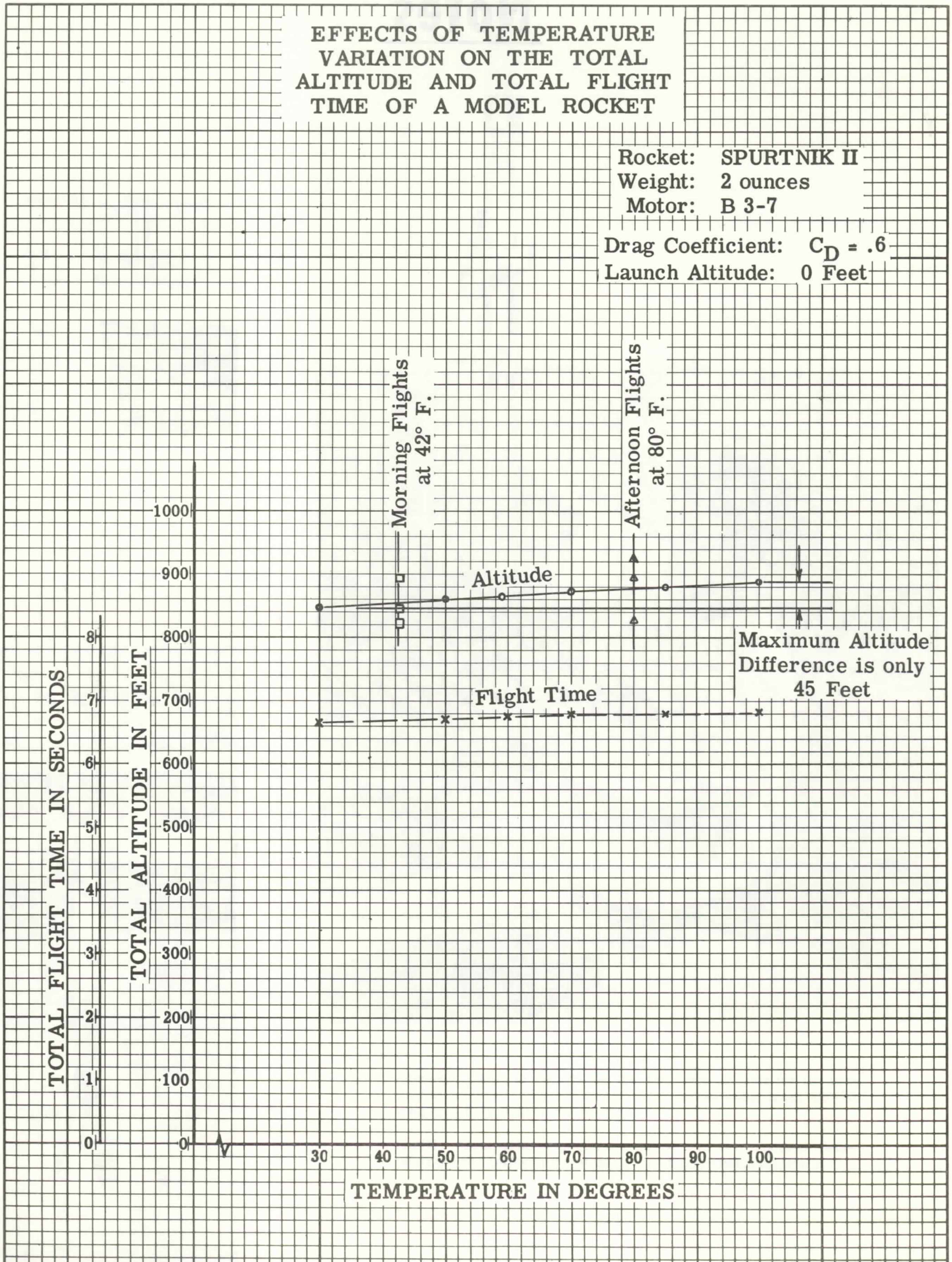
Note: 1) It appears that using a C_D of .6 would be valid after looking at both graphs.

2) All the "A" motor data was discarded since inadequate variation with C_D is shown on the graph.

An error of just .1 sec. high or low would cause a C_D shift of .165 as shown for test point 3.

Also an error of just 10 feet high or low would cause a C_D shift of .11 as shown for test point 3 on the altitude graph.

Figure 14


EFFECTS OF TEMPERATURE VARIATION ON THE TOTAL ALTITUDE AND TOTAL FLIGHT TIME OF A MODEL ROCKET
Rocket: SPURTNIK II
Weight: 2 ounces
Motor: B 3-7
Drag Coefficient: C_D = .6
Launch Altitude: 0 Feet
Morning Flights at 42° F.
Afternoon Flights at 80° F.
Altitude
Flight Time
Maximum Altitude Difference is only 45 Feet
TOTAL FLIGHT TIME IN SECONDS
TOTAL ALTITUDE IN FEET
TEMPERATURE IN DEGREES
0
1
2
3
4
5
6
7
8
0
100
200
300
400
500
600
700
800
900
1000
30
40
50
60
70
80
90
100

MODEL ROCKETRY

ESTES

SCIENTIFIC SPACE AGE HOBBY